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General Form of Domination Polynomial for Two Types of Graphs Associated to Dihedral Groups

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Abstract A domination polynomial is a type of graph polynomial in which its coefficients represent the number of dominating sets in the graph. There are many researches being done on the domination polynomial of some common types of graphs but not yet for graphs associated to finite groups. Two types of graphs associated to finite groups are the conjugate graph and the conjugacy class graph. A graph of a group G is called a conjugate graph if the vertices are non-central elements of G and two distinct vertices are adjacent if they are conjugate to each other. Meanwhile, a conjugacy class graph of a group G is a graph in which its vertices are the non-central conjugacy classes of G and two distinct vertices are connected if and only if their class cardinalities are not coprime. The conjugate and conjugacy class graph of dihedral groups can be expressed generally as a union of complete graphs on some vertices. In this paper, the domination polynomials are computed for the conjugate and conjugacy class graphs of the dihedral groups.

Keywords Domination polynomial; conjugate graph; conjugacy class graph, dihedral group

Mathematics Subject Classification 05C25, 05C31, 05C69.

1 Introduction

A simple graph, Γ consists of non-empty set of vertices, V and a set of edges, E, which is the unordered pair of vertices from V. For vertices $u, v \in V$, u and v are adjacent if they are connected by an edge, $e = (u, v) \in E$. Open neighborhood of v is defined to be the set of all vertices adjacent to v, without v, denoted by $N(v) = \{u \in V : (u, v) \in E, u \neq v\}$. The set $N(v) \cup \{v\}$ is called closed neighborhood, denoted by $N[v]$ [1]. A complete graph, K_n, is a type of common graph in which it contains n vertices and each pair of distinct vertices is connected by an edge [2].

By using the properties of graphs, the algebraic properties of groups are studied. Two types of graphs associated to groups that will be considered in this research are the conjugate graph and the conjugacy class graph. A conjugate graph Γ_G^{conj} of a group G is defined as the graph whose vertex set is the non-central elements of G and two distinct vertices are adjacent if they are conjugate [3]. A conjugacy class graph Γ_G^{cl} of a group G is a graph whose vertex set is the non-central conjugacy classes of G and two distinct vertices $cl(a)$ and $cl(b)$ are adjacent if their class cardinalities are not coprime that is $gcd(|cl(a)|, |cl(b)|) > 1$ [4].

Graph polynomial is a well-developed area that is useful to analyze and study the properties of a graph. A domination polynomial is a graph polynomial, containing coefficients that represent the number of dominating sets in the graph [5]. Domination polynomials are useful in solving the communication network problem, the facility location problem and the land surveying and routings problem. The domination polynomial is used to measure new network reliability for some particular kind of service networks (in [6]). Dominating set has been applied in bioinformatics research through the domain decomposition algorithm in the assignment of structural domains in complex protein structures (in [7]).

In this research, the domination polynomials are obtained for the conjugate graph and conjugacy class graph of dihedral groups of order $2n$ with the group representation

$$D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1}\rangle,$$

where $n \geq 3$, $n \in \mathbb{N}$.

2 Preliminaries

Some preliminaries related to group theory and graph theory that are useful in this research are presented in this section. Firstly, the following are some basic concepts of the domination polynomial of a graph.

Definition 1 [8] Dominating Set, Domination Number

The dominating set of a graph Γ is a subset $S \subseteq V$ in which every vertex in $V - S$ is adjacent to at least one vertex in S, or equivalently $N[S] = V$. The minimum cardinality of a dominating set in Γ is called the domination number $\delta(\Gamma)$.

Definition 2 [5] Domination Polynomial

The domination polynomial of a graph Γ, denoted by $D(\Gamma, x)$, is the polynomial whose coefficient on x^k is given by the number of dominating sets of order k in Γ. So

$$D(\Gamma; x) = \sum_{k=\delta(\Gamma)}^{|V(\Gamma)|} d_k x^k$$

in which d_k is the number of dominating sets of size k and $\delta(\Gamma)$ is the domination number of the graph Γ.

From previous researches, some properties related to the domination polynomial are presented as in the following.

Theorem 1 [8] *Let Γ_1 and Γ_2 be two disjoint graphs. Then the domination polynomial of the union of both graphs can be expressed as follows:*

$$D(\Gamma_1 \cup \Gamma_2; x) = D(\Gamma_1; x) \cdot D(\Gamma_2; x).$$

Theorem 2 [8] *The domination polynomial of a complete graph, K_n on n vertices is*

$$D(K_n; x) = (1 + x)^n - 1.$$

Theorem 3 [9] *The domination polynomial of the union of m complete graphs is*

$$D(\bigcup_{i=1}^{m} K_{n_i}; x) = [(1 + x)^{n_i} - 1]^m.$$

Next, we state some basic definitions and results from group theory and graph theory that are important in this research.

Theorem 4 [10] *Let D_{2n} be dihedral groups of order $2n$, then the conjugacy classes of D_{2n} are expressed as follows:*

- *For odd n:*
  $\{1\}, \{a, a^{-1}\}, \{a, a^{-2}\}, \ldots, \{a^{\frac{n-1}{2}}, a^{-\frac{(n-1)}{2}}\}$ *and* $\{a^i b : 0 \leq i \leq n - 1\}$.
- *For even n:*
  $\{1\}, \{a^{\frac{n}{2}}\}, \{a, a^{-1}\}, \{a, a^{-2}\}, \ldots, \{a^{\frac{n-2}{2}}, a^{-\frac{(n-2)}{2}}\}, \{a^{2i} b : 0 \leq i \leq \frac{n-2}{2}\}$ *and*
  $\{a^{2i+1} b : 0 \leq i \leq \frac{n-2}{2}\}$.

The conjugacy classes of dihedral group as presented by Samaila *et al*. [10] are being used in the computation for this research to obtain the general form for the conjugate graph of dihedral group, so that, the domination polynomial can be constructed. The general form of the conjugacy class graph of dihedral group has been obtained by Mahmoud *et al*. [11] as stated in the following.

Theorem 5 [11] *Let D_{2n} be dihedral groups of order $2n$, then the conjugacy classes of D_{2n} are:*

$$\Gamma^{cl}_{D_{2n}} = \begin{cases} K_{\frac{n-1}{2}} \cup K_1 & ; \ n \text{ is odd} \\ K_{\frac{n+2}{2}} & ; \ n \text{ and } \frac{n}{2} \text{ are even} \\ K_{\frac{n-2}{2}} \cup K_2 & ; \ n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

3 Main Results

This section contains three parts. The first part presents the general form of the conjugate graph of D_{2n}.The second part is on the domination polynomial of the conjugate graph of D_{2n} while the last part is on the domination polynomial of the conjugacy class graph of D_{2n}.

3.1 The Conjugate Graph of the Dihedral Groups

Theorem 6 *Let D_{2n} be the dihedral groups of order $2n$, $n \geq 3$, $n \in \mathbb{N}$. The conjugate graph of D_{2n} can be presented as follows:*

$$\Gamma^{conj}_{D_{2n}} = \begin{cases} \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_2\right) \cup K_n & ; \ n \text{ is odd} \\ \left(\bigcup_{i=1}^{\frac{n-2}{2}} K_2\right) \cup \left(\bigcup_{i=1}^{2} K_{\frac{n}{2}}\right) & ; \ n \text{ is even.} \end{cases}$$

Proof Suppose that D_{2n} is a dihedral group of order $2n$ with $\Gamma^{conj}_{D_{2n}}$ as its conjugate graph. When n is odd, based on Theorem 4, the conjugacy classes in D_{2n} are $\{1\}$, $\{a, a^{-1}\}$, $\{a, a^{-2}\}$, …, $\{a^{\frac{n-1}{2}}, a^{-\frac{(n-1)}{2}}\}$ and $\{a^i b : 0 \leq i \leq n-1\}$. The vertex set of $\Gamma^{conj}_{D_{2n}}$ contains all non-central elements of D_{2n} such that $V(\Gamma^{conj}_{D_{2n}}) = \{a, a^{-1}, a^2, a^{-2}, \dots, a^{n-1}, a^{-(n-1)}, b, ab, \dots, a^{n-1}b\}$. For the vertices of the form $a^{\pm k}$, $k = 1, 2, \dots, \frac{n-1}{2}$, they always belong to conjugacy classes of cardinality two, hence there are $\frac{n-1}{2}$ complete graphs of two vertices. For the vertices of the form $a^i b$, $0 \leq i \leq n-1$, they belong to a conjugacy class of cardinality $(n-1)+1 = n$, hence there is one complete graph of n vertices. Thus, we obtain the conjugate graph of D_{2n}, $\Gamma^{conj}_{D_{2n}} = \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_2\right) \cup K_n$. Meanwhile, when n is even, the conjugacy classes in D_{2n} are $\{1\}$, $\{a^{\frac{n}{2}}\}$, $\{a, a^{-1}\}$, $\{a, a^{-2}\}$, …, $\{a^{\frac{n-2}{2}}, a^{-\frac{(n-2)}{2}}\}$, $\{a^{2i}b : 0 \leq i \leq \frac{n-2}{2}\}$ and $\{a^{2i+1}b : 0 \leq i \leq \frac{n-2}{2}\}$. The vertex set of $\Gamma^{conj}_{D_{2n}}$ is $V(\Gamma^{conj}_{D_{2n}}) = \{a, a^{-1}, a^2, a^{-2}, \dots, a^{n-1}, a^{-(n-1)}, b, ab, \dots, a^{n-1}b\}$. For vertices $a^{\pm k}$, $k = 1, 2, \dots, \frac{n-2}{2}$, they belong to conjugacy classes of cardinality two, hence there are $\frac{n-2}{2}$ complete graphs of two vertices. Meanwhile, vertices $a^{2i}b$, $0 \leq i \leq \frac{n-2}{2}$ belong to a conjugacy class of cardinality $\frac{n-2}{2} + 1 = \frac{n}{2}$ and vertices $a^{2i+1}b$, $0 \leq i \leq \frac{n-2}{2}$ also belong to a conjugacy class of cardinality $\frac{n-2}{2} + 1 = \frac{n}{2}$. Hence, there are two complete graphs, each of $\frac{n}{2}$ vertices. Therefore, the conjugate graph of D_{2n} is $\Gamma^{conj}_{D_{2n}} = \left(\bigcup_{i=1}^{\frac{n-2}{2}} K_2\right) \cup \left(\bigcup_{i=1}^{2} K_{\frac{n}{2}}\right)$. □

Example 1 Let D_{10} be the dihedral group of order 10 and $\Gamma^{conj}_{D_{10}}$ be its conjugate graph. The vertex set of $\Gamma^{conj}_{D_{10}}$ contains the noncentral elemets of D_{10}, that is

$$V\left(\Gamma^{conj}_{D_{10}}\right) = \{a, a^2, a^3, \ a^4, b, ab, a^2b, a^3b, a^4b\}.$$

The conjugate graph of D_{10} is $\left(\bigcup_{i=1}^{2} K_2\right) \cup K_5$, the union of two complete graphs of two vertices and a complete graph of five vertices as illustrated in Figure 1.

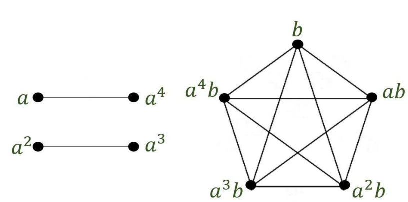

Figure 1: The Conjugate Graph of D_{10}, $K_2 \cup K_2 \cup K_5$

3.2 The Domination Polynomial of the Conjugate Graph of Dihedral Groups

Theorem 7 *Suppose that D_{2n} be dihedral groups of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$, then the domination polynomial of the conjugate graph of D_{2n} can be expressed as follows:*

$$D(\Gamma^{conj}_{D_{2n}}) = \begin{cases} \left(2x + x^2\right)^{\frac{n-1}{2}}\left((1+x)^n - 1\right) & ; \quad n \text{ is odd} \\ \left(2x + x^2\right)^{\frac{n-2}{2}}\left((1+x)^{\frac{n}{2}} - 1\right)^2 & ; \quad n \text{ is even.} \end{cases}$$

Proof Let D_{2n} be a dihedral group of order $2n$ and $\Gamma^{conj}_{D_{2n}}$ be its conjugate graph. When n is odd, from Theorem 6, we have the conjugate graph of dihedral groups as $\left(\bigcup_{i=1}^{\frac{n-1}{2}} K_2\right) \cup K_n$. By Theorem 1, Theorem 2 and Theorem 3, we can compute the domination polynomial of $\Gamma^{conj}_{D_{2n}}$ as in the following:

$$\begin{aligned} D(\Gamma^{conj}_{D_{2n}}; x) &= D\Big(\Big(\bigcup_{i=1}^{\frac{n-1}{2}} K_2\Big) \cup K_n; x\Big) \\ &= D\Big(\bigcup_{i=1}^{\frac{n-1}{2}} K_2; x\Big) \cdot D\Big(K_n; x\Big) \\ &= \Big[(1+x)^2 - 1\Big]^{\frac{n-1}{2}} \cdot \Big[(1+x)^n - 1\Big] \\ &= \Big(2x + x^2\Big)^{\frac{n-1}{2}}\Big((1+x)^n - 1\Big). \end{aligned}$$

When n is even, the conjugate graph of dihedral groups is $\left(\bigcup_{i=1}^{\frac{n-2}{2}} K_2\right) \cup \left(\bigcup_{i=1}^{2} K_{\frac{n}{2}}\right)$. By Theorem 1, Theorem 2 and Theorem 3, the domination polynomial of $\Gamma^{conj}_{D_{2n}}$ is computed:

$$\begin{aligned} D(\Gamma^{conj}_{D_{2n}}; x) &= D\Big(\Big(\bigcup_{i=1}^{\frac{n-2}{2}} K_2\Big) \cup \Big(\bigcup_{i=1}^{2} K_{\frac{n}{2}}\Big); x\Big) \\ &= D\Big(\bigcup_{i=1}^{\frac{n-1}{2}} K_2; x\Big) \cdot D\Big(\bigcup_{i=1}^{2} K_{\frac{n}{2}}; x\Big) \\ &= \Big[(1+x)^2 - 1\Big]^{\frac{n-2}{2}} \cdot \Big[(1+x)^{\frac{n}{2}} - 1\Big]^2 \\ &= \Big(2x + x^2\Big)^{\frac{n-2}{2}}\Big((1+x)^{\frac{n}{2}} - 1\Big)^2. \end{aligned}$$

□

3.3 The Domination Polynomial of the Conjugacy Class Graph of Dihedral Groups

Theorem 8 *Suppose that D_{2n} be dihedral groups of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$, then the domination polynomial of the conjugacy class graph of D_{2n} can be expressed as follows:*

$$D(\Gamma^{cl}_{D_{2n}}) = \begin{cases} x(1+x)^{\frac{n-1}{2}} - x & ; \; n \text{ is odd} \\ \Big(1+x\Big)^{\frac{n+2}{2}} - 1 & ; \; n \text{ and } \frac{n}{2} \text{ are even} \\ \Big(2x+x^2\Big)\Big((1+x)^{\frac{n-2}{2}} - 1\Big) & ; \; n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

Proof Let D_{2n} be a dihedral group of order $2n$ and $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph. When n is odd,from Theorem 5, we have the conjugacy class graph of dihedral groups as $K_{\frac{n-1}{2}} \cup K_1$. By Theorem 1 and Theorem 2, the domination polynomial of $\Gamma^{cl}_{D_{2n}}$ is computed as follows:

$$\begin{aligned} D(\Gamma^{cl}_{D_{2n}};x) &= D\Big(K_{\frac{n-1}{2}} \cup K_1;x\Big) \\ &= D\Big(K_{\frac{n-1}{2}};x\Big) \cdot D\Big(K_1;x\Big) \\ &= \Big[(1+x)^{\frac{n-1}{2}} - 1\Big] \cdot \Big[(1+x) - 1\Big] \\ &= x\Big(1+x\Big)^{\frac{n-1}{2}} - x. \end{aligned}$$

When n and $\frac{n}{2}$ are even, the conjugacy class graph graph of dihedral groups is expressed as $K_{\frac{n+2}{2}}$. By Theorem 1 and Theorem 2, the domination polynomial of $\Gamma^{cl}_{D_{2n}}$ is computed as follows:

$$D(\Gamma^{cl}_{D_{2n}};x) = D\Big(K_{\frac{n+2}{2}};x\Big) = \Big(1+x\Big)^{\frac{n+2}{2}} - 1.$$

When n is even and $\frac{n}{2}$ is odd, the conjugacy class graph graph of dihedral groups is $K_{\frac{n-2}{2}} \cup K_2$. The domination polynomial of $\Gamma^{cl}_{D_{2n}}$ can be computed as follows:

$$\begin{aligned} D(\Gamma^{cl}_{D_{2n}};x) &= D\Big(K_{\frac{n-2}{2}} \cup K_2;x\Big) \\ &= D\Big(K_{\frac{n-2}{2}};x\Big) \cdot D\Big(K_2;x\Big) \\ &= \Big[(1+x)^{\frac{n-2}{2}} - 1\Big] \cdot \Big[(1+x)^2 - 1\Big] \\ &= \Big(2x+x^2\Big)\Big((1+x)^{\frac{n-2}{2}} - 1\Big). \end{aligned}$$ □

4 Conclusion

In this paper, the graphs of dihedral groups are in the form of the union of some complete graphs. The degree of domination polynomial associated to dihedral groups is equal to the number of vertices in the graph. The domination polynomial of the conjugate graph of the dihedral groups, D_{2n} can be expressed depending on two cases of n. When n is odd, the domination polynomial of the conjugate graph of dihedral groups is of degree $2n-1$ and when n is even, the domination polynomial is of degree $2n-2$. Meanwhile, the domination polynomial of the conjugacy class graph of the dihedral groups, D_{2n} are established depending on three cases of n. When n is odd, the degree of the domination polynomial of the conjugacy class graph of dihedral groups is $\frac{n+1}{2}$. When n is even, eventhough the polynomials are expressed based on the cases $\frac{n}{2}$ is even or odd, the degree of the domination polynomial of the conjugacy class graph of dihedral groups for both cases is $\frac{n+2}{2}$.

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