

Bounds on the Action Degree of Groups

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Abstract The commutativity degree is an invariant used to measure the probability that two arbitrarily chosen group elements commute. This concept was generalized by many authors in different ways. One of these generalizations involves the action degree of finite groups, which is the probability that a random group element fixes a random element from a nonempty set. In this paper, we present some properties and important inequalities about the action degree and give a relationship between the action degree of a finite group and that of the kernel subgroup of this action. In addition, we show that the action degree play an important role to determine some information about the cardinality of the set of all fixed points. On the other hand, we introduce in this paper a new definition for the action degree of finitely generated groups. This definition generalizes another recent definition of the commutativity degree of finitely generated groups. Furthermore, some boundaries with respect to this degree are presented.

Keywords Group theory; action degree; commutativity degree of groups; finite groups; infinite groups.

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1 Introduction

There has been a growing interest in recent years on the use of the probability in group theory. One of the most important aspects which have been studied extensively is the probability that a selected random pair of elements (x, y) in a group G commute, denoted as $cd(G)$. In fact, the determination of the commutativity degree of a group yields a wealth of information about the group's structure and properties. This concept was introduced in 1968 by Erdos and Turan [1], where they worked on symmetric groups.

Gustafson [2] showed that this probability can be computed using conjugacy classes and MacHale [3] computed it using 0,1-Table. Numerous researches have been done on the commutativity degree and many results have been achieved. In 1975, Sherman [4] defined the

probability that a random element from a group G fixes a random element from a non-empty set S . We refer to this probability as the action degree of groups, $ad_S(G)$. Sherman [4] studied the probability that an automorphism fixes a group element, which gives commutativity degree in a special case. He considered only the case for which G is a finite abelian group and A is its group of automorphisms.

Omer *et al.* [5] generalized the commutativity degree by introducing the probability of an element g in G to fix a set X and G acts on the set of all subset of commuting elements in G of size 2. Moreover, the probability is used on different group structures namely dihedral group, quaternion group and alternating group. The probability was found for some symmetric groups in [6] in the case that a group acts on a set regularly and by conjugation. In 2015, El-sanfraz *et al.* [7] found the probability that a group element fixes a set under regular action for metacyclic 2-groups of positive type of nilpotency class of at least three.

In the first part of this paper, the general definition of action degree (the probability that a random element from a group G fixes a random element from a non-empty set S) is considered where some inequalities and bounds on the action degree of finite groups are determined.

In 2017, Antolin *et al.* [8] defined the commutativity degree of finitely generated groups and proved that there is a generalization of Gustafson's result on the bound of commutativity degree of finite groups. Inspired by their work, we generalize the definition of the action degree of finite groups and define the action degree of finitely generated groups. Furthermore, some bounds and limiting conditions on this degree are obtained.

2 Some Basic Concepts

The aim of this section is to briefly recall some definitions and preparatory results that are needed throughout this article.

In the following we recall the formal definitions of the commutativity degree and some of its generalizations which are related to this research and we begin with the concept of commutativity degree.

Definition 1 [2]

Let G be a group, the commutativity degree is the probability that two random elements (x, y) in G commute, defined as:

$$cd(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

The action degree of finite groups is defined as follows:

Definition 2 [4]

The probability that a random element from a group G fixes a random element from the non-empty set S is defined as:

$$ad_S(G) = \frac{|\{(x, s) \in G \times S \mid xs = s\}|}{|S| \cdot |G|}.$$

Remark:

Let G be a finite group acts on a finite set S .

1. The probability $ad_S(G) = 1$ if and only if $gs = s$ for all $g \in G$ and $s \in S$.
2. If G acts on itself by conjugation, $gs = gsg^{-1}$, then in this case, $ad_S(G) = cd(G)$, the commutativity degree of G .

It is useful to recall some basic concepts regarding the action of groups. Given a finite group G acting on a finite set S , the stabilizer of an element $s \in S$ is the subgroup G_s of G defined by $G_s = \{g \in G \mid gs = s\}$. By definition of the stabilizer, it is clear that the action degree can be defined as:

$$ad_S(G) = \frac{\sum_{s \in S} |G_s(G)|}{|S| \cdot |G|}.$$

The orbit of an element s in S is the subset $o(s)$ of G defined as $o(s) = \{gs \mid g \in G\}$. The relation \sim on S defined by $s \sim t$ if $s = gt$ for some $g \in G$ is an equivalent relation. Furthermore, the equivalent class of any $s \in S$ is the orbit $o(s)$. The number of distinct equivalent classes of this relation is denoted by $k_s(G)$.

The following is a result by Sherman [4] that can simplify the computation of the action degree. It is also needed to prove Corollary 1.

Proposition 1 [4]

Let G be a finite group acts on a finite set S . Then $ad_s(G) = \frac{k_s(G)}{|S|}$, where $k_s(G)$ denotes the number of distinct equivalent classes.

The focus in Sherman's research [4] was on the probability that an automorphism fixes a group element. He considered $P_A(G)$, where G is a finite abelian group and A is its group of automorphisms where he obtained some bounds and limiting conditions on this probability. In this paper, we provide some bounds and relations between the action degree of groups, the set of fixed points of the action and the kernel of action. Meanwhile, the class equation of group action is used to prove some of these relations. We recall these concepts in the following.

If G is a finite group that acts on a finite set S , the set

$$S_0(G) = \{s \in S \mid gs = s \text{ for all } g \in G\}$$

is called the fixed point set of the action. It may also be viewed as the union of all single element orbits. Suppose a group G is acting on a finite set S . If $\{o(s_1), o(s_2), \dots, o(s_k)\}$ is the set of distinct orbits of G on S . Then

$$|S| = \sum_{i=1}^k |o(s_i)|.$$

If $I = \{1 \leq i \leq n \text{ such that } |o(s_i)| > 1\}$, then

$$|S| = |S_0(G)| + \sum_{i=1}^k |o(s_i)|.$$

Let G be a finite group that acts on a finite set S , the set

$$Z_s(G) = \{g \in G \mid gs = s \ \forall s \in S\}$$

is known as the kernel of action and it is a normal subgroup of G .

One main objective of this research is to determine some bounds on the probability that a random element from a group G fixes a random element from the non-empty set S that was presented by Sherman [4]. As mentioned earlier, we refer to this probability by the action degree of groups.

The other main objective of this article is to extend the work on action degree of finite groups to the finitely generated groups. Thus, the following part presents some basic concepts and earlier studies on the degree of commutativity of finitely generated groups.

Let G be a group and S a generating set for G . The set of inverses of elements of S is denoted by S^{-1} . The length function $l_S : G \rightarrow \mathbb{N}$ gives the smallest integer n such that there exist $s_1, s_2, \dots, s_n \in S \cup S^{-1}$ with $g = s_1 s_2 \dots s_n$. It is then often written $|g| = n$. The notion of the growth of a finitely generated group was introduced by Schwarz [9] and independently by Milnor *et al.* [10]. The definition is given as follows: Let G be a finitely generated group and S a finite set of generators for G . The growth function of the group G with respect to S , denoted as $B_S(n)$ is the set of elements in G that can be written as a word on S having length at most n . If S and T are two finite generating sets for G , then the growth functions with respect to S and T are equivalent. So for each group G the growth type, defined to be the equivalence class of one of its growth functions is distinct and independent of any generating set.

Some bounds on the action degree of finitely generated groups in the current research are only applied to groups of subexponential growth. Accordingly, this concept needs to be clarified. By Grigorchuk [11], the limit $\lim_{n \rightarrow \infty} |B_S(n)|^{\frac{1}{n}}$ always exists and it is called the exponential growth rate of G with respect to S . Moreover, this limit depends on the set of generators S but the fact of being 1 or larger than 1 does not depend on S . Therefore, it is an invariant of the group G . A group is said to have exponential growth when its exponential growth rate is larger than one, and subexponential growth when the exponential growth rate is one.

In 2017, Antolin *et al.* [8] defined the commutativity degree of finitely generated groups as follows:

Definition 3 [8] *Let G be a finitely generated group and X a finite generating set. The degree of commutativity of G with respect to X , denoted $cd_X(G)$, is defined as:*

$$cd_X(G) = \limsup_{n \rightarrow \infty} \frac{|\{(g, h) \in (B_X(n))^2 \mid gh = hg\}|}{|B_X(n)|^2}$$

where $B_X(n) = \{g \in G : l_X(g) \leq n\}$ for every generating set X of G .

The next lemma is needed to prove some bounds on the action degree of finitely generated groups.

Lemma 1 [12] *Let G be a finitely generated group with subexponential growth, and let X be a finite generating set for G . For every finite index subgroup H of G and every $g \in G$, we have:*

$$\lim_{n \rightarrow \infty} \frac{|gH \cap B_X(n)|}{|B_X(n)|} = \frac{1}{[G : H]}.$$

The present paper introduces a similar definition of the action degree of finitely generated groups. Further, some useful bounds and inequalities regarding this probability are provided.

3 Results and Discussion

The main results of this paper are presented in this section where some inequalities and limiting conditions on the action degree of finite groups are provided. Moreover, the definition and some properties of the action degree of finitely generated groups are presented.

3.1 Action Degree of Finite Groups

Given a finite group G acting on a finite set S , we define a set $A_S(G)$ that consists of all ordered pairs (x, s) in $G \times S$ where the element $x \in G$ fixes $s \in S$, precisely, $A_S(G) = \{(x, s) \in G \times S \mid xs = s\}$.

Therefore

$$ad_S(G) = \frac{|A_S(G)|}{|S| \cdot |G|}.$$

Throughout this paper, we use this formula or the formula of the stabilizers of elements to prove our results.

Lemma 2 *Let G be a finite group that acts on a finite set S . Then*

$$ad_S(G) \geq \frac{1}{|G|}.$$

Proof Since $1s = s$ for all $s \in S$, then

$$\{1\} \times S \subseteq A_S(G).$$

Therefore

$$|S| \leq |A_S(G)|.$$

Dividing by $|S| \cdot |G|$, we get the result. \square

Proposition 2 *Let G be a finite group that acts on a finite set S and H a subgroup of G . Then:*

$$ad_S(H) \leq [G : H] ad_S(G).$$

Proof For all $s \in S$, we have

$$G_s(H) \subseteq G_s(G).$$

Then,

$$\sum_{s \in S} |G_s(H)| \leq \sum_{s \in S} |G_s(G)|.$$

From this inequality and since $|H| = |G| [G : H]^{-1}$, we get

$$\frac{1}{|H|} \sum_{s \in S} |G_s(H)| \leq \frac{[G : H]}{|G|} \sum_{s \in S} |G_s(G)|.$$

That means

$$ad_S(H) \leq [G : H] ad_S(G). \quad \square$$

Applying Proposition 1, we obtain the following corollary:

Corollary 1 *Let G be a finite group that acts on a finite set S and H a subgroup of G . Then*

$$k_s(H) \leq [G : H] k_s(G).$$

This corollary is a generalization of a result proved by Ernest [13] in the case when $k(G)$ is the number of conjugacy classes of G .

Proposition 3 *Let H and K be two finite groups acting respectively on finite sets S_1 and S_2 such that $S_1 \cap S_2 = \emptyset$. Then*

$$ad_{S_1 \times S_2}(H \times K) = ad_{S_1}(H) \cdot ad_{S_2}(K).$$

Proof First recall that the group $H \times K$ acts naturally on $S_1 \times S_2$ via:

$$(h, k)(s_1, s_2) = (hs_1, ks_2).$$

We know that

$$|H \times K| = |H| \cdot |K|$$

and

$$|(S_1 \times S_2)| = |S_1| \cdot |S_2|.$$

To prove

$$ad_{S_1 \times S_2}(H \times K) = ad_{S_1}(H) \cdot ad_{S_2}(K),$$

we need to show that:

$$|A_{S_1 \times S_2}(H \times K)| = |A_{S_1}(H)| \cdot |A_{S_2}(K)|.$$

Consider the following function

$$\psi : A_{S_1 \times S_2}(H \times K) \rightarrow A_{S_1}(H) \times A_{S_2}(K)$$

defined by:

$$\psi((h, k), (s_1, s_2)) = ((h, s_1), (k, s_2)).$$

Remark that this function is well-defined and it is a bijection. Then

$$\begin{aligned} |A_{S_1 \times S_2}(H \times K)| &= |A_{S_1}(H) \times A_{S_2}(K)| \\ &= |A_{S_1}(H)| \cdot |A_{S_2}(K)|. \end{aligned}$$

So,

$$\begin{aligned} ad_{S_1 \times S_2}(H \times K) &= \frac{|A_{S_1 \times S_2}(H \times K)|}{|S_1 \times S_2| \cdot |H \times K|} \\ &= \frac{|A_{S_1}(H)| \cdot |A_{S_2}(K)|}{|S_1| \cdot |S_2| \cdot |H| \cdot |K|} \\ &= \left(\frac{|A_{S_1}(H)|}{|S_1| \cdot |H|} \right) \left(\frac{|A_{S_2}(K)|}{|S_2| \cdot |K|} \right) \\ &= ad_{S_1}(H) \cdot ad_{S_2}(K). \end{aligned}$$

□

Before proceeding to the next theorem, recall that the kernel of action

$$Z_s(G) = \{g \in G \mid gs = s \ \forall s \in S\},$$

is a normal subgroup of G . In addition, $G/Z_s(G)$ acts on the set S by $(gH)s = gs$ and this action is well defined.

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is a normal subgroup of G . In addition, $G/Z_s(G)$ acts on the set S by $(gH)s = gs$ and this action is well defined.

Theorem 1 *Let G be a finite group that acts on a finite set S and $H = Z_s(G)$. Then $ad_S(G) = ad_S(G/H)$.*

Proof Since $gs = s$ if and only if $gHs = s$, we have:

$$|A_S(G)| = |H| \cdot |A_S(G/H)|.$$

Then

$$\begin{aligned} ad_S(G/H) &= \frac{|A_S(G/H)|}{|G/H| \cdot |S|} \\ &= \frac{|A_S(G)|}{|H| \cdot |G/H| \cdot |S|} \\ &= \frac{|A_S(G)|}{|G| \cdot |S|} \\ &= ad_S(G). \end{aligned}$$

□

Proposition 4 *Let G be a finite group that acts on a finite set S . Then*

$$ad_S(G) \geq \frac{|Z_s(G)|}{|G|}.$$

Proof Recall that

$$A_S(G) = \{(x, s) \in G \times S \mid xs = s\}.$$

Clearly, for all $g \in Z_s(G)$, we have $\{g\} \times S \subseteq A_S(G)$ which shows that:

$$Z_s(G) \times S \subseteq A_S(G).$$

Therefore

$$\begin{aligned} ad_S(G) &\geq \frac{|Z_s(G) \times S|}{|S| \cdot |G|} \\ &= \frac{|Z_s(G)| \cdot |S|}{|S| \cdot |G|} \\ &= \frac{|Z_s(G)|}{|G|}. \end{aligned}$$

□

Proposition 5 *Let G be a finite group that acts on a finite set S . Then*

$$ad_S(G) \leq \frac{1}{2} + \frac{|S_0(G)|}{2|S|}.$$

Proof Consider the class equation:

$$|S| = |S_0(G)| + \sum_{i=|S_0(G)|+1}^{k_s(G)} |o(s_i)|.$$

Then

$$|S| \geq |S_0(G)| + 2(k_s(G) - |S_0(G)|).$$

Therefore

$$|S| \geq 2k_s(G) - |S_0(G)|.$$

Solving for $k_s(G)$ yields

$$k_s(G) \leq \frac{|S| + |S_0(G)|}{2}.$$

Therefore

$$ad_S(G) \leq \frac{1}{2} + \frac{|S_0(G)|}{2|S|}. \quad (\square)$$

Proposition 6 *Let G be a finite group that acts on a finite set S . Then*

$$ad_S(G) \geq \frac{|S_0(G)|}{|S|}. \quad (1)$$

Proof Let t be a fixed point of this action. Then $(x, t) \in A_S(G)$ for all $x \in G$. Hence $G \times S_0(G) \subseteq A_S(G)$.

From this we can show that:

$$ad_S(G) \geq \frac{|S_0(G)|}{|S|}. \quad \square$$

Corollary 2 *If $ad_S(G) > \frac{1}{2}$, then there is at least one fixed point.*

Proof Assume that $ad_S(G) > \frac{1}{2}$. By using the inequality (1), we get

$$\frac{|S_0(G)|}{|S|} > 0.$$

This shows that $|S_0(G)| > 0$, and hence the set $S_0(G)$ is not empty. \square

Remark that if the action has a fixed point, then $ad_S(G)$ is not necessary greater than $\frac{1}{2}$, which means that the opposite of Corollary (2) is not always true. For an example, let G be a group acting on itself by conjugation. Then the action of G on itself in this case is the commutativity degree of G and $sg = gsg^{-1} = s \quad \forall g \in G$. Then $gs = sg \quad \forall g \in G$. Therefore $s \in Z(G)$ is a fixed point of this action while the action degree (commutativity degree in this case) of the group can be less than $\frac{1}{2}$ as in the symmetric group S_4 for instance.

Theorem 2 Let G be a finite group that acts on a finite set S . Let $a = \frac{|Z_s(G)|}{|G|}$ and $b = \frac{|S_0(G)|}{|S|}$. Then

$$ad_S(G) \geq a + b - ab.$$

Proof We know that:

$$(Z_s(G) \times S) \cup (G \times S_0(G)) \subseteq A_S(G).$$

So

$$|A_S(G)| \geq |Z_s(G) \times S| + |G \times S_0(G)| - |Z_s(G) \times S_0(G)|.$$

Therefore

$$|A_S(G)| \geq |Z_s(G)| \cdot |S| + |G| \cdot |S_0(G)| - |Z_s(G)| \cdot |S_0(G)|.$$

Finally

$$ad_S(G) \geq \frac{|Z_s(G)|}{|G|} + \frac{|S_0(G)|}{|S|} - \left(\frac{|Z_s(G)|}{|G|} \right) \left(\frac{|S_0(G)|}{|S|} \right). \quad \square$$

In the following, the definition of the action degree of finitely generated groups is presented along with some bounds and inequalities.

3.2 Action Degree of Finitely Generated Groups

So far, this paper has focused on finite groups. The following part moves on to finitely generated groups. We begin with the definition of the action degree of finitely generated groups.

Definition 4 Let G be a finitely generated group and X a finite generating set of G . Assume that G acts on a finite set S . We define the action degree of G with respect to X , denoted as $ad_S(G, X)$ as:

$$ad_S(G, X) = \limsup_{n \rightarrow \infty} \frac{|\{(g, s) \in B_X(n) \times S \mid gs = s\}|}{|B_X(n)| \cdot |S|}$$

where $B_X(n) = \{g \in G : l_X(g) \leq n\}$ for every generating set X of G .

If G acts on itself by conjugation $gs = gsg^{-1}$, then:

$$cd_X(G) = \limsup_{n \rightarrow \infty} \frac{|\{(g, h) \in (B_X(n))^2 \mid gh = hg\}|}{|B_X(n)|^2}$$

and this is the degree of commutativity of infinite groups presented by Antolin *et al.* in [8].

Proposition 7 If G is a finite group, then $ad_S(G, X) = ad_S(G)$ for every generating set of G .

Proof Assume that G is finite, then there exists n_0 such that for all $n \geq n_0$, $B_X(n) = G$.
 So

$$\frac{|\{(g, s) \in B_X(n) \times S \mid gs = s\}|}{|B_X(n)| \cdot |S|} = \frac{|\{(g, s) \in G \times S \mid gs = s\}|}{|G| \cdot |S|},$$

for all $n \geq n_0$. Passing to limit when $n \rightarrow \infty$, we get $ad_S(G, X) = ad_S(G)$. □

Note that if X and Y are two generating sets of G , then $ad_S(G, X) = ad_S(G, Y)$. This follows directly from the fact that the growth functions with respect to X and Y are equivalent. Now, let us determine some inequalities on the action degree of finitely generated groups.

Theorem 3 Let $S_0(G) = \{s \in S \mid gs = s \text{ for all } g \in G\}$. Then

$$ad_S(G, X) \geq \frac{|S_0|}{|S|}.$$

Proof Consider the set

$$A_S(G, X)_n = \{(g, s) \in B_X(n) \times S \mid gs = s\}.$$

Clearly

$$B_X(n) \times S_0(G) \subseteq A_S(G, X)_n.$$

So

$$|A_S(G, X)_n| \geq |B_X(n) \times S_0(G)| = |B_X(n)| \cdot |S_0(G)|.$$

Therefore, for all $n \geq 1$.

$$\frac{|A_S(G, X)_n|}{|S| \cdot |B_X(n)|} \geq \frac{|B_X(n)| \cdot |S_0(G)|}{|S| \cdot |B_X(n)|} = \frac{|S_0(G)|}{|S|}.$$

Passing to limit when $n \rightarrow \infty$, we obtain the result. □

Theorem 4 Let G be a finitely generated group that acts on a finite set S and let X be a finite generating set for G . Suppose that G has subexponential growth. Then for every finite index normal subgroup H of G contained in $Z_s(G)$,

$$ad_S(G, X) \leq ad_S(G/H).$$

Proof Since H is a finite index normal subgroup of G , then

$$ad_S(G/H) = \frac{|A_S(G/H)|}{|S| \cdot [G : H]}.$$

Then the set $A_S(G/H)$ can be written as:

$$A_S(G/H) = \bigcup_{i=1}^r \{(g_i H, s_i)\},$$

for some $g_i \in G$ and $s_i \in S$. Consider the set

$$A_S(G, X)_n = \{(g, s) \in B_X(n) \times S \mid gs = s\}.$$

Remark that for every element $(g, s) \in A_S(G, X)_n$, we have:

$$(g, s) \in (gH \cap B_X(n)) \times \{s\}.$$

Since (gH, s) is an element in $A_S(G/H)$, we obtain $(g, s) \in (g_iH \cap B_X(n)) \times \{s_i\}$, for some $i = 1, \dots, r$. This shows that

$$A_S(G, X)_n \subseteq \bigcup_{i=1}^r (g_iH \cap B_X(n)) \times \{s_i\}.$$

Therefore

$$|A_S(G, X)_n| \leq \sum_{i=1}^r |g_iH \cap B_X(n)|.$$

Hence

$$\begin{aligned} \frac{|A_S(G, X)_n|}{|S| \cdot |B_X(n)|} &\leq \frac{\sum_{i=1}^r |g_iH \cap B_X(n)|}{|S| \cdot |B_X(n)|} \\ &\leq \frac{1}{|S|} \sum_{i=1}^r \frac{|g_iH \cap B_X(n)|}{|B_X(n)|}. \end{aligned}$$

Passing to limit when $n \rightarrow \infty$ we get by using Lemma 1:

$$\begin{aligned} ad_S(G, X) &\leq \frac{1}{|S|} \sum_{i=1}^r \lim_{n \rightarrow \infty} \frac{|g_iH \cap B_X(n)|}{|B_X(n)|} \\ &\leq \frac{1}{|S|} \sum_{i=1}^r \frac{1}{[G : H]} \\ &= \frac{|A_S(G/H)|}{|S| \cdot [G : H]} \\ &= ad_S(G/H). \end{aligned} \quad \square$$

4 Conclusion

In this paper, the probability that an element of a group fixes a set is further studied for finite groups. Several results concerning the action degree are presented along with some limiting conditions and inequalities. Moreover, the work on action degree of finite groups is extended to finitely generated groups which generalizes previous results on the commutativity degree of finitely generated groups.

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