Nonlinear Partial Differential Equations Model Related to Oxidation Pond Treatment System: A Case Study of mPHO at Taman Timor Oxidation Pond, Johor Bahru

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Abstract

This study presents a mathematical model examining wastewater pollutant removal through an oxidation pond treatment system. This model was developed to describe the reaction between microbe-based product mPHO (comprising Phototrophic bacteria (PSB)), dissolved oxygen (DO) and pollutant namely chemical oxygen demand (COD). It consists of coupled advection-diffusion-reaction equations for the microorganism (PSB), DO and pollutant (COD) concentrations, respectively. The coupling of these equations occurred due to the reactions between PSB, DO and COD to produce harmless compounds. Since the model is nonlinear partial differential equations (PDEs), coupled, and dynamic, computational algorithm with a specific numerical method, which is implicit Crank-Nicolson method, was employed to simulate the dynamical behaviour of the system. Furthermore, numerical results revealed that the proposed model demonstrated high accuracy when compared to the experimental data.

Keywords Oxidation pond; nonlinear PDEs; PSB; implicit Crank-Nicolson.

Mathematics Subject Classification 35B65, 65N99, 92B05.

1 Introduction

Oxidation pond technique has been acknowledged as the most popular approach among small communities due to its low construction and operating costs [1]. The construction and maintenance costs of this treatment are inexpensive compared to other recognized treatment systems

including microbial fuel cell (MFC), membrane bioreactor (MBR), and rotating biological contactor (RBC). The core procedure of an oxidation pond treatment process is the degradation of contaminants and organic matter in two conditions; where oxygen is present (aerobic) or absence (anaerobic). At each stage, existing microorganisms are used to breakdown either organic or inorganic substances of influent and to reduce organic material into more concise forms, which are carbon dioxide, water, and cell biomass.

Oxidation pond chosen for the pilot scale study is an exposed oxidation pond located at Taman Timor Oxidation Pond, Tampoi, Johor, as it has been experimentally studied by J-Biotech for three months period to observe the effects of microbe-based product (mPHO) in treating sewage. Nonetheless, there is no specific study done on this product until now. Briefly, the size of this pond is estimated about 1,909 square metres with a depth of 1.5 metres and total water volume of 2,864.13 cubic metres or 2,864,125.13 litres (refer to Figure 2 and Figure 3). However, the input and output flow rate of wastewater as well as the volume of rain may not change the volume of water in the pond as the wastewater is discharged due to overflow. In order to intensify the effectiveness of oxidation pond technique and to accelerate the population of *Phototrophic* bacteria (PSB) in the pond, mPHO containing mainly PSB have been added regularly within three months period of study between 13 November 2013 to 12 February 2014.

The product mPHO is made from selected species of PSB (refer Figure 4) manufactured by J-Biotech. About 1,375 litres of mPHO were applied to the pond throughout three months of treatment. Samples were collected at two points, which are CP1 (influent and application of mPHO) and CP2 (effluent) (refer to Figure 5). Comparison of data taken at both points CP1 and CP2 demonstrated that mPHO has a good effect in reducing the concentration of pathogenic bacteria (*E. coli* and *Coliform*), BOD, COD and other pollutants as the PSB and dissolved oxygen (DO) concentration increases.
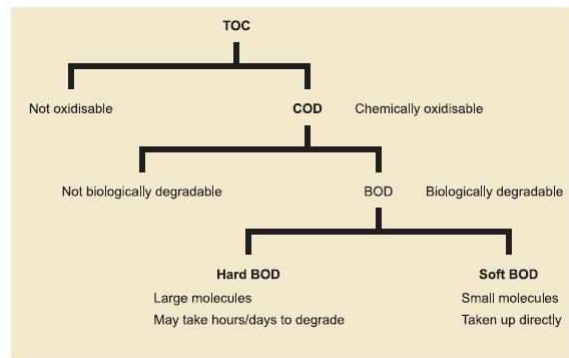

Figure 1: Relationship Between the Organic Carbon in Sewage [2]

2   Literature Review

There are several types of mathematical modelling that are widely used in wastewater treatment process. Some examples of mathematical models related to rivers and ponds are also stated in the sections below. Findings from this study demonstrated that these models were further improved using the advancement in computing by considering the dynamical structure of biological system.

Figure 2: Aerial View of Taman Timor Oxidation Pond, Johor Bahru [3]

Figure 3: Physical Condition of Taman Timor Oxidation Pond, Johor Bahru [3]

Wastewater treatment process models consider various types of wastewater problem and select proper modes of treatment as well as the design of the equipment required. Furthermore, the theories involved in the specific process such as chemical kinetics, material and energy balances are discussed. Wastewater process models can be successfully used in planning and developing several applications such as river, pond and wetland.

Mathematical modelling in solving actual problems such as water pollution has been developed since 1925 by Streeter and Phelps [4]. One of the issues of concern is to scrutinize the quality of water accessible in rivers and stabilization ponds. One of the earliest mathematical modelling of water quality given in Beck and Young [4] by using Streeter-Phelps Equation to study the relationship between BOD and DO on the River Cam, Eastern England. The model used a linear first order differential equation as

$$\frac{dD}{dt} = k_1 \; L_t - k_2 \; D, \tag{1}$$

where D is the saturation oxygen deficit, which can be derived from the dissolved oxygen concentration at saturation minus the actual dissolved oxygen concentration (g m^{-3}), k_1 is the deoxygenation rate (day^{-1}), k_2 is the reaeration rate (day^{-1}), L_t is the oxygen demand remaining at time t, and t is the elapsed time, usually in day.

This model explains how BOD and DO can vary along with observing period. Many studies have been conducted to predict the effluent quality of river and stabilization ponds. The

Figure 4: Beneficial Microbe-based Product (mPHO) Produced by J-Biotech company [3]
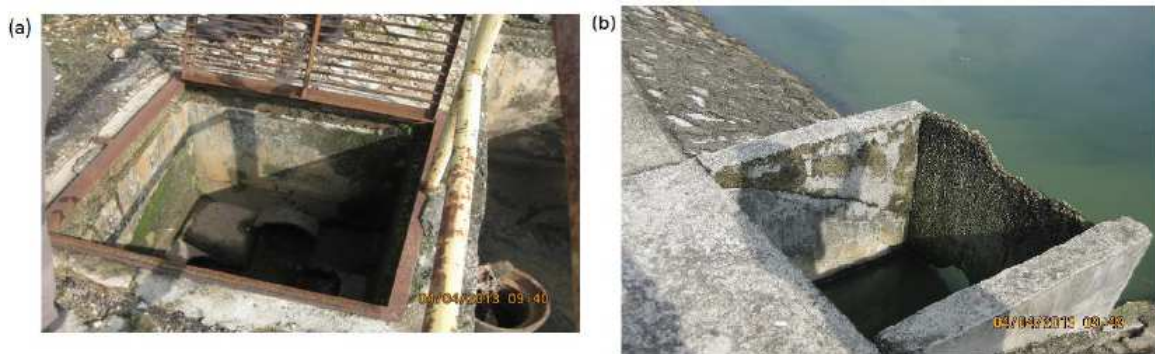

Figure 5: (a) Location of CP1 (Influent) (b) Location of CP2 (Effluent) [3]

Streeter-Phelps equation has been frequently used as a reference, while ordinary differential equations were used as the basis in relating parameters to each other. Through this model, significant relationship between parameters involved in the ecosystem is obtained. For instance, the pollutant against DO, algae against DO, and bacteria against algae.

There are also other mathematical models developed for specific problems that occur in a given locality. A study has been conducted on Tha Chin river stream in Thailand considering the effects of pollutant with dissolved oxygen [5]. This study proposed the model of two-dimensional coupled advection-dispersion equations for both state variables, respectively. The relationship between state variables was used considering that the interactions occurred between materials contaminated with oxygen will produce harmful substances. In this model, pollutant and DO concentrations were permitted to fluctuate along the length of stream and dealt as homogeneous over the cross-segment of river subject to Dobbin's criterion [6]. For simplification, the model was reduced to steady state solutions and analytically solved for simple cases.

Besides, there is another study developing mathematical model to predict the specific growth rate and biomass concentration of microbes in wastewater treatment [7]. This study suggests that the rate of bacterial growth can be defined as first order reaction

$$r_g = \frac{\mu_m\, S}{K_s + S},\tag{2}$$

where r_g is the rate of bacterial growth (time^{-1}), μ_m is the maximum specific growth rate

(time^{-1}), S is concentration of growth limiting substrate in the effluent (mass (unit volume)$^{-1}$ time^{-1}), K_s is the half velocity constant (mass (unit volume)$^{-1}$ time^{-1}). This finding presents Michaelis-Menten term in the hypothesis of growth model with the view that the growth rate of enzyme will take the same form. In biochemistry, Michaelis-Menten term expression can be considered as among the best-known models for enzyme kinetics. This model is associated with German biochemist Leonor Michaelis and Canadian physician Maud Menten. The model describes the rate of enzymatic reactions by relating the reaction rate v with substrate S using the formula of

$$v = \frac{V_{\max}[S]}{K_m + [S]},$$

where V_{max} is the maximum rate attained by the system referred to as saturating substrate concentrations (refer Figure 6). Meanwhile, the Michaelis constant K_m is the substrate concentration that takes half the value of V_{max}.

Apart from that, an ecological study has been conducted proposing a mathematical model to obtain input/output relationships between upstream and downstream of a river for DO and BOD parameters [8]. The appropriate example is the Beck modified Khanna Bhutiani (BMKB) model, which is developed to study the coexisting interaction occurs between DO and BOD. The study of wastewater took place at river Suswa, India. Precisely, the following model

$$\begin{cases} \lambda_1(S) = C_1 \ \lambda_1(S-1) + C_2 \ U_1(S-1) + C_3 \ e_1(S), \\ \lambda_2(S) = C_4 \ \lambda_1(S-1) + C_5 \ U_1(S-1) - C_6 \ e_1(S), \end{cases} \tag{3}$$

was used where $\lambda_1(S)$ is the value of DO (mg/l) to be obtained in particular season, $\lambda_2(S)$ is the value of BOD (mg/l), $\lambda_1(S-1)$ is DO/BOD observation of same place in previous season, $U_1(S-1)$ is DO/BOD observation of upstream in previous season, $e_1(S)$ is the error of possibility while constants $(c_1, c_2, \ldots, c_6)$ are described as the specific rates of the aforementioned term. The results were achieved by computing DO divided by BOD of the same upstream in the past season, which remains a single output solution. The model has been proved by the water quality information of the samples gathered from river Suswa in various seasons.
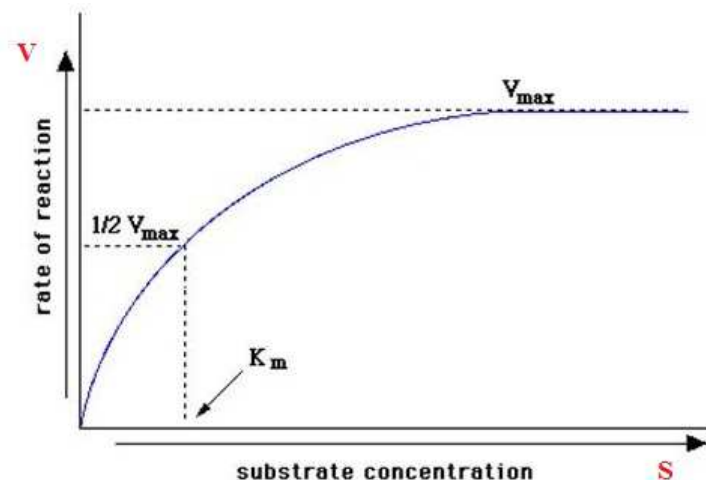

Figure 6: Michaelis-Manten Kinetics [12]

A more complex model has been considered discussing on the variations of COD, DO, ammonia, phosphorus, bacteria and algae concentrations with time and the dimensions of the pond [9–11]. This model predicts the correlation between those aforementioned variables at the effluent to measure the quality of the stabilization pond (natural pond). A two-dimensional hydraulic model was employed taking into consideration the dispersed flow and diffusion in horizontal and vertical directions, respectively. The pilot scale study of this model focused around the accumulated data from a full-scale lake in Turkey. The model can be utilized to redesign new outline for the lakes, thus enhancing the pro-fluent nature of the existing lakes.

Meanwhile, for this study, the advection-diffusion-reaction equation model proposed by Pimpunchat *et al.* [5] was extended to three state variables in this study, which are phototrophic bacteria (PSB), chemical oxygen demand (COD) and DO. Currently, there are several mathematical models available for simulating oxidation pond process where some important parameters are considered such as bacteria (cleansing agent), pollutants and dissolved oxygen (DO). However, previous results did not provide a good approximation on the required parameters.

3   Application of mPHO at Taman Timor Oxidation Pond

This section presents the application of mPHO product by J-Biotech Company at Taman Timor oxidation pond, Skudai, Johor. The treatment was conducted within three months period from 14 November 2014 to 13 February 2015 [3]. The location for applying mPHO and sampling points are presented in Figure 3.4. In the first week, 25 litres mPHO was applied each day with 2 days break in the weekend followed by adding 30 litres mPHO each day for the second week. This phase is considered as the first phase of application. During the third until twelve-th week, 10 litres of mPHO was applied each day with 2 days break in the weekend (refer to Table 1 and Table 2). Phase 1 and Phase 2 represented a continuous treatment of mPHO application within 91 days. All the amounts of mPHO given during the study were applied by the company using trial and error method. Meanwhile, the two days break during the weekend is considered as the regular holiday period. Hence, mathematical modelling was utilised in this study to assist in explaining further treatment process and aid in managing a more efficient treatment process.

4   Advection-Diffusion-Reaction Equation Model (Transport of Pollutant)

To represent the oxidation pond treatment system, the PDEs model was used considering the dimensions of the pond (x) as well as time observed (t). Therefore, the length of pond associated with variables x must be included as the independent variables for the model. This model has to be considered by two independent variables, which are x and t. This will allow the wastewater flows through the pond that contains several substances to be described by both advection and diffusion terms. Advection is the transport of substances through fluid. Mathematically, it can be written as the partial differential equation that governs the motion of a conserved scalar field as it is advected by a known velocity vector field [13]. One example of visualized phenomenon is the transport of crude oil dumped into a river. As the river flows, crude oil will move downstream in an oscillating behaviour via advection where the water's movement transports the oil. Meanwhile, the diffusion term is known as the second order

Table 1: Treatment Schedule for mPHO Application at Taman Timor Oxidation Pond (Phase 1) [3]

Application / Activities			Frequency	Location
Pre-Sampling			3 times	Both Point (CP 1 & CP 2)
Treatment Approaches	Day (Date)	Application / Activities	Quantity /Frequency	Location
Intensive (Phase 1)	1 (14/11/2013)	Sampling	2 locations	Both Point (CP 1 & CP 2)
		mPHO	25 liters	CP 1 (Influent)
	2 (15/11/2013)	mPHO	25 liters	CP 1 (Influent)
	3 (16/11/2013)	mPHO	25 liters	CP 1 (Influent)
	4 (17/11/2013)	Sampling	2 locations	Both Point (CP 1 & CP 2)
		mPHO	25 liters	CP 1 (Influent)
	5 (18/11/2013)	mPHO	25 liters	CP 1 (Influent)
	(19-20/11/2013) Break for 2 days			
	8 (21/11/2013)	Sampling	2 locations	Both Point (CP 1 & CP 2)
		mPHO	30 litres	CP 1 (Influent)
	9 (22/11/2013)	mPHO	30 litres	CP 1 (Influent)
	10 (23/11/2013)	mPHO	30 litres	CP 1 (Influent)
	11 (24/11/2013)	Sampling	2 locations	Both Point (CP 1 & CP 2)
		mPHO	30 litres	CP 1 (Influent)
	12(25/11/2013)	mPHO	30 litres	CP 1 (Influent)

PDEs. Cross sectional area (A) of a pond is also considered as one of the important factors for this model. Additionally, this factor has been previously employed by Pimpunchat [5] for a system of PDEs comprising DO and pollutants. This extended model is made to introduce another state variable which is PSB as the cleansing agent used in the process of degrading pollutants. Variables and parameters used in this mathematical model are stated in Table 4.14 and Table 4.15. Table 4.16 demonstrates the equation of all process rates considered in the Mathematical Model.

Meanwhile, the unknown constants of the model which is stated in Table 4.15 can also be determined using the parameter estimation (least squares) method in *Mathematica* routine `NonlinearModelFit` based on the experimental data at CP2. Hence, a system consists of three second order PDEs with coupled-equation for the aforementioned state variables is constructed and listed in equations (4–6).

Table 2: Treatment Schedule for mPHO Application at Taman Timor Oxidation Pond (Phase 2) [3]

Treatment Approaches	Cycles (week)	Application / Activities	Quantity (Litre)	Location
		Sampling	1	CP 1 & CP 2
Continuous (Phase 2)	1st (28/11/2013-4/12/2013)	Application mPHO	100	CP 1 (Influent)
	2nd (5/12/2013-11/12/2013)	Application mPHO	100	CP 1 (Influent)
	3rd (12/12/2013-18/12/2013)	Application mPHO	100	CP 1 (Influent)
	4th (19/12/2013-25/12/2013)	Application mPHO	100	CP 1 (Influent)
	5th (2/1/2014-8/1/2014)	Application mPHO	100	CP 1 (Influent)
	7th (9/1/2014-15/1/2014)	Application mPHO	100	CP 1 (Influent)
	8th (16/1/2014-22/1/2014)	Application mPHO	100	CP 1 (Influent)

$$\frac{\partial (A\,M(x,t))}{\partial t} = D_M \frac{\partial^2 (A\,M(x,t))}{\partial x^2} - \upsilon \frac{\partial (A\,M(x,t))}{\partial x} +$$
$$c_1 \frac{P(x,t)}{k_P + P(x,t)} A\,X(x,t)M(x,t) + \mu_b M(x,t)\left(1 - \frac{M(x,t)}{k_m}\right) \tag{4}$$

$$\frac{\partial (A\,P(x,t))}{\partial t} = D_P \frac{\partial^2 (A\,P(x,t))}{\partial x^2} - \upsilon \frac{\partial (A\,P(x,t))}{\partial x} - c_1 \frac{P(x,t)}{k_P + P(x,t)} A\,X(x,t)M(x,t), \tag{5}$$

$$\frac{\partial (A\,X(x,t))}{\partial t} = D_X \frac{\partial^2 (A\,X(x,t))}{\partial x^2} - \upsilon \frac{\partial (A\,X(x,t))}{\partial x} -$$
$$c_1 \frac{P(x,t)}{k_P + P(x,t)} A\,X(x,t)M(x,t) + k_L(\alpha - X(x,t)), \tag{6}$$

Table 3: Description of State Variables Used in the Mathematical Model

Symbol	Description	Unit
$M(x,\,t)$	Concentration of PSB in the pond where x varies from 0 to 54 metres and t varies from initial time up to 91 days	g m^{-3}
$P(x,\,t)$	Concentration of COD in the pond	g m^{-3}
$X(x,\,t)$	Concentration of DO in the pond	g m^{-3}
$U(t)$	Amount of mPHO applied to the pond according to schedule	m^3 day^{-1}
$M_i(t)$	Input of PSB from the inlet source that varies with time	g m^{-3}
$P_i(t)$	Input of COD from the inlet source that varies with time	g m^{-3}
$X_i(t)$	Input of DO from the inlet source that varies with time	g m^{-3}

Table 4: Process Rates

Symbol	Description	Formula	Unit
B	Breakdown of organic/ inorganic matter	$k_1 \dfrac{P(x,t)}{k_P + P(x,t)} A\ X(x,t)B(x,t)$	g m^{-3}
F	Mass transfer of oxygen	$k_L(\alpha - X(x,t))$	g m^{-3}
L	Logistic growth of bacteria	$\mu_b\ M(x,t)\left(1 - \dfrac{M(x,t)}{k_m}\right)$	g m^{-3}

where $0 \leq t \leq 90$ day, $0 \leq x \leq 54$ meter. At initial time and length of $t = 0$, and $x = 0$, the value of PSB is 0.00036 g m^{-3} with 116.7 g m^{-3} COD and 2.764 g m^{-3} DO. The initial conditions presented below were obtained by quadratic interpolation of the data at CP1 and CP2 when $t = 0$.

Initial Conditions:

$$
\begin{cases}
M(x,0) = \dfrac{1}{54^2}(0.001 - 0.00036)x^2 + 0.00036; \\[2mm]
P(x,0) = \dfrac{1}{54^2}(65 - 116.7)x^2 + 116.7; \\[2mm]
X(x,0) = \dfrac{1}{54^2}(2.8185 - 2.764)x^2 + 2.764.
\end{cases}
\tag{7}
$$

Boundary Conditions:

$$
\begin{cases}
\left.\dfrac{\partial M(x,t)}{\partial x}\right|_{x=0} = 0; & \left.\dfrac{\partial P(x,t)}{\partial x}\right|_{x=0} = 0; & \left.\dfrac{\partial X(x,t)}{\partial x}\right|_{x=0} = 0; \\[2mm]
M(0,t) = M_i(t); & P(0,t) = P_i(t); & X(0,t) = X_i(t).
\end{cases}
\tag{8}
$$

Table 5: Parameter Values Used in the Mathematical Model

Symbol	Description	Value	Unit	Source
m_0	Rate of PSB addition within the pond	19.1	g m^{-3}	MISG 2014 Report [3]
p_0	Concentration of COD in one m^3 of sewage from input CP1	50.0	g m^{-3}	MISG 2014 Report [3]
v_s	Average amount of sewage from CP1	290	m^3 day^{-1}	MISG 2014 Report [3]
v_p	Volume of the pond	2864	m^3	MISG 2014 Report [3]
v	Velocity of wastewater in the pond	7.54	m day^{-1}	MISG 2014 Report [3]
A	Cross section area of oxidation pond	53	m^2	MISG 2014 Report [3]
α	Saturated oxygen concentration	3.4	g m^{-3}	Calibrated
μ_b	Maximum growth rate for PSB	31	m^2 day^{-1}	Calibrated
D_M	Diffusion coefficient of PSB in the x direction	50	m^2 day^{-1}	Calibrated
D_P	Diffusion coefficient of COD in the x direction	50	m^2 day^{-1}	Calibrated
D_X	Diffusion coefficient of DO in the x direction	50	m^2 day^{-1}	Calibrated
k_L	Mass transfer rate for oxygen from air to water	4	m^2 day^{-1}	Calibrated
k_p	Half saturation oxygen demand concentration for COD decay	560	g m^{-3}	Calibrated
c_1	Degradation rate coefficient for COD	10	m^3 g^{-1} day^{-1}	Calibrated
k_m	Half-saturation coefficient for growth of PSB	1.7×10^{-4}	g m^{-3}	Calibrated

The boundary conditions $M(0, t)$, $P(0, t)$ and $X(0, t)$ can be calculated through linear interpolation of given data at CP1 and by considering there is no out flux for all state variables at CP1, which is at $x = 0$ at all time.

By using Crank-Nicolson algorithm as already mentioned earlier, equations (4-6) can be discretized into

$$\frac{A}{\Delta t}(M_i^{j+1} - M_i^j) = \frac{D_M}{2\,(\Delta x)^2}(M_{i+1}^j - 2\,M_i^j + M_{i-1}^j + M_{i+1}^{j+1} - 2\,M_i^{j+1} + M_{i-1}^{j+1})$$

$$- \frac{\upsilon}{4\Delta x}(M_{i+1}^j - M_{i-1}^j + M_{i+1}^{j+1} - M_{i-1}^{j+1}) + c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j\,M_i^j + \mu_b M_i^j\left(1 - \frac{M_i^j}{k_m}\right). \quad (9)$$

$$\frac{A}{\Delta t}(P_i^{j+1} - P_i^j) = \frac{D_P}{2\,(\Delta x)^2}(P_{i+1}^j - 2\,P_i^j + P_{i-1}^j + P_{i+1}^{j+1} - 2\,P_i^{j+1} + P_{i-1}^{j+1})$$

$$- \frac{\upsilon}{4\Delta x}(P_{i+1}^j - P_{i-1}^j + P_{i+1}^{j+1} - P_{i-1}^{j+1}) - c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j\,M_i^j. \quad (10)$$

$$\frac{A}{\Delta t}(X_i^{j+1} - X_i^j) = \frac{D_X}{2\,(\Delta x)^2}(X_{i+1}^j - 2\,X_i^j + X_{i-1}^j + X_{i+1}^{j+1} - 2\,X_i^{j+1} + X_{i-1}^{j+1})$$

$$- \frac{\upsilon}{4\Delta x}(X_{i+1}^j - X_{i-1}^j + X_{i+1}^{j+1} - X_{i-1}^{j+1}) - c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j\,M_i^j + k_L(\alpha - X_i^j). \quad (11)$$

Equations (9-11) can be rewritten as

$$-\frac{A}{\Delta t}M_i^j - \frac{D_M}{2\,(\Delta x)^2}(M_{i+1}^j - 2\,M_i^j + M_{i-1}^j) + \frac{\upsilon}{4\Delta x}(M_{i+1}^j - M_{i-1}^j)$$

$$- c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j\,M_i^j + \mu_b M_i^j\left(1 - \frac{M_i^j}{k_m}\right)$$

$$= -\frac{A}{\Delta t}M_i^{j+1} + \frac{D_M}{2\,(\Delta x)^2}(M_{i+1}^{j+1} - 2\,M_i^{j+1} + M_{i-1}^{j+1}) - \frac{\upsilon}{4\Delta x}(M_{i+1}^{j+1} - M_{i-1}^{j+1}). \quad (12)$$

$$- \frac{A}{\Delta t}P_i^j - \frac{D_P}{2\,(\Delta x)^2}(P_{i+1}^j - 2\,P_i^j + P_{i-1}^j) + \frac{\upsilon}{4\Delta x}(P_{i+1}^j - P_{i-1}^j) + c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j\,M_i^j$$

$$= -\frac{A}{\Delta t}P_i^{j+1} + \frac{D_P}{2\,(\Delta x)^2}(P_{i+1}^{j+1} - 2\,P_i^{j+1} + P_{i-1}^{j+1}) - \frac{\upsilon}{4\Delta x}(P_{i+1}^{j+1} - P_{i-1}^{j+1}). \quad (13)$$

$$-\frac{A}{\Delta t}X_i^j - \frac{D_X}{2\,(\Delta x)^2}(X_{i+1}^j - 2\,X_i^j + X_{i-1}^j) + \frac{\upsilon}{4\Delta x}(X_{i+1}^j - X_{i-1}^j) + c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j\,M_i^j$$

$$= -\frac{A}{\Delta t}X_i^{j+1} + \frac{D_X}{2\,(\Delta x)^2}(X_{i+1}^{j+1} - 2\,X_i^{j+1} + X_{i-1}^{j+1}) - \frac{\upsilon}{4\Delta x}(X_{i+1}^{j+1} - X_{i-1}^{j+1}). \quad (14)$$

Equations (12-14) can be rearranged to follow Crank-Nicolson scheme by

$$\left(-\frac{D_M}{2(\Delta x)^2} - \frac{\upsilon}{4\Delta x}\right)M_{i-1}^j + \left(-\frac{A}{\Delta t} + \frac{D_M}{(\Delta x)^2} - c_1\frac{P_i^j}{k_P + P_i^j}A\,X_i^j + \mu_b\left(1 - \frac{M_i^j}{k_m}\right)\right)M_i^j$$

$$\left(-\frac{D_M}{2(\Delta x)^2} + \frac{\upsilon}{4\Delta x}\right)M_{i+1}^j = \left(\frac{D_M}{2(\Delta x)^2} + \frac{\upsilon}{4\Delta x}\right)M_{i-1}^{j+1} + \left(-\frac{A}{\Delta t} + \frac{D_M}{(\Delta x)^2}\right)M_i^{j+1}$$

$$+ \left(\frac{D_M}{2(\Delta x)^2} - \frac{\upsilon}{4\Delta x}\right)M_{i+1}^{j+1} \quad (15)$$

$$\left(-\frac{D_P}{2(\Delta x)^2} - \frac{v}{4\Delta x}\right) P_{i-1}^j + \left(-\frac{A}{\Delta t} + \frac{D_P}{(\Delta x)^2} + c_1 \frac{X_i^j \, M_i^j}{k_P + P_i^j} A\right) P_i^j$$

$$\left(-\frac{D_P}{2(\Delta x)^2} + \frac{v}{4\Delta x}\right) P_{i+1}^j = \left(\frac{D_P}{2(\Delta x)^2} + \frac{v}{4\Delta x}\right) P_{i-1}^{j+1} + \left(-\frac{A}{\Delta t} + \frac{D_P}{(\Delta x)^2}\right) P_i^{j+1}$$

$$+ \left(\frac{D_P}{2(\Delta x)^2} - \frac{v}{4\Delta x}\right) P_{i+1}^{j+1} \tag{16}$$

$$\left(-\frac{D_X}{2(\Delta x)^2} - \frac{v}{4\Delta x}\right) X_{i-1}^j + \left(-\frac{A}{\Delta t} + \frac{D_X}{(\Delta x)^2} + c_1 \frac{P_i^j \, M_i^j}{k_P + P_i^j} A\right) X_i^j$$

$$\left(-\frac{D_X}{2(\Delta x)^2} + \frac{v}{4\Delta x}\right) X_{i+1}^j = \left(\frac{D_X}{2(\Delta x)^2} + \frac{v}{4\Delta x}\right) X_{i-1}^{j+1} + \left(-\frac{A}{\Delta t} + \frac{D_X}{(\Delta x)^2}\right) X_i^{j+1}$$

$$+ \left(\frac{D_X}{2(\Delta x)^2} - \frac{v}{4\Delta x}\right) X_{i+1}^{j+1}. \tag{17}$$

Lastly, equations (15-17) can be written into a linear equation matrix system as follows

$$
\begin{pmatrix}
b_0 & c_0 & 0 & 0 & 0 & 0 \\
a_1 & b_1 & c_1 & 0 & 0 & 0 \\
0 & a_2 & b_2 & c_2 & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & a_{imax} & b_{imax} \\
-- & -- & -- & -- & -- & -- \\
b_0 & c_0 & 0 & 0 & 0 & 0 \\
a_1 & b_1 & c_1 & 0 & 0 & 0 \\
0 & a_2 & b_2 & c_2 & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & a_{imax} & b_{imax} \\
-- & -- & -- & -- & -- & -- \\
b_0 & c_0 & 0 & 0 & 0 & 0 \\
a_1 & b_1 & c_1 & 0 & 0 & 0 \\
0 & a_2 & b_2 & c_2 & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & a_{imax} & b_{imax}
\end{pmatrix}
\begin{pmatrix}
M_0^j \\ M_1^j \\ M_2^j \\ \vdots \\ M_{imax-1}^j \\ M_{imax}^j \\ --- \\ P_0^j \\ P_1^j \\ P_2^j \\ \vdots \\ P_{imax-1}^j \\ P_{imax}^j \\ --- \\ X_0^j \\ X_1^j \\ X_2^j \\ \vdots \\ X_{imax-1}^j \\ X_{imax}^j
\end{pmatrix}
=
\begin{pmatrix}
Y1_0^j \\ Y1_1^j \\ Y1_2^j \\ \vdots \\ Y1_{imax-1}^j \\ Y1_{imax}^j \\ --- \\ Y2_0^j \\ Y2_1^j \\ Y2_2^j \\ \vdots \\ Y2_{imax-1}^j \\ Y2_{imax}^j \\ --- \\ Y3_0^j \\ Y3_1^j \\ Y3_2^j \\ \vdots \\ Y3_{imax-1}^j \\ Y3_{imax}^j
\end{pmatrix}
\tag{18}
$$

where

$$Y1_i^j = \left(\frac{D_M}{2(\Delta x)^2} + \frac{v}{4\Delta x}\right) M_{i-1}^{j+1} + \left(-\frac{A}{\Delta t} + \frac{D_M}{(\Delta x)^2}\right) M_i^{j+1} + \left(\frac{D_M}{2(\Delta x)^2} - \frac{v}{4\Delta x}\right) M_{i+1}^{j+1},$$

$$Y2_i^j = \left(\frac{D_P}{2(\Delta x)^2} + \frac{v}{4\Delta x}\right) P_{i-1}^{j+1} + \left(-\frac{A}{\Delta t} + \frac{D_P}{(\Delta x)^2}\right) P_i^{j+1} + \left(\frac{D_P}{2(\Delta x)^2} - \frac{v}{4\Delta x}\right) P_{i+1}^{j+1},$$

$$Y3_i^j = \left(\frac{D_X}{2(\Delta x)^2} + \frac{v}{4\Delta x} \right) X_{i-1}^{j+1} + \left(-\frac{A}{\Delta t} + \frac{D_X}{(\Delta x)^2} \right) X_i^{j+1} + \left(\frac{D_X}{2(\Delta x)^2} - \frac{v}{4\Delta x} \right) X_{i+1}^{j+1}.$$

Equation (18) with the given initial and boundary conditions in equation (7) and equation (8) was solved using *Mathematica* routine `NDSolve` with Crank-Nicolson method as earlier discussed.

5   Numerical Results

The results are depicted in Figures 7-15. Figure 7 and Figure 8 showed the 3D and contour plot of the solution of PSB along the pond (x) and time (t). Figure 9 and Figure 10 describe the 3D and contour plot of pollutant (COD). Meanwhile, Figure 11 and Figure 12 present the 3D and contour plot of DO. Lastly, the comparison between the simulated Model and the experimental data at CP2 for all three state variables was shown in Figures 13-15 for PSB, COD and DO, respectively.

Table 6: Values of Calibrated Constants and Values Specified in Literature

Symbol	Model Calibration	Value	Unit	Source
μ_b	0.6	$2-10$	day^{-1}	Tchobanoglous and
k_L	4	2.67	m^2 day^{-1}	Burton [14]
D_X	50	45	m^2 day^{-1}	
D_M	50	45	m^2 day^{-1}	Beran and Kargi [9]
D_P	50	45	m^2 day^{-1}	
k_m	0.00017	$0-0.1$	g m^{-3}	Ahmed and Mayo [15]

The simulated model demonstrated a result from all state variables supporting the experimental data given at CP2. As can be seen, the amount of DO in the pond increased as time increases. The variation of PSB in the pond showed that the concentration of PSB has continuously increased until the end of treatment period. Lastly, an increase in COD concentration was observed throughout the treatment period. This was due to the increased amount of COD that goes into the oxidation pond as time increases, which was displayed from the data provided by J-Biotech.

6   Conclusion

In summary, this study has proposed a mathematical model for oxidation pond treatment system. The PDEs model comprising advection-diffusion-reaction equations for transporting pollutants. This mathematical model was considered as the best model in terms of complexity along with a good result compared to experimental data.
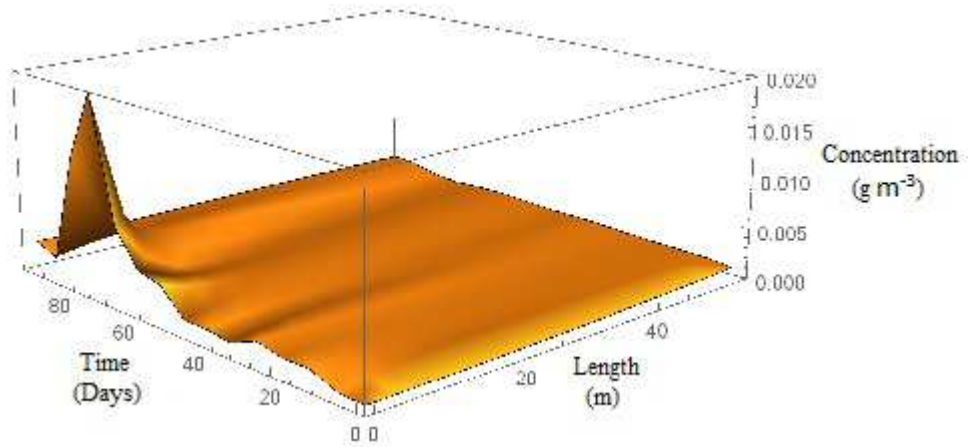
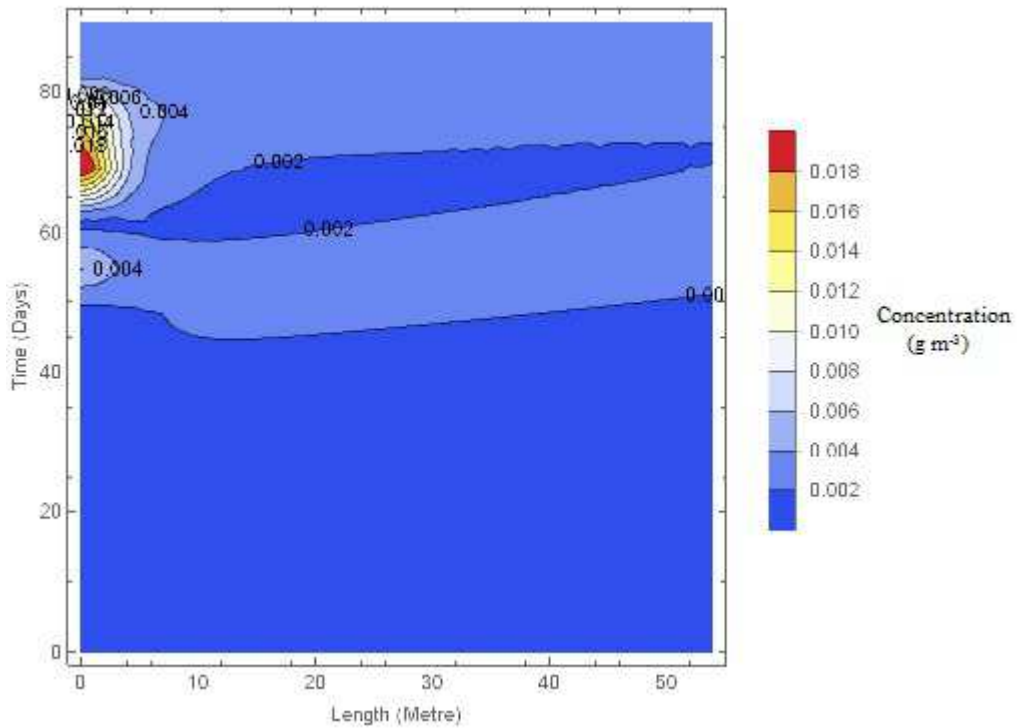Figure 7: 3D Graph of PSB Based on the Mathematical Model

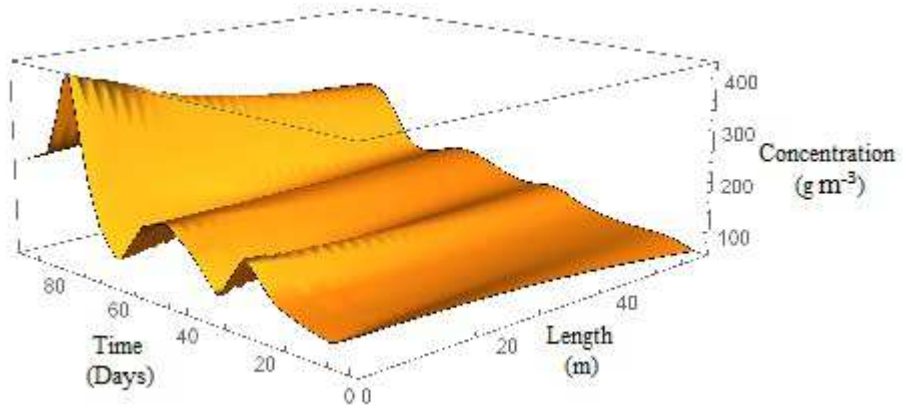Figure 8: Contour Plot of PSB Based on the Mathematical Model

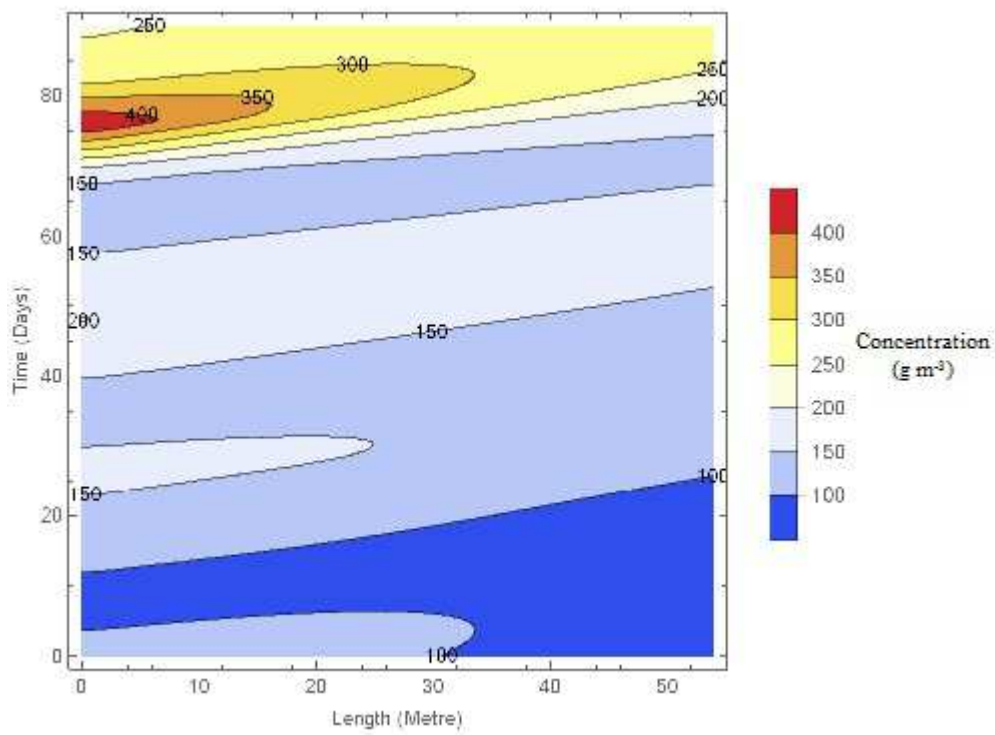Figure 9: 3D Graph of COD Based on the Mathematical Model

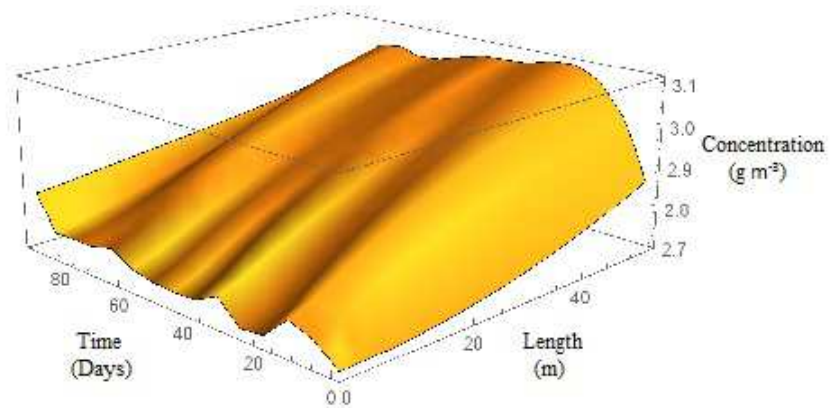Figure 10: Contour Plot of COD Based on the Mathematical Model

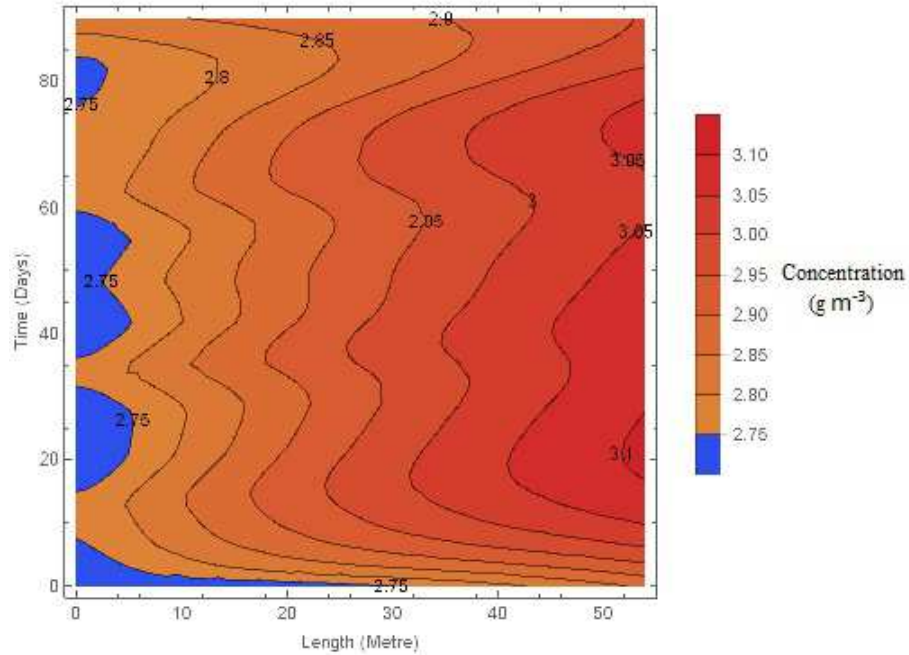Figure 11: 3D Graph of DO Based on the Mathematical Model

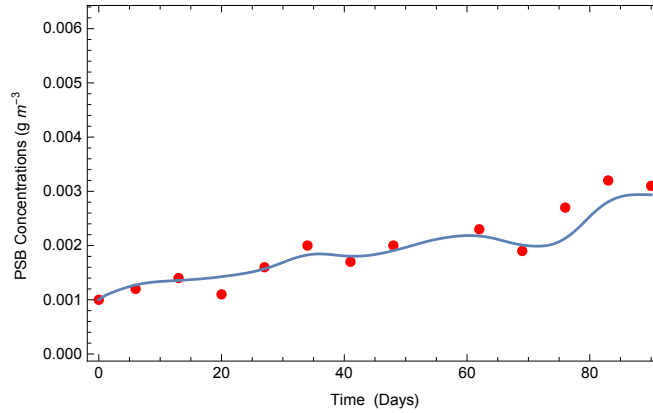Figure 12: Contour Plot of DO Based on the Mathematical Model

Figure 13: Graphical Comparison between Experimental Data and Simulated Model for PSB (The blue curve represents the numerical results and red dots are the experimental data ($R^2 = 0.90$))
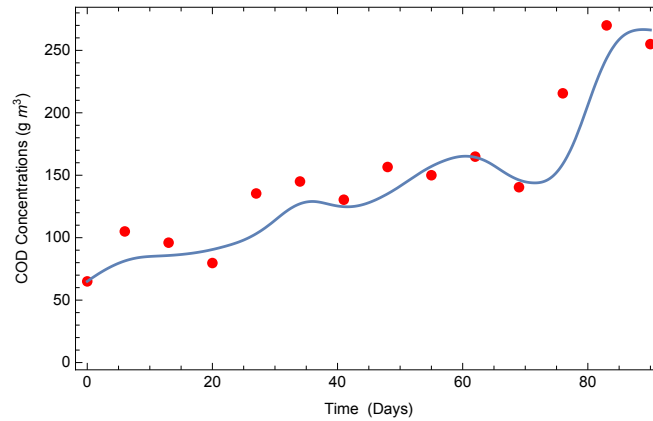

Figure 14: Graphical Comparison between Experimental Data and Simulated Model for COD (The blue curve represents the numerical results and red dots are the experimental data ($R^2 = 0.86$))
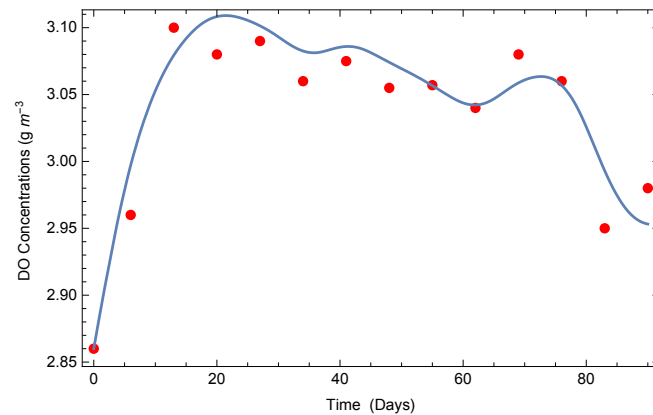

Figure 15: Graphical Comparison between Experimental Data and Simulated Model for DO (The blue curve represents the numerical results and red dots are the experimental data ($R^2 = 0.92$))

Acknowledgment

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