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On Some Graphs of Finite Metabelian Groups of Order Less Than 24

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Abstract In this work, a conjugacy class graph is represented by Γ and G represents non-abelian metabelian group. Conjugacy class graph of a group is the graph associated with the conjugacy classes of the group. Its vertices are the non-central conjugacy classes of the group, and two distinct vertices are joined by an edge if their cardinalities are not coprime. A group is referred to as metabelian if there exits an abelian normal subgroup in which the factor group is also abelian. It has been proven earlier that 25 non-abelian metabelian groups which have order less than 24, which are considered in this work, exist. In this article, the conjugacy class graphs of non-abelian metabelian groups of order less than 24 are determined as well as examples of some finite groups associated to other graphs are given.

Keywords Connected graph; Disconnected graph; Conjugacy class; Metabelian groups; Conjugacy class graph.

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1 Introduction

Group theory play a remarkable role of simplifying many complicated systems of real life problems over many decades. The use of symmetry in solving very complicated Hamiltonian problems and quantum mechanics problems that lead to the famous group of the Schrodinger equation is one out of many applications of group theory to other concepts. This and other related outcomes gave birth to more of relating group theory with many other fields of research to get more generalizing results and solve many other real-life problems. In this paper we consider the idea and term of metabelian groups that was initially used to prove theorems associated with algebraic knot theory, number theory, as well as the foundations of geometry [1]. Groups that are close to being abelian are referred to as metabelian groups, in a way that every abelian group is metabelian but not the other way round [1]. This closeness is revealed in the particular structure of their commutator subgroups.

This paper is formed into three parts. The first part discusses the classification of all metabelian groups of order less than 24, and different kinds of graphs, whereas the second part gives some earlier and recent research that are related to the metabelian groups, conjugacy class graph and other related graphs. In the last part, we introduce the results on the conjugacy class graphs of non-abelian metabelian groups of order less than 24.

The non-abelian metabelian groups of order less than 24 were classified into 25 groups by Abdul Rahman [2] in 2010 and stated in Table 1.

In this paper, the 25 groups in the classification will be referred as groups of type (1) to type (25).

1.1 Different Kind of Graphs

In this subsection, we provide different kind of graphs that have been used by several researchers starting with complete graph, connected graph and disconnected graph.

Definition 1 [3] ***Complete and Disconnected Graphs***
A graph is said to be a complete graph, if it is a simple graph such that all pair of vertices are joined by an edge, and any complete graph that has n number of vertices is represented by K_n. On the other hand, a graph is connected if for every pair of vertices x and y, there is a path from x to y, while a disconnected graph on the other hand is made up of connected pieces called components, or isolated vertices in some cases, usually denoted by $\overline{K_n}$.

Definition 2 [3] ***Non-Commuting Graph***
A non-commuting graph, denoted by Δ_G^N is a graph whose vertices are non-central elements of G i.e. $G - Z(G)$ in which the vertices are adjacent if they do not commute. In other words, for any two distinct vertices x and y be joint whenever $xy \neq yx$.

The following example illustrates the non-commuting graph of non-abelian metabelian group of type (1).

Example 1 Let G be a non-abelian metabelian group of type (1), $D_3 \cong S_3 = \langle a,\ b :\ a^3 = b^2 = 1, bab = a^{-1}\rangle$. The non-commuting graph, $\Delta_{D_3}^N$ is given as follows: The elements of D_3 is $D_3 = \{1, a, a^2,\ b,\ ab,\ a^2b\}$ and $Z(D_3) = \{1\}$. The set of vertices of $\Delta_{D_3}^N$ is equal to $G - Z(G)$. Thus $|V(D_3)| = 5$. Based on vertices adjacency of non-commuting graph, $\Delta_{D_3}^N\ = K_3 \cup \{a\} \cup \{a^2\}$.

Definition 3 [4] ***Commuting Graph***
The commuting graph, denoted by Δ_G, is a graph whose vertices are non-central elements of G i.e. $V(G) = G - Z(G)$ in which two vertices are adjacent if they commute. That is, if $h \neq g$ and $|h, g| \in Z(G)$.

The following example illustrates the commuting graph of non-abelian metabelian group of type (1).

Table 1: Non-abelian Metabelian Groups of Order Less than 24 Classified into 25 Groups

Type	**Non-Abelian Metabelian Groups**
1	$D_3 \cong S_3 = \langle a,\ b :\ a^3 = b^2 = 1, bab = a^{-1}\rangle$
2	$D_4 = \langle a, b : a^4 = b^2 = 1, bab = a^{-1}\rangle$
3	$Q_3 = \langle a, b : a^4 = 1, b^2 = a^2, aba = b\rangle$
4	$D_5 = \langle a, b : a^5 = b^2 = 1, bab = a^{-1}\rangle$
5	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^3 = 1, aba = a\rangle$
6	$\mathrm{A}_4 = \langle a, b, c : a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac\rangle$
7	$D_6 = \langle a, b : a^6 = b^2 = 1, bab = a^{-1}\rangle$
8	$D_7 = \langle a, b : a^7 = b^2 = 1, bab = a^{-}\rangle$
9	$D_8 = \langle a, b : a^8 = b^2 = 1, bab = a^{-1}\rangle$
10	Quasi-dihedral group, $= \langle a, b : a^8 = b^2 = 1, bab = a^3\rangle$
11	$Q_8 = \langle a, b : a^8 = 1, a^4 = b^2, aba = b\rangle$,
12	$D_4 \times \mathbb{Z}_2 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, bab = a^{-1}\rangle$
13	$Q_3 \times \mathbb{Z}_2 = \langle a, b, c : a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3b, ac = ca, bc = cb\rangle$
14	Modular$-16 = \langle a, b : a^8 = b^2 = 1, ab = ba^5\rangle$
15	$B = \langle a, b : a^4 = b^4 = 1, ab = ba^3\rangle$
16	$K = \langle a, b, c : a^4 = b^2 = c^2 = 1, bab = a, ac = ca\rangle$
17	$G_{4,4} = \langle a, b : a^4 = b^4 = abab = 1, ab^3 = ba^3\rangle$
18	$D_9 = \langle a, b : a^9 = b^2 = 1, bab = a^{-1}\rangle$
19	$S_3 \times \mathbb{Z}_3 = \langle a, b, c : a^3 = b^2 = c^3 = 1, b = a^{-1}, ac = ca, bc = cb\rangle$
20	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \langle a, b, c : a^2 = b^3 = c^3 = 1, ba = cb, bab = a, cac = a\rangle$
21	$D_{10} = \langle a, b : a^{10} = b^2 = 1, bab = a^{-1}\rangle$
22	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^5 = 1, ba = ab^2\rangle$
23	$\mathrm{Z}_4 \rtimes \mathbb{Z}_5 = \langle a, b : a^4 = b^5 = 1, bab = a\rangle$
24	$\mathbb{Z}_7 \rtimes \mathrm{Z}_3 = \langle a, b : a^3 = b^7 = 1, ba = ab^2\rangle$
25	$D_{11} = \langle a, b : a^{11} = b^2 = 1, bab = a^{-1}\rangle$

Example 2 Let G be a non-abelian metabelian group of type (1), $D_3 = \langle a,\ b:\ a^3 = b^2 = 1, bab = a^{-1}\rangle$. The commuting graph of D_3, is given as follows: Based on the vertices adjacency of commuting graph, the commuting graph of the said group is $\Delta_G = K_2 \cup \{b\} \cup \{ab\} \cup \{a^2 b\}$.

Definition 4 [5] ***Conjugate graph***
The conjugate graph, denoted by Γ_G^c, is a graph whose vertices are non-central elements of G i.e. $V(G) = G \backslash Z(G)$ in which two vertices are adjacent if they are conjugate.

The following example illustrates the conjugate graph of non-abelian metabelian group of type (1).

Example 3 Let G be a non-abelian metabelian group of type (1), $D_3 = \langle a,\ b:\ a^3 = b^2 = 1, bab = a^{-1}\rangle$. The conjugate graph of D_3 is given as follows: Based on vertices adjacency of conjugate graph, the conjugate graph of D_3 is obtained to be $\Gamma^c_{D_3} = K_2 \cup K_3$.

Definition 5 [6] ***Conjugacy Class Graph***
The conjugacy class graph, denoted by Γ_G, is a graph whose vertices are non-central conjugacy classes of G i.e. $|V(G)| = K(G) - |Z(G)|$ where $K(G)$ is the number of conjugacy classes in G and G is a finite group and let $Z(G)$ be the center of G. Two vertices are adjacent if their cardinalities are not coprime (i.e. have common factor).

The following example illustrates the conjugacy class graph of non-abelian metabelian group of type (1).

Example 4 Let G be a non-abelian metabelian group of type (1), $D_3 = \langle a,\ b:\ a^3 = b^2 = 1, bab = a^{-1}\rangle$. The conjugacy class graph, Γ_{D_3}, is given as follows: Based on vertices adjacency of conjugacy class graph of D_3 is given as Γ_{D_3} is $\overline{K_2}$.

In this work, the conjugacy class graph is found for the non-abelian metabelian groups of order less than 24.

2 Preliminaries

In this part, we provide some related works to non-abelian metabelian groups and conjugacy class graph. Starting with the conjugacy classes of non-abelian metabelian groups of order less than 24.

The conjugacy classes of metabelian groups of order less than 24 were determined by Sarmin *et al.* [7]. The followings are some propositions that are needed in this paper.

Proposition 1 [7] *Let G be a non-abelian metabelian group of type* (1), $D_3 = \langle a,\ b : a^3 = b^2 = 1, bab = a^{-1}\rangle$*. Then the number of conjugacy classes of G is three, represented as follows:* $\mathrm{cl}\,(e) = \{e\}$, $\mathrm{cl}\,(a) = \{a, a^2\}$, $\mathrm{cl}\,(b) = \{b, ab, a^2 b\}$.

Remark 2.1 [7] *Since S_3 is isomorphic to D_3, the number of conjugacy classes in S_3 is the same as the number conjugacy classes in Proposition 1.*

The following are the conjugacy classes of non-abelian metabelian groups of type (2), (3), (4), and (16) respectively.

Proposition 2 [7] *Let G be a non-abelian metabelian group of type* (2)*,* $D_4 = \langle a, b : a^4 = b^2 = 1, bab = a^{-1}\rangle$*. Then, the number of conjugacy classes in G is equal to five, presented as:* $\mathrm{cl}\,(e) = \{e\}\,,\ \mathrm{cl}\,(a) = \{a, a^3\}\,,\ \mathrm{cl}\,(b) = \{b, a^2b\}\,,\ \mathrm{cl}\,(a^2) = \{a^2\}\,,$ *and* $\mathrm{cl}\,(a^3b) = \left\{ab,\ a^3b\right\}.$

Remark 2.2 [7] *The number of conjugacy classes of non-abelian metabelian group of type* (3)*,* $Q = < a, b\ :\ a^4 = 1,\ a^2 =\ b^2, aba = b >$ *is the same as the number of conjugacy classes given in Proposition 2*

Proposition 3 [7] *Let G be a non-abelian metabelian group of type* (4)*,* $D_5 = \langle a, b : a^5 = b^2 = 1, bab = a^{-1}\rangle$*. Then, the number of conjugacy classes of G is equal to four, presented as* $\mathrm{cl}\,(e) = \{e\}\,, \mathrm{cl}\,(a) = \{a,\ \ a^4\}\,, \mathrm{cl}(b) = \{b,\ ab, a^2b,\ a^3b,\ a^4b\},$ *and* $\mathrm{cl}(a^2) = \{a^2,\ \ a^3\}.$

Remark 2.3 [7] *The number of conjugacy classes of non-abelian metabelian group of type* (6)*,* $A_4 = < a, b\ \ : a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac >$ *is the same as the number of conjugacy classes given in Proposition 3.*

The conjugacy classes of metabelian groups of types (5), (7), (8), (22) and (24) are given in the following propositions and remarks.

Proposition 4 [7] *Let G be non-abelian metabelian group of type* (5)*,* $\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^3 = 1, aba = a\rangle$*. Then, the number of conjugacy classes in G is equal to six which are:* $\mathrm{cl}\,(e) = \{e\}\,, \mathrm{cl}\,(b) = \{b,\ \ a^3ba\}\,, \mathrm{cl}\,(a) = \{a,\ \ ab, ba\}\,, \mathrm{cl}\,(a^2) = \{a^2\}\,, \mathrm{cl}\,(a^2b) = \{a^2b,\ aba\}$ *and* $\mathrm{cl}\,(a^2ba) = \{a^2ba,\ a^3,\ a^3b\}.$

Remark 2.4 [7] *The non-abelian metabelian group of type* (7)*,* $D_6 = < a, b\ \ :\ \ a^6 =\ b^2 = 1, bab =\ a^{-1} >$ *and type* (24)*,* $Fr_{21} \cong\ \mathbb{Z}_7 \rtimes \mathbb{Z}_3 =< a, b\ \ :\ \ a^3 =\ b^7 = 1, ba = a\ b^2 >$ *have the same number of non-central conjugacy classes given in Proposition 4.*

Proposition 5 [7] *Let G be a non-abelian metabelian group of type* (8)*,* $D_7 =< a, b\ \ :\ \ a^7 = b^2 = 1, bab =\ a^{-1} >$*. Then, the conjugacy classes of G are* $\mathrm{cl}\,(e) = \{e\}\,, \mathrm{cl}\,(a) = \{a,\ \ a^6\} = \mathrm{cl}\,(a^6)\,, \mathrm{cl}\,(b) = \left\{b,\ \ ab,\ a^2b,\ a^3b,\ a^4b,\ a^5b,\ a^6b\right\}, \mathrm{cl}\,(a^2) = \{a^2,\ \ a^5\}\,,$ *and* $\mathrm{cl}\,(a^3) = \left\{a^3,\ a^4\right\}.$ *Hence the number of conjugacy classes of G,* $K\,(G) = 5.$

Remark 2.5 [7] *The number of conjugacy classes of non-abelian metabelian group of type* (22) *is the same as the number of conjugacy classes given in Proposition 5.*

Next is the conjugacy classes of non-abelian metabelian groups of types (9) to (18) and (20) to (23).

Proposition 6 [7] *Let G be a non-abelian metabelian group of type* (9)*,* $D_8 = \langle a, b : a^8 = b^2 = 1, bab = a^{-1}\rangle$*. Then, the number of conjugacy classes of G is seven, represented as* $\mathrm{cl}\,(e) = \{e\}\,,\ \mathrm{cl}\,(a) = \{a,\ a^7\}\,,\ \mathrm{cl}\,(b) = \{b,\ a^2b,\ a^4b,\ a^6b\}\,,\ \mathrm{cl}\,(ab) = \{ab,\ a^3b,\ a^5b,\ \ a^7b\}\,,$ $\mathrm{cl}\,(a^2) = \{a^2,\ a^6\} = \mathrm{cl}\,(a^6)\,, \mathrm{cl}\,(a^3) = \{a^3,\ a^5\}\,,$ *and* $\mathrm{cl}\,(a^4) = \{a^4\}\,.$

Remark 2.6 [7] *The number of non-central conjugacy classes of non-abelian metabelian group of type* (10)*,* (11) *and type* (12) *are the same the as that of* D_8 *as given in Proposition 6*

Proposition 7 [7] *Let G be a non-abelian metabelian group of type* (13)*, $Q_3 \times \mathbb{Z}_2 \cong \langle a, b, c : a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3b, ac = ca, bc = cb \rangle$. Then, the conjugacy classes of G are:* $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, a^3\}$, $\mathrm{cl}(b) = \{b, \ a^2b\}$, $\mathrm{cl}(c) = \{c\}$, $\mathrm{cl}(ab) = \{ab, \ a^3b\}$, $\mathrm{cl}(ac) = \{ac, \ a^3c\}$, $\mathrm{cl}(bc) = \{bc, \ a^2bc\}$, $\mathrm{cl}(abc) = \{abc, \ a^3bc\}$, $\mathrm{cl}(a^2) = \{a^2\}$, *and* $\mathrm{cl}(a^2c)$. *Thus, the number of conjugacy classes of G, denoted as $K(G) = 10$.*

Remark 2.7 [7] *The number of non-central conjugacy classes of non-abelian metabelian groups of type* (15)*,* (16) *and* (17)*, namely B, K, and $G_{4,\ 4}$ are the same number of non-central conjugacy classes as given in Proposition 7.*

Proposition 8 [7] *Let G be a non-abelian metabelian group of type* (18)*, $D_9 = \langle a, b : a^9 = b^2 = 1, bab = a^{-1} \rangle$. Then, the number of conjugacy classes of G is six, represented as:* $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, \ a^8\}$, $cl(b) = \{b, \ ab, \ a^2b, \ a^3b, \ a^4b, \ a^5b, a^6b, \ a^7b, \ a^8b\}$, $\mathrm{cl}(a^3) = \{a^3, \ a^6\}$, *and* $\mathrm{cl}(a^4) = \{a^4, \ a^5\}$.

Remark 2.8 [7] *The number of non-central conjugacy classes of the non-abelian metabelian groups of type* (14)*,* (20) *and* (23) *which are Modular-*16*, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and $\mathbb{Z}_4 \rtimes \mathbb{Z}_5$ respectively, are the same to that of the number of conjugacy classes as given in Proposition 8.*

The following propositions are the conjugacy classes of non-abelian metabelian groups of types (19), (21) and (25).

Proposition 9 [7] *Let G be a non-abelian metabelian group of type* (19)*, $S_3 \times \mathbb{Z}_3 \cong \langle a, b, c : a^3 = b^2 = c^3 = 1, b = a^{-1}, ac = ca, bc = cb \rangle$. Then, the conjugacy classes of G are given as follows:* $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, \ a^2\}$, $\mathrm{cl}(b) = \{b, \ ab, a^2b\}$, $\mathrm{cl}(c) = \{c\}$, $\mathrm{cl}(ac) = \{ac, \ a^2c\}$, $\mathrm{cl}(bc) = \{bc, \ abc, \ a^2bc\}$, $\mathrm{cl}(c^2) = \{c^2\}$, $\mathrm{cl}(a^2c) = \{a^2c, \ a^2c^2\}$, *and* $\mathrm{cl}(bc^2) = \{bc^2, \ abc^2, a^2bc^2\}$. *Thus, the number of conjugacy classes of G, $K(G) = 9$.*

Proposition 10 [7] *Let G be a non-abelian metabelian group of type* (21)*, $D_{10} = \langle a, b : a^{10} = b^2 = 1, bab = a^{-1} \rangle$. Then, the conjugacy classes of G are proved to be:* $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, \ a^9\}$, $\mathrm{cl}(b) = \{b, \ a^2b, \ a^4b, a^6b, a^8b,\}$, $\mathrm{cl}(ab) = \{ab, \ a^3b, a^5b, a^7b, a^9b\}$, $\mathrm{cl}(a^2) = \{a^2, \ a^8\}$, $\mathrm{cl}(a^3) = \{a^3, \ a^7\}$, $\mathrm{cl}(a^4) = \{a^4, \ a^6\}$, *and* $\mathrm{cl}(a^5) = \{a^5\}$. *Then, the number of conjugacy classes of G, $K(G) = 8$.*

Proposition 11 [7] *Let G be a metabelian group of type* (25)*, $D_{11} = \langle a, b : a^{11} = b^2 = 1, bab = a^{-1} \rangle$. Then, the number of conjugacy classes of G is seven, represented as follows:* $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, \ a^{10}\}$, $\mathrm{cl}(b) = \{b, \ ab, \ a^2b, \ a^3b, a^4b, \ a^5b, \ a^6b, \ a^7b, \ a^8b, \ a^9b, a^{10}b\}$, $\mathrm{cl}(a^2) = \{a^2, \ a^9\}$, $\mathrm{cl}(a^3) = \{a^3, \ a^8\}$, $\mathrm{cl}(a^4) = \{a^4, \ a^7\}$, *and* $\mathrm{cl}(a^5) = \{a^5, \ a^6\}$. *Thus, the number of conjugacy classes of G, $K(G) = 7$.*

Here, we provide some works related to the conjugacy class graphs stated as follows:

In 2003, Beltran and Felipe [8] have studied the conjugacy class graph of p-solvable groups in which the vertices are primes conjugacy classes. They also proved that the diameter of this graph is at most three which considered to be the best bound for the diameter of conjugacy class graph of p-solvable groups. A year later, in [9], they studied the structure and arithmetic properties of p-regular class sizes in p-solvable groups that have a disconnected conjugacy class graph. In 2002, Bianchi *et al.* [10] studied the regularity of the graph related to conjugacy classes and provided some results and later in 2013, Moradipour *et al.* [11] used the result of graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-proups.

In the next part, we provide our results on conjugacy class graphs of non-abelian metabelian groups of order less than 24.

3 Results and Discussion

This part presents our results on the conjugacy class graphs of non-abelian metabelian groups of order less than 24. Throughout this section, we denote Γ_G for the conjugacy class graph of a group G.

Theorem 1 *Let G be a non-abelian metabelian group of type* (1), $G = D_3 = \langle \mathrm{a},\ \mathrm{b} :\ \mathrm{a}^3 = b^2 = 1, bab = a^{-1}\rangle$. *Then, the conjugacy class graph of G represented as Γ_G is $\overline{K_2}$.*

Proof Suppose $G = D_3$, based on Proposition 1, there are three conjugacy classes including $cl\,(e)$. Thus, the number of vertices in Γ_G is two. Since the greatest common divisor of the sizes of the two conjugacy classes is one, thus Γ_G consists of two isolated vertices. Hence, Γ_G is $\overline{K_2}$. □

Remark 3.1 *The conjugacy class graph of non-abelian metabelian group of S_3 is the same as the conjugacy class graph in Theorem 1 by Remark 2.1, since D_3 is isomorphic to S_3.*

Theorem 2 *Let G be a non-abelian metabelian group of type* (2). $G = D_4 = \langle a, b : a^4 = b^2 = 1, bab = a^{-1}\rangle$. *Then, $\Gamma_G = K_3$.*

Proof Suppose $G = D_4$, based on Proposition 2, the center of G, $Z\,(G) = \{e, a^2\}$, then, the non-central conjugacy classes of G is three. Hence $|V\,(G)| = 3$ likewise the greatest common divisors between the sizes of any two conjugacy classes is proved to be greater than one. Therefore, Γ_G consists of one complete graph of K_3 as claimed. □

Remark 3.2 *The conjugacy class graph of non-abelian metabelian group of type* (3) *is the same as to the conjugacy class graph in Theorem 2.*

Theorem 3 *Let G be a non-abelian metabelian group of type* (4), $G = D_5 = \langle a, b : a^5 = b^2 = 1, bab = a^{-1}\rangle$. *Then, $\Gamma_G = K_2 \bigcup \{\mathrm{cl}\,(b)\}$.*

Proof Suppose $G = D_5$, based on Proposition 3, the center of G, $Z\,(G) = \{e\}$, then, the number of non-central conjugacy class of G is equal to three. Therefore, the number of vertices in Γ_G is equal three. Based on vertices adjacency of conjugacy class graph, thus Γ_G consists of one complete graph K_2 and one isolated vertex, namely $\mathrm{cl}\,(\mathrm{ab})$. □

Remark 3.3 *The conjugacy class graph of A_4 is the same as conjugacy class graph in Theorem 3. By Remark 2.3 they have the same number of non-central conjugacy classes so also the size of the conjugacy classes are the same.*

Theorem 4 *Let G be a non-abelian metabelian group of type* (5), $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_4 =< a, b : a^4 = b^3 = 1, aba = a >$. *Then, the conjugacy class graph,* $\Gamma_G = K_2 \cup K_2$.

Proof Suppose $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, according to Proposition 4, the center of G $Z(G) = \{e, a^2\}$, the number of non-central conjugacy class of G is equal to four and this is equal to the number of vertices in G. Based on the vertices adjacency of conjugacy class graph, G consist of two complete components of K_2. □

Remark 3.4 *The non-abelian metabelian groups of types* (7) *and* (24), *namely* D_6 and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ *with order 12 and 21 respectively, are having the same number of non-central conjugacy classes and the size of conjugacy classes are the same given by Remark 4. Thus, the conjugacy class graph of* D_6 and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ *are the same as is Theorem 4.*

Theorem 5 *Let G be a non-abelian metabelian group of type* (8), $G \cong D_7 = < a, b : a^7 = b^2 = 1, bab = a^{-1} >$. *Then,* $\Gamma_G = K_3 \cup \{\mathrm{cl}(b)\}$.

Proof Suppose $G = D_7$, and based on Proposition 5, the center of G, denoted as $Z(G) = \{e\}$, then, the number of non-central conjugacy class of G is equal to four which is equal to the number of vertices in Γ_G. According to the vertices adjacency of conjugacy class graph, Γ_G consists of one complete graph of K_3 with an isolated vertex, namely $\mathrm{cl}(b)$. Thus $\Gamma_G = K_3 \cup \{\mathrm{cl}(b)\}$ as desired. □

Remark 3.5 *The conjugacy class graph in Theorem 5 has the same number of complete components with non-abelian metabelian group of type* (22), *namely* $Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

Theorem 6 *Let G be a non-abelian metabelian group of type* (9), $D_8 = \langle a, b : a^8 = b^2 = 1, bab = a^{-1} \rangle$. *Then* $\Gamma_G = K_5$.

Proof Suppose $G = D_8$, based on Proposition 6, the center of G, $Z(G) = \{e, a^4\}$, then, the number of non-central conjugacy classes of G is equal to five which is equal to the number of vertices in Γ_G. Since the greatest common divisor between the sizes between any two conjugacy classes is greater than one. Therefore, there is a complete graph of K_5. □

Remark 3.6 *The non-abelian metabelian groups of types* (9), (11) *and* (12) *have the same number of non-central conjugacy classes by Remark 2.6, hence they have the same conjugacy class graphs to that of the graphs in Theorem 6.*

Theorem 7 *Let G be a non-abelian metabelian group of type* (13), $G \cong \mathcal{Q} \times \mathbb{Z}_2 = < a, b, c : a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3b, ac = ca, bc = cb >$. *Then* $\Gamma_G = K_6$.

Proof Suppose $G = Q \times \mathrm{Z}_2$, and according to Proposition 7, the center of G, $Z(G) = \{e, c, a^2, a^2c\}$, then, the number of non-central conjugacy class of G is equal to 6 and it is equal to the number of vertices in Γ_G. Since the greatest common divisor between the sizes of any two conjugacy classes is greater than one, therefore Γ_G consists of one complete graph of K_6. □

Remark 3.7 *The conjugacy class graph of non-abelian metabelian groups of types* (15)*, type* (16) *and type* (17) *is similar to the conjugacy class graph in Theorem 7.*

Theorem 8 *Let G be a non-abelian metabelian group of type* (18)*, $G \cong D_9 = < a, b : a^9 = b^2 = 1, bab = a^{-1} >$. Then $\Gamma_{D_9} = K_4 \cup \{cl\,(b)\}$.*

Proof Suppose $G = D_9$, from Proposition 8, the center of G, $Z\,(G) = \{e\}$, then, the number of non-central conjugacy class of D_9 is equal to five which is equal to the number of vertices in Γ_G. Since the greatest common divisors between the sizes of any two of the four conjugacy classes is greater than one, thus Γ_G consists of one complete graph of K_4 with one isolated vertex, namely cl (b). □

Remark 3.8 *The non-abelian metabelian groups of Modular-*16*, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and $\mathbb{Z}_4 \rtimes \mathbb{Z}_5$, have the same conjugacy class graph as in Theorem 8. Since the sizes of non-central conjugacy classes of these groups are the same.*

Theorem 9 *Let G be a non-abelian metabelian group of type* (19)*, $G = S_3 \times \mathbb{Z}_3 = < a, b, c : a^3 = b^2 = c^3 = 1, bab = a^{-1}, ac = ca, bc = cb >$. Then, $\Gamma_G = K_3 \cup K_3$.*

Proof Suppose $G = S_3 \times \mathrm{Z}_3$, by Proposition 9, the center of G, $Z\,(G) = \{e, c, c^2\}$, then, the number of non-central conjugacy class of G is equal to six and this is equal to the number of vertices in Γ_G. According to the vertices adjacency of conjugacy class graph, Γ_G consists of two complete components of K_3, as claimed. □

Theorem 10 *Let G be a non-abelian metabelian group of type* (21)*, $G \cong D_{10} = < a, b : a^{10} = b^2 = 1, bab = a^{-1} >$. Then $\Gamma_G = K_4 \cup K_2$.*

Proof Suppose $G = D_{10}$, based on Proposition 10, the center of G, $Z\,(G) = \{e\}$, then, the number of non-central conjugacy classes of non-abelian metabelian group G is equal to six which is equal to the number of vertices in Γ_G. Since the greatest common divisor of the sizes of any two conjugacy classes is greater than one. Alternatively, four conjugacy classes have size two and two conjugacy classes have size five. Hence, Γ_G consist of two complete components of K_4 and K_2 i.e. $\Gamma_G = K_4 \cup K_2$. □

Theorem 11 *Let G be a non-abelian metabelian group of type* (25)*, $G \cong D_{11} = < a, b : a^{10} = b^2 = 1, bab = a^{-1} >$. Then $\Gamma_G = K_5 \cup \{\mathrm{cl}(b)\}$.*

Proof Suppose $G = D_{11}$, from Proposition 11, the center of G, $Z\,(G) = \{e\}$, the number of non-central conjugacy classes of non-abelian metabelian group G is equal to six thus $|\mathrm{V}\,(\mathit{\Gamma}_G)| = 6$. Based on the adjacency of conjugacy class graph, Γ_G hence it has one complete components of K_5 and an isolated vertex, namely cl (b). □

4 Conclusion

In this paper, it has been proven that the conjugacy class graph of non-abelian metabelian groups of types (2) and type (3) is a complete component graph of K_3. Whereas, the conjugacy class graph of non-abelian metabelian groups of types (4) and type (6) consists of one complete components with an isolated vertex. The conjugacy class graph of $K_2 \cup K_2$ corresponds to the conjugacy class graph of non-abelian metabelian groups of types (5), type (7) and type (24).

In addition, the non-abelian metabelian groups of type (8) and type (22) have the same conjugacy class graph, which is K_3 and an isolated vertex. However, the non-abelian metabelian groups of type (9), type (11), type (14) and type (12) have a complete graph of K_5. While K_6 is the conjugacy class graph that corresponds to non-abelian metabelian groups of type (13), type (15), type (16) and type (17). The conjugacy class graphs of non-abelian metabelian groups of type (14), type (18), type (20) and type (23) are the same, namely K_6.

The remaining conjugacy class graph $K_3 \cup K_3$ is for the group of type (19), and the union of component conjugacy class graph $K_4 \cup K_2$ is for the group of type (21). Moreover, the conjugacy class graph of non-abelian metabelian group of type (25) consists of one complete component of K_5 and an isolated vertex.

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