MATEMATIKA, 2017, Volume 33, Number 2, 165–175

Solution for nonlinear Duffing oscillator using variable order variable stepsize block method

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Abstract Real life phenomena found in various fields such as engineering, physics, biology and communication theory can be modeled as nonlinear higher order ordinary differential equations, particularly the Duffing oscillator. Analytical solutions for these differential equations can be time consuming whereas, conventional numerical solutions may lack accuracy. This research propose a block multistep method integrated with a variable order step size (VOS) algorithm for solving these Duffing oscillators directly. The proposed VOS Block method provides an alternative numerical solution by reducing computational cost (time) but without loss of accuracy. Numerical simulations are compared with known exact solutions for proof of accuracy and against current numerical methods for proof of efficiency (steps taken).

Keywords Duffing Oscillator; multistep method; ordinary differential equations.

AMS mathematics subject classification 34A12, 34A34, 65L06, 65L05

1 Introduction

Solving higher order ordinary differential equations (ODEs) directly via multistep method have been researched by authors many authors [1–6]. In [7], Suleiman proposed a divided difference formulation with variable order stepsize (VOS) capability. Techniques and strategies suggested in [7] allows for order and stepsize change subjected to certain criteria. Based on ideas in [7], a two-point explicit and implicit block divided difference formulation was established in [8] and then implemented into a fully implicit backward difference formulation by Majid [4]. Ibrahim in [5] then derived a block backward differentiation formulae for solving stiff ODEs. In the current research, a predictor-corrector VOS algorithm is established in backward difference form for solving nonlinear duffing differential equations.

The backward difference formulation established, requires calculating the integration coefficients only once in contrast to the divided difference formulation which calculates the integration coefficients at every step size change. A recurrence relationship between explicit and implicit integration coefficients and coefficients of different orders is obtained and coded to reduce the amount of programming lines.

Systems of nonlinear higher order ordinary differential equations are found in phenomena in various fields such as physics, engineering and communication theory ranging from electrical circuits to modern telecommunications. The limitless applications of nonlinear higher order ordinary differential equations has made it the subject of interest of many researchers, particularly the Duffing oscillator.

The general form of the Duffing oscillator a second order non-linear initial value ordinary differential equation (ODE)

$$y''(t) + \delta y'(t) + \alpha y(t) + \beta y^3(t) = \gamma \sin \omega t, \tag{1}$$

with the constants $\delta, \alpha, \beta, \gamma$ and ω as a parameter.

Higher order ODEs such as the Duffing oscillator were previously reduced to a system of differential equations and solved using conventional numerical methods. The current work proposes to solve the Duffing differential equation directly using a variable order variable step multistep method, a two point block predictor-corrector (PeCe) formulation.

2 Predict-evaluate correct-evaluate backward difference mode with variable order variable step size

To formulate a two point PeCe block variable order stepsize backward difference algorithm, elements such as explicit-implicit integration coefficients and order-stepsize strategy are necessary.

2.1 Deriving the explicit-implicit integration coefficients

Consider a higher order ordinary differential equation

$$y^{(d)} = f(x, \widetilde{Y}), \tag{2}$$

with the d^{th} order ODE and the initial value condition given by $\widetilde{Y}(\alpha) = \widetilde{\eta}$ where

$$\widetilde{Y}(x) = (y, y', \ldots, y^{(d-1)}),$$
$$\widetilde{\eta} = (\eta, \eta^{'}, \ldots, \eta^{(d-1)}),$$

in the interval $\alpha \leq x \leq \beta$.

Integrating $y^{(d)}$ by $1, 2, 3, \ldots, d$ number of times and interpolating $(y, y', \ldots, y^{(d-1)})$ by the Newton-Gregory backwards difference polynomial

$$P_n(x) = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} -s \\ i \end{pmatrix} \nabla^i f_n, \qquad s = \frac{x - x_n}{h},$$

for the predictor whereas

$$P_{n+r}(x) = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} -s \\ i \end{pmatrix} \nabla^i f_{n+r}, \qquad s = \frac{x - x_{n+r}}{h}, \qquad r = 1, 2,$$

for the corrector where $j = 0, 1, \ldots, d$. Let r denote the number of blocks, thus providing Predictor:

$$y^{(d-j)}(x_{n+r}) = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}(x_n) + \int_{x_n}^{x_{n+r}} \frac{(x_{n+r} - x)^{d-1}}{(d-1)!} \sum_{i=0}^{k} (-1)^i \begin{pmatrix} -s \\ i \end{pmatrix} \nabla^i f_n dx, \tag{3}$$

Corrector:

$$y^{(d-j)}(x_{n+r}) = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}(x_n) + \int_{x_n}^{x_{n+r}} \frac{(x_{n+r}-x)^{d-1}}{(d-1)!} \sum_{i=0}^{k} (-1)^i \binom{-s}{i} \nabla^i f_{n+r}\, dx, \tag{4}$$

which can be rewritten as

Predictor:

$$y^{(d-j)}(x_{n+r}) = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}(x_n) + \int_0^r \frac{(r-s)^{d-1}}{(d-1)!} \sum_{i=0}^{k} (-1)^i \binom{-s}{i} \nabla^i f_n ds,$$

Corrector:

$$y^{(d-j)}(x_{n+r}) = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}(x_n) + \int_{-r}^0 \frac{(-s)^{d-1}}{(d-1)!} \sum_{i=0}^{k} (-1)^i \binom{-s}{i} \nabla^i f_{n+r}\, ds,$$

with coefficients denoted by the following integrals
Explicit:

$$\gamma_{r,j,i} = (-1)^i \int_0^r \frac{(r-s)^{d-1}}{(d-1)!} \binom{-s}{i} ds,$$

Implicit:

$$\gamma^*_{r,j,i} = (-1)^i \int_{-r}^0 \frac{(-s)^{d-1}}{(d-1)!} \binom{-s}{i} ds.$$

Finally, the variable order stepsize predictor-corrector backward difference algorithm has the following form

Predictor:

$$y^{(d-j)}(x_{n+r}) = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}(x_n) + h^j \sum_{i=0}^{k-1} \gamma_{r,j,i} \nabla^i f_n,$$

Corrector:

$$y^{(d-j)}(x_{n+r}) = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}(x_n) + h^j \sum_{i=0}^{k-1} \gamma^*_{r,j,i} \nabla^i f_{n+r}.$$

By mathematical induction, the relationship between integration coefficients of different orders is obtained as follows
Explicit coefficients:

$$\gamma_{r,d,0} = \gamma_{r,d-1,1}, \quad \gamma_{r,d,k} = \gamma_{r,d-1,k+1} - \sum_{i=0}^{k-1} \left(\frac{\gamma_{r,d,i}}{k-i+1} \right), \quad k = 1, 2, \ldots.$$

Implicit coefficients:

$$\gamma^*_{r,d,0} = \gamma^*_{1,d-1,1}, \quad \gamma^*_{r,d,k} = \gamma^*_{r,d-1,k+1} - \sum_{i=0}^{k-1} \left(\frac{\gamma^*_{r,d,i}}{k-i+1} \right), \quad k = 1, 2, \dots .$$

In similar manner to [9], the following recursive relationship between the explicit and implicit integration coefficients is obtained.

$$\sum_{i=0}^{\infty} \gamma^*_{r,j,i} t^i = (1-t)^r \sum_{i=0}^{\infty} \gamma_{r,j,i} t^i .$$

The next section details the order and step size strategy.

3 Order and step size

When implementing a variable order stepsize algorithm, the order and stepsize selection is crucial. In a VOS algorithm, the reliability of the method relies on the acceptance criteria where as the efficiency of the algorithm relies order and step size strategy. This is because a VOS multistep method depends on the back values stored. The order of a VOS algorithm can be increased depending on the previous back values stored and decreased by discarding the necessary amount of back values. An unbiased order strategy proposed by [7] adopted the selection criteria as suggested in [10].

Because of issues involving the stability and convergence of VOS techniques, Shampine and Gordon [10] recommends the restrictions on the ratio of successive steps. This is to ensure stability. Consider the current step size as h and the final step size as h_{end}. By multiplying a safety factor of R with h for a conventional estimate of h_{end} such that $h_{end} = Rh$ reduces the number of rejected steps.

In this research, a modified doubling and halving step change techniques is implemented based on the step size change algorithm introduced in [11] for Adam-Bashforth and Adams-Moulton based method in backward difference form (see Algorithm 1). Finally, the next section proceeds with the error estimate.

4 Error estimation

In this section, an estimation for the local error of each integration step is obtained similar to the approach suggested in [12]. Our estimation begins by denoting the predictor as follows

$$^{pr}y^{(d-j)}_{n+r} = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}_n + h^j \sum_{i=0}^{k-1} \gamma_{r,j,i} \nabla^i f_n . \tag{5}$$

By applying a $P_k E C_{k+1} E$ algorithm, the corrector is denoted as follows

$$^{cr}y^{(d-j)}_{n+r} = \sum_{i=0}^{j-1} \frac{(rh)^i}{i!} y^{(d-j+i)}_n + h^j \sum_{i=0}^{k-1} \gamma^*_{r,j,i} \nabla^i f_{n+r} \tag{6}$$

For computational purposes, the corrector is written in term of the predictor as follows

$$^{cr}y^{(d-j)}_{n+r} =^{pr} y^{(d-j)}_{n+r} + \gamma^*_{r,j,i} \nabla^i_{pr} f_{n+r}$$

Algorithm 1 Integration coefficients

```
Begin
  Block := 2;
  Temp := Temp1 := 1;
  For b := 0, to Block step 1
    Begin
      For j := 1, to 12 step 1
        Begin
          If (j = 0)
            Begin
              γ_{b,0,j} := Temp1;
            End
          Else
            Begin
              Temp1 = Temp1 × ((j−1)∗B)/j;
              γ_{b,0,j} := Temp1;
            End
        End
      For m := 1, to D[I] step 1
        Begin
          For j := 0, to 15 − m step 1
            Begin
              If (m = 1)
                Begin
                  Temp:= 1;
                End
              Else
                Begin
                  Temp:= γ_{b,m−1,j+1};
                End
              For t := 0, to j − 1 step 1
                Begin
                  Temp:=Temp− γ_{b,m,t}/(j+1−t);
                End
              γ_{b,m,j} :=Temp;
              If (j = 0)
                Begin
                  γ*_{b,m,0} := γ_{b,m,0};
                End
              Else
                Begin
                  γ*_{b,m,j} := γ_{b,m,j} − γ_{b,m,j−1};
                End
            End
        End
    End
End
```

with ∇^i_{pr}, as the i-th backward difference of the predictor where $j = 0, 1, \ldots, d$ and $i = 0, 1, \ldots, k$.

By Milne error estimate, the local truncation error (LTE) can be written as the following formulation

$$\widetilde{E}^{(j)}_{n+r,k} = h^j \gamma^*_{r,j,i} \nabla^k_{pr} f_{n+r}.$$

Selection of a suitable p for $\widetilde{E}^{(d-p)}_k$ to control order and step size can be found in [13]. The asymptotic validity can be established using

$$\widetilde{E}^{(d-p)}_{n+r,\ k+1} = h^{d-p} \gamma^*_{r,\ d-p,\ k+1} \nabla^{k+1} f_{n+r}, \qquad r = 1, 2.$$

5 Numerical results

Current simulation with numerical approximation for the Duffing oscillator can be obtained from works such as [14–17] and many others. The 2-Point Block Variable Order Stepsize (2PBVOS) was tested with numerous nonlinear Duffing oscillators of different orders and parameters. Results were then compared with current and conventional numerical methods. Problem 1 and 2 are non homogeneous second order differential equations where as Problem 3 is a Duffing Oscillator without any known solution. And finally, Problem 4 is a fourth order nonlinear differential equation which was intended to add a certain level of difficulty.

STEPS: total steps,
MAXE: the overall maximum error,
AVER: the average error,
MTD: the method used
2PBVOS: 2-Point Block Variable Order Stepsize
DI: direct integration,
VOSBD: VOS backward difference,
SHPM: standard homotopy perturbation,
SNM: standard numerical.

Problem 1 The equation $y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3t}$ for $0 \le x \le 100$ was obtained from [18] with initial value conditions $y(0) = \frac{1}{2}, \quad y'(0) = -\frac{1}{2}$ and $y(x) = \frac{1}{2}e^{-t}$ as the exact solution.

Problem 2 The equation $y''(x)+y(x)+y'(x)+y^2(x)y'(x) = 2\cos x - \cos^3 x$ for $0 \le x \le 100$ was obtained from [19] with initial value conditions $y(0) = 0, \quad y'(0) = 1$ and $y(x) = \sin x$ as the exact solution.

Problem 3 The equation $y''(x) + y(x) + y^3(x) = 0$ for $0 \le x \le 5$ was obtained from [20] with initial value conditions $y(0) = 1, \quad y'(0) = 0$ and without any known exact solution.

Problem 4 The equation $y''''(x) + 5y''(x) + 4y(x) - \frac{1}{6}y^3(x) = 0$ for $0 \le x \le 14$ was obtained from [21] with initial value conditions $y(0) = 0, \quad y'(0) = 1.91103, \quad y''(0) = 0, \quad y'''(0) = -1.15874$ and $y(x) = 2.1906 \sin 0.9x - 0.02247 \sin 2.7x + 0.000045 \sin 4.5x$ as the exact solution.

Table 1: Comparison of total steps and accuracy for Problems 1 and 2

TOL	MTD	Problem 1 STEPS	Problem 1 MAXE	Problem 1 AVER	Problem 2 STEPS	Problem 2 MAXE	Problem 2 AVER
10^{-2}	DI	156	4.54154(−2)	1.12922(−2)	254	8.49079(−2)	2.04794(−2)
	VOSBD	154	5.54428(−2)	9.04569(−3)	217	1.07600(−1)	3.03894(−2)
	2PBVOS	219	9.97057(−1)	1.13013(−1)	163	5.90439(−1)	6.87560(−2)
10^{-4}	DI	169	3.28968(−4)	8.48859(−5)	332	1.54704(−3)	4.72630(−4)
	VOSBD	215	4.90227(−4)	6.47465(−5)	284	1.24649(−3)	1.92899(−4)
	2PBVOS	150	2.67777(−3)	1.68773(−4)	186	3.18495(−3)	7.36456(−4)
10^{-6}	DI	173	2.19533(−5)	2.80453(−6)	382	4.24089(−5)	1.56849(−5)
	VOSBD	236	1.31698(−5)	1.91707(−6)	330	1.28039(−5)	3.26641(−6)
	2PBVOS	160	1.18496(−5)	1.32048(−6)	279	1.22989(−5)	3.85559(−6)
10^{-8}	DI	204	1.48896(−7)	3.10007(−8)	651	7.93605(−7)	1.55130(−7)
	VOSBD	224	1.82416(−7)	2.22705(−8)	499	7.27324(−7)	1.46368(−7)
	2PBVOS	192	1.95831(−7)	2.73453(−8)	609	3.19819(−7)	1.91880(−8)
10^{-10}	DI	317	1.18565(−9)	3.94604(−10)	772	7.83863(−9)	2.03721(−9)
	VOSBD	224	1.12450(−9)	2.04848(−10)	702	9.05773(−9)	1.01381(−9)
	2PBVOS	217	2.80516(−9)	1.17721(−9)	1376	2.80516(−9)	1.28110(−10)

Table 2: Numerical result for Problem 3

x	2PBVOS			SHPM	SNM
	$Tol = 1 \times 10^{-1}$	$Tol = 1 \times 10^{-5}$	$Tol = 1 \times 10^{-10}$		
0.5	7.68843(-1)	7.68802(-1)	7.68802(-1)	7.68766(-1)	7.68802(-1)
1.0	2.34320(-1)	2.33694(-1)	2.33692(-1)	2.33680(-1)	2.33692(-1)
2.0	-8.65820(-1)	-8.59353(-1)	-8.59349(-1)	-8.9323(-1)	-8.59349(-1)
3.5	-2.91355(-1)	-9.30087(-2)	-9.30110(-2)	-9.30340(-2)	-9.30130(-2)
5.0	3.19402(-1)	9.47105(-1)	9.47130(-1)	9.47107(-1)	9.47130(-1)

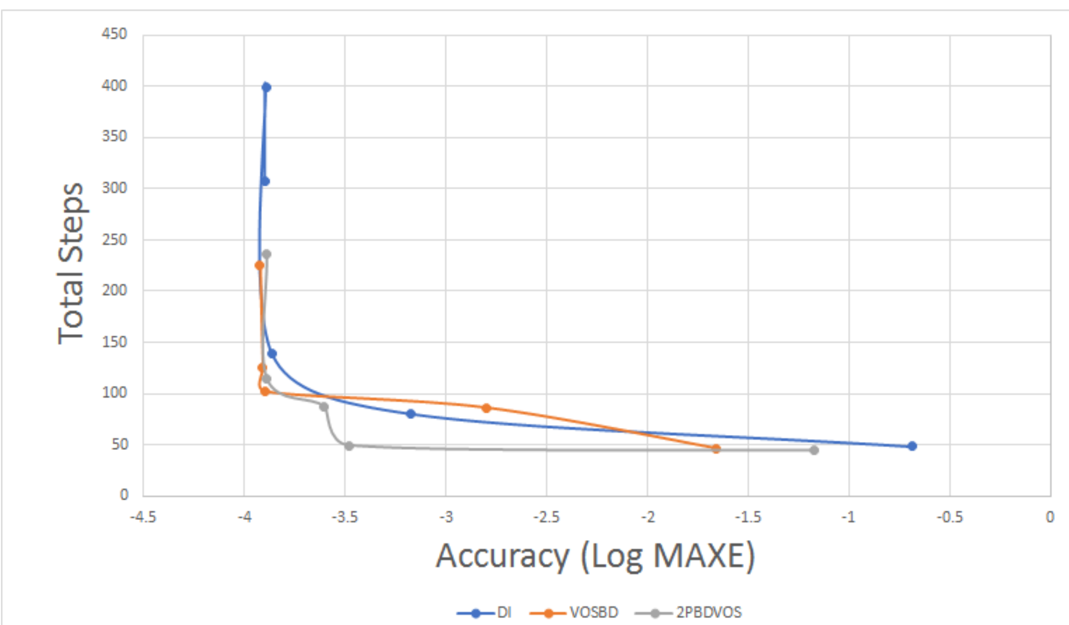

Figure 1: Accuracy of DI, VOSBD and 2PBVOS method for Problem 1

Table 3: Comparison of total steps and accuracy for Problem 4

		Problem 4		
TOL	MTD	STEPS	MAXE	AVER
10^{-2}	DI	48	2.06305(−1)	3.75201(−2)
	VOSBD	46	2.19901(−2)	4.59352(−3)
	2PBVOS	44	6.68423(−2)	1.58501(−2)
10^{-4}	DI	80	6.74572(−4)	2.44601(−4)
	VOSBD	86	1.58280(−3)	3.94428(−4)
	2PBVOS	49	3.28712(−4)	6.40115(−5)
10^{-6}	DI	139	1.37722(−4)	4.02719(−5)
	VOSBD	102	1.26373(−4)	3.15140(−5)
	2PBVOS	87	2.49624(−4)	5.12886(−5)
10^{-8}	DI	399	1.28150(−4)	3.03717(−5)
	VOSBD	126	1.23095(−4)	2.94248(−5)
	2PBVOS	114	1.29053(−4)	3.80954(−5)
10^{-10}	DI	308	1.27015(−4)	3.42987(−5)
	VOSBD	225	1.19540(−4)	3.24947(−5)
	2PBVOS	236	1.28890(−4)	4.07321(−5)

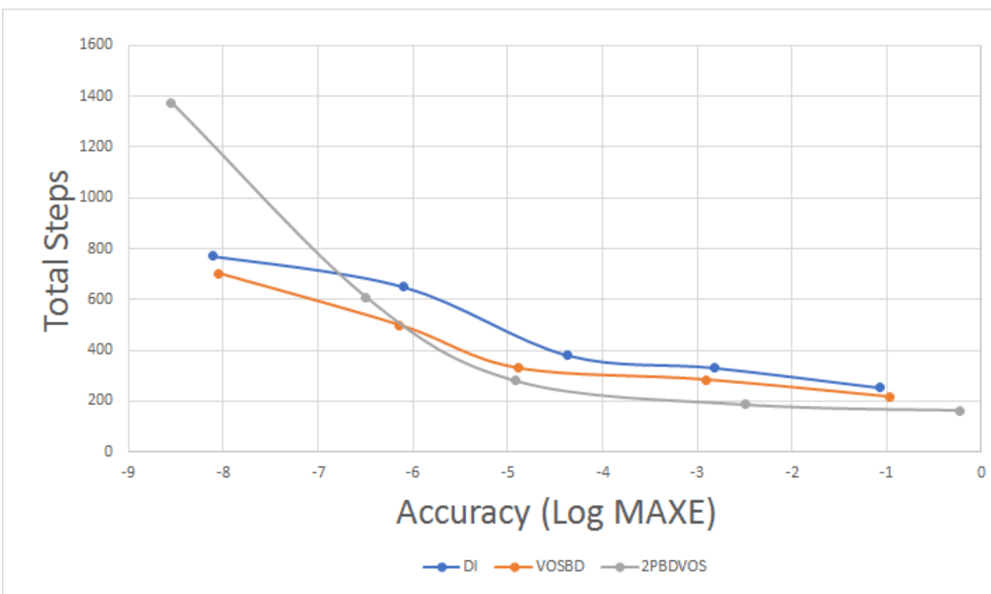

Figure 2: Accuracy of DI, VOSBD and 2PBVOS method for Problem 2

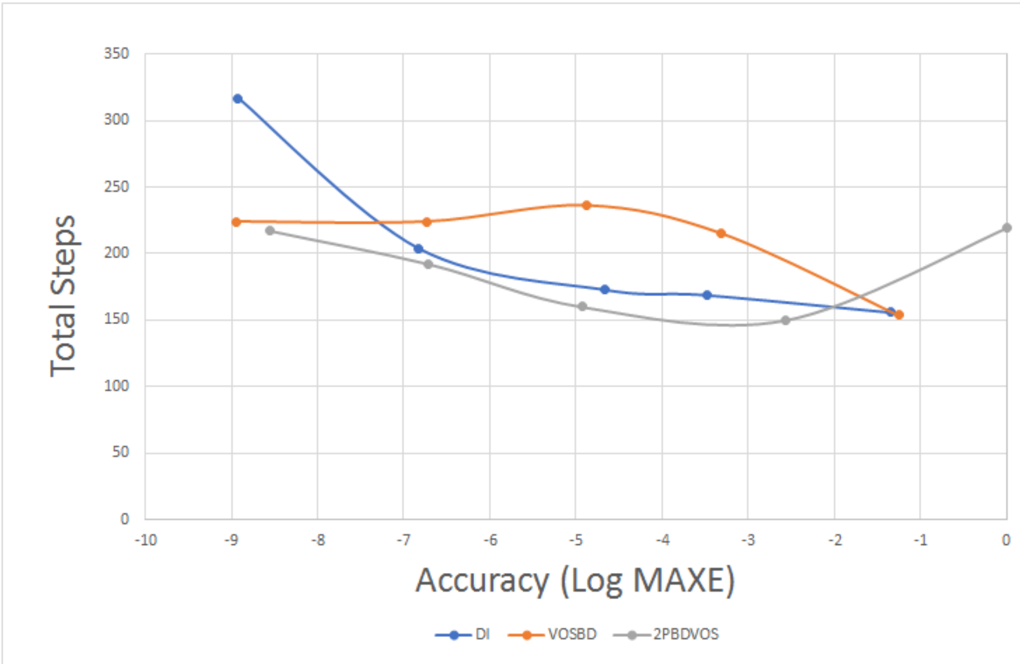

Figure 3: Accuracy of DI, VOSBD and 2PBVOS method for Problem 3

5.1 Discussion and conclusion

Table 1 displays the comparison of numerical results between three VOS methods, DI, VOS and 2PBVOS. The DI method established by Suleiman [3] is the benchmark for most VOS algorithm. In this research, the DI method acts as the standard of efficiency. An overall review of Table 1 so the competitive nature of the 2PBVOS method against the VOSBD and well within the range of efficiency provided by the DI method. For Problem 1, it is apparent that the 2PBVOS requires the least number of steps for each tolerance (with the exception when TOL=10^{-2}) while maintaining a level of accuracy similar to DI and VOSBD. On the other hand, results for Problem 2 shows the advantage of the 2PBVOS in terms of total step for larger tolerances, when TOL is between 10^{-2} and 10^{-6}. For finer tolerances (TOL 10^{-8} and 10^{-10}), the 2PBVOS method is seen to be more accurate but, with the cost of compromising the number steps.

Problem 3 is a Duffing oscillator without any known exact solution. This problem was selected to test the accuracy of the 2PBVOS method. The approximated solution obtained by the 2PBVOS method is compared with conventional methods. Table 2 provides the approximated solution for the SHPM, SNM and 2PBVOS (tolerances 10^{-1}, 10^{-5} and 10^{-10}). Results show that the 2PBVOS becomes more accurate solution when a finer TOL is used.

A higher order Duffing oscillator (order 4) is considered to observe the capability of the 2PBVOS method when dealing with more difficult problems. The numerical results in Table

3 again features the comparison between the DI, VOSBD and 2PBVOS methods. And once again, the 2PBVOS is proven to require the least number of steps without lost of accuracy.

Figure 1 to 3 illustrates efficiency of the methods. Here, efficiency of the methods is defined by the undermost curve of the provided graphs. The figures clearly show that the 2PBVOS has the undermost curve of all 3 methods with the exception of a few accuracy. In conclusion, it is apparent that the 2PBVOS method is a viable option for solving Duffing oscillators. The 2PBVOS also has the added advantage of being parallel programmable which will reduce computational cost even more.

Acknowledgments

This research has been supported by Ministry of Education (MoE) under the Fundamental Research Grant Scheme (FRGS), project number USIM/FRGS/FEM/055002/51517 and Universiti Sains Islam Malaysia (USIM) under the Short Term Grant Scheme, project number PPP/FST-0117/051000/11417.

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