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Sensitive Dependence on Initial Conditions for an Example of Markov Map: Skewed Doubling Map

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Abstract Markov map is one example of interval maps where it is a piecewise expanding map and obeys the Markov property. One well-known example of Markov map is the doubling map, a map which has two subintervals with equal partitions. In this paper, we are interested to investigate another type of Markov map, the so-called skewed doubling map. This map is a more generalized map than the doubling map. Thus, the aims of this paper are to find the fixed points as well as the periodic points for the skewed doubling map and to investigate the sensitive dependence on initial conditions of this map. The method considered here is the cobweb diagram. Numerical results suggest that there exist dense of periodic orbits for this map. The sensitivity of this map to initial conditions is also verified where small differences in initial conditions give different behaviour of the orbits in the map.

Keywords Markov map; discrete dynamical system, skewed doubling map; sensitive dependence on initial conditions.

Mathematics Subject Classification 39A11

1 Introduction

Markov map can be used to study the long-term behavior of a discrete-time process such as in finance, economics and population processes. One example of the well-known Markov map is the doubling map. This map is a representation of the real-world problems that deals for two events with equal probabilities considering this map has two equal partitions. This paper will look at a more generalized case than the doubling map where there are also cases where two events do not necessarily happen with equal probabilities. The objectives of this paper are to find the fixed points as well as periodic points for the skewed doubling map and to show the existence of the sensitive dependence on initial conditions in the skewed doubling map.

This paper is organized as follows. In this section, we review some basic definitions in dynamical systems such as orbit, fixed point and periodic point. The Markov map with its properties is also discussed here. In Section 2, we explain briefly about the doubling map as our motivation for this research. The cobweb method and the generalized map (the skewed doubling map) is described in Section 3. We elaborate our numerical results in Section 4.

In this paper, we consider a dynamical system in the form of the discrete time map

$$x_{n+1} = T(x_n),$$

where $x \in \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$. This map is generated by iteration of map T.

Definition 1 (Orbit, Fixed Point and Periodic Point [1]) *Let $T : \mathbb{R}^n \to \mathbb{R}^n$. The* orbit *of a point $x_0 \in \mathbb{R}^n$ is*

$$\{T^n(x_0)\}_{n=0}^{\infty} = \{x_0, T(x_0), T(T(x_0)), \ldots, T^n(x_0), \ldots\},$$

where T^n denotes the nth iterate of T, i.e.,

$$T^n(x_0) = T(T(T(\ldots T(T(x_0)) \ldots))), \quad n \geq 0.$$

The point $x_0 \in \mathbb{R}^n$ is a fixed point for $T : \mathbb{R}^n \to \mathbb{R}^n$ if:

$$T(x_0) = x_0.$$

The point $x_0 \in \mathbb{R}^n$ is a periodic point of period n for $T : \mathbb{R}^n \to \mathbb{R}^n$ if:

$$T^n(x_0) = x_0,$$

for some $n > 0$. Therefore we say n is the period of x_0 and $\{x_0, T(x_0), T^2(x_0), \ldots, T^{n-1}(x_0)\}$ is a period-n *orbit of T.*

1.1 Markov Maps

A Markov map consists of finite partitions of subintervals; is a very useful tool in the theory of dynamical systems, where it allows one to use the methods of symbolic dynamics. We follow the notations from Pollicott and Yuri [2] and Jenkinson and Pollicott [3]. In here we discuss the Markov map's properties for both doubling and skewed doubling maps followed by discussion of the invariant measures for both maps.

Let $I = [0, 1]$ be an interval. Let also $\mathcal{I} = \{I_i\}_{i=1}^{k}$ be a *partition* of the interval I into a finite number of closed sub-intervals $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, k$, with endpoints $0 = x_0 < x_1 < \cdots < x_k = 1$.

Definition 2 (Markov Map [2]) *We consider a map $T : I \to I$ which are continuously differentiable functions C^1 and monotone for each open intervals int$(I_i) = (x_{i-1}, x_i)$ and in order for T to be a Markov map, therefore it must satisfy the following properties:*

1. *Piecewise expanding: There exists $|T'(x)| > \lambda > 1$ for all $x \in [0, 1]$.*

2. *Markov property: If $T(\mathrm{int}(I_i)) \cap \mathrm{int}(I_j) \neq \emptyset$, then $T(\mathrm{int}(I_i)) \supset \mathrm{int}(I_j)$ for $i, j = 1, \ldots, k$.*

For this piecewise expanding Markov interval map, we can define a $k \times k$ matrix A with entries either 0 or 1 as in the following definition.

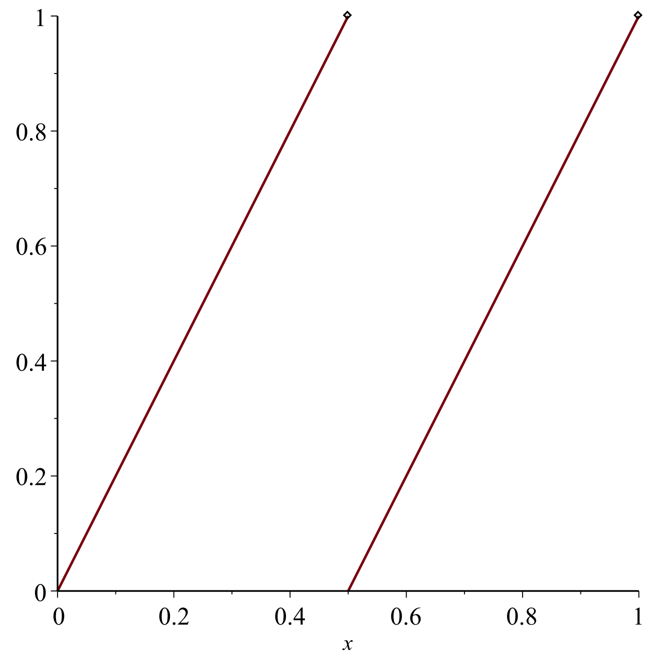

Figure 1: The Doubling Map T with Equal Partition

Definition 3 (Transition Matrix [2]) *A transition matrix A is defined by*

$$A(i,j) = \begin{cases} 1 & \textit{if } T(int(I_i)) \cap int(I_j) \neq \emptyset, \\ 0 & \textit{if } T(int(I_i)) \cap int(I_j) = \emptyset. \end{cases} \tag{1}$$

The first condition in (1) can also be written as $T(\text{int}(I_i)) = \bigcup_{j:A(i,j)=1} I_j$. In this case, we call $\mathcal{I}$ the *Markov partition* for T, and this Markov partition is not unique since any *refinement* $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{I}$ is also a Markov partition [3]. Note that if $\mathcal{I}, \mathcal{J}$ are partitions, then $\mathcal{I} \bigvee \mathcal{J} = \{I_i \cap J_j : I_i \in \mathcal{I}, J_j \in \mathcal{J}\}$.

2 Background Motivation: The Doubling Map

Doubling map is an example of Markov map which is also a discrete dynamical system. The explanation in this section is referred to Pollicott and Yuri's book [2]. We consider the one-dimensional doubling map $T : [0, 1) \to [0, 1]$ defined by

$$T(x) = 2x \pmod 1, \tag{2}$$

with the associated partitions $\{I_1, I_2\}$ where $I_1 = [0, 1/2)$ and $I_2 = [1/2, 1]$. Note that we can also write (2) as follows:

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2, \\ 2x - 1 & \text{if } 1/2 \leq x \leq 1. \end{cases} \tag{3}$$

We show the graph of doubling map in Figure 1.

Note that this map satisfies property (i) in Definition 2 where it is piecewise expanding since the slope for every partition is 2, i.e. $T'(x) = 2 > 1$. This map also satisfies the

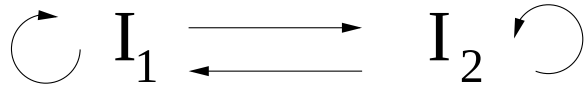

Figure 2: The Transition Graph for Doubling Map T.

Markov property since the images of I_1 and I_2 is equal to the union of the two partitions, i.e. $T(0,1/2) = T(1/2,1) = (0,1) \supset (0,1/2) \cup (1/2,1)$. For this map, the symbolic dynamics is captured by the *transition graph* given schematically in Figure 2. Hence, the transition matrix for this map is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4}$$

3 Methodology

To verify the sensitiveness of dependence on initial conditions in a Markov map, we may use the cobweb plot method. In this section, we discuss the construction of the cobweb diagram and explain how this diagram can be used to show the sensitive dependence on initial condition.

3.1 Cobweb Plot to Show Sensitive Dependence on Initial Conditions

To plot the cobweb diagram, first we sketch the graph of the function T together with the diagonal line $T(x) = x$. Then we represent the orbits under map f by a cobweb or staircase diagram as follows [4]:

1. Draw vertical line $T(x) = x$ at the initial condition.
2. Draw horizontal line from here back to $T(x) = x$.
3. Repeat.

Cobweb plot is particularly useful to study the behaviour of orbits of initial conditions of the discrete map. In this paper, we choose some nearby initial conditions to show that the map is sensitive to the initial conditions. In dynamical systems, especially in the subject of chaos, small perturbations in initial conditions may lead to major changes of the behaviour of the orbits.

Definition 4 (Alligood *et al.* [1]) *Let T be a map on $\mathbb{R}$. A point x_0 has sensitive dependence on initial conditions if there is a nonzero distance d such that some points arbitrarily near x_0 are eventually mapped at least d units from the corresponding image of x_0. More precisely, there exists $d > 0$ such that any neighbourhood N of x_0 contains a point x such that $|T^k(x) - T^k(x_0)| \geq d$ for some nonnegative integer k.*

3.2 Generalized Model: Skewed Doubling Map

In this section, we consider the generalization of the doubling map (3) to a *skewed doubling map*. It is named such due to the division of the interval into $[0, s)$ and $[s, 1]$ which is not

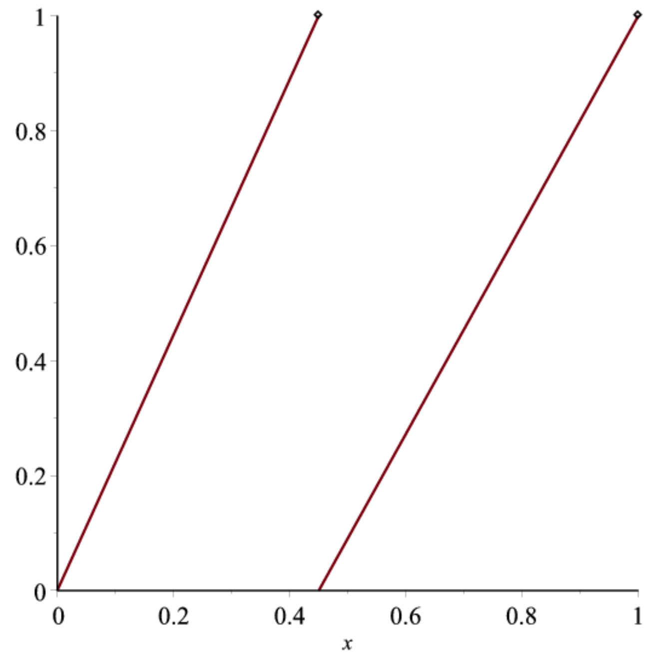

Figure 3: The Skewed Doubling Map T_s in (5) with Unequal Partition, for $s = 0.45$.

necessarily symmetric; and the usual doubling map is the special case for $s = 1/2$. This skewed doubling map has been studied by some authors including Georgiou *et al.* [5]. In 2016, Roslan and Ashwin [6] have considered this map as the base map for their skew product dynamical system where this map is also called as piecewise expanding linear map.

We consider the one-dimensional map $T_s : [0, 1] \to [0, 1]$

$$T_s(x) = \begin{cases} \dfrac{x}{s} & \text{if } 0 \le x < s, \\ \dfrac{x-s}{1-s} & \text{if } s \le x \le 1, \end{cases} \tag{5}$$

where we assume that $s \in [0, 1]$ and the corresponding partition is $\{[0, s), [s, 1]\}$. This map is also a piecewise expanding map since its derivatives are bounded away from 1, i.e. $T_s'(x) = 1/s > 1$ for interval $[0, s)$ and $T_s'(x) = 1/(1 - s) > 1$ for interval $[s, 1]$. It also satisfies the Markov property such that $T(\text{int}I_i) \supset I_1 \cup I_2$ for $i = 1, 2$. In addition, this map shares the same symbolic dynamics as T and therefore has the same transition matrix as in (4) and graph as in Figure 2.

However, this map has different behaviour compared to T in terms of the orbit's visiting in certain interval. For the map T, the proportion of the orbit in T that lies in each interval is the same, i.e. $(1/2, 1/2)$, whereas in T_s the proportion of the orbit lies in $[0, s)$ is s and the proportion of the orbit lies in $[s, 1]$ is $1 - s$ where $s \neq 1/2$. The skewed doubling map is shown in Figure 4. We will use this map to show that the sensitiveness to initial conditions exists.

4 Main Results: Cobweb Plot for the Skewed Doubling Map

In this section, we discuss our numerical results obtained using Maple. From previous section, we have applied the cobweb method on the skewed doubling map. We recall that in Section 3.1, a line $T(x) = x$ is drawn on the map. In fact, from the intersection of the line with map, we obtain two fixed points, which are at $x = 0$ and $x = 1$. We show this intersection in Figure 4(a). Meanwhile Figures 4(b)-(f) show the behaviour of orbits for different initial conditions chosen for the skewed doubling map.

Besides fixed points, this map also has periodic points. From Definition 1, we have $T^n(x_0) = x_0$. This means that after certain number of period n, the values of the orbit are equal to the values of its initial conditions. There are in fact infinitely many periodic orbits for this map. We can also say that this map has dense periodic orbits. For instances, the periodic orbits for skewed doubling map are shown in Figure 4(c) and Figure 4(f). For the evolution of orbits that do not return to the original initial conditions, the orbits may lead to chaotic behaviour. Some examples of this kind of orbits are shown in Figures 4(b),(d) and (e).

To show the sensitive dependence on initial conditions, some initial conditions x_0 are chosen and the orbits $T^n(x)$ are computed for successive values of k. For examples for this map, we choose $x_0 = 0.1$ and $x_0 = 0.10255$. Although these two points are close to each other, the behaviour of the orbits of the two points are totally different. The orbit for $x_0 = 0.1$ is chaotic while for $x_0 = 0.10255$ the orbit is periodic. The results of the orbits for these points are shown in Table 4. This table shows the beginning of two separate orbits whose initial conditions differ by 0.00255. We can see that the two points begin as close together, will eventually move apart. We also show the sensitive dependence on initial conditions for $x_0 = 0.2$, $x_0 = 0.26$ and $x_0 = 0.269$ in Table 4.

Table 1: The Comparison of the Orbits of Nearly Equal Initial Conditions $x_0 = 0.1$ and $x_0 = 0.10255$ Under the Skewed Doubling Map for the First 10 Iterates which Correspond to Figures 4(b) and (c). $x_0 = 0.10255$ Generates Periodic Orbit of Period-3.

n	x_n	x_n
0	$x_0 = 0.1$	$x_0 = 0.10255$
1	0.2222	0.22788
2	0.4938	0.50640
3	0.0797	0.10255
4	0.1771	0.22788
5	0.3935	0.50640
6	0.8745	0.10255
7	0.7718	0.22788
8	0.5850	0.50640
9	0.2455	0.10255
10	0.5455	0.22788

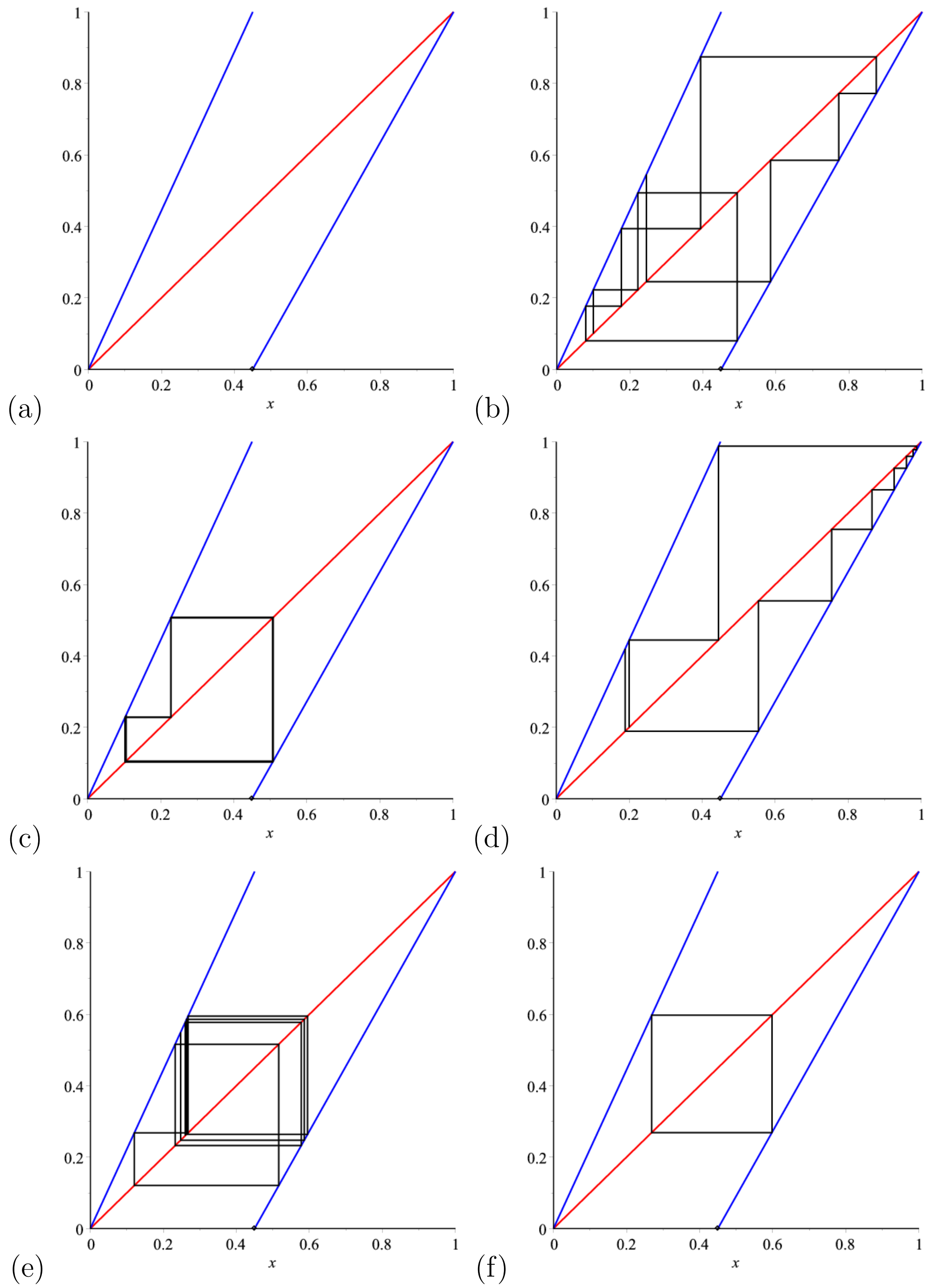

Figure 4: (a) The Intersection of the Skewed Doubling Map (i.e. the Two Right Skewed Lines) with the Diagonal Line which Indicates that There are Two Fixed Points of Period One; at 0 and 1. (b) Chaotic Orbit for Initial Condition $x_0 = 0.1$. (c) Periodic Orbit for Initial Condition $x_0 = 0.10255$ with Period 3. (d) Chaotic Orbit for Initial Condition $x_0 = 0.2$. (e) Chaotic Orbit for Initial Condition $x_0 = 0.26$. (f) Periodic Orbit for Initial Condition $x_0 = 0.269$ with Period 2.

Table 2: The Comparison of the Orbits of Nearly Equal Initial Conditions $x_0 = 0.2$, $x_0 = 0.26$ and $x_0 = 0.269$ Under the Skewed Doubling Map for the First 10 Iterates which Correspond to Figures 4(d), (e) and (f). $x_0 = 0.269$ Generates Periodic Orbit of Period-2.

n	x_n	x_n	x_n
0	$x_0 = 0.2$	$x_0 = 0.26$	$x_0 = 0.269$
1	0.4444	0.5778	0.598
2	0.9877	0.2323	0.269
3	0.9776	0.5163	0.598
4	0.9592	0.1205	0.269
5	0.9258	0.2678	0.598
6	0.8651	0.5951	0.269
7	0.7547	0.2637	0.598
8	0.5540	0.5861	0.269
9	0.1891	0.2474	0.598
10	0.4202	0.5498	0.269

5 Conclusion

In this paper, we have studied a type of Markov map which is the skewed doubling map. It has been shown that this map contains fixed points as well as periodic points. In addition, from Definition 4, our numerical results suggest that the skewed doubling map has sensitive dependence on initial conditions since the nearby initial conditions chosen lead to major differences in the journey of the orbits. In fact, this sensitivity may lead to the route of chaos.

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