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Flow Past a Pair of Separated Porous Spheres using Brinkman Model

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Abstract In this work, we obtained an analytical solution to the problem of viscous fluid flow past a pair of porous separated spheres. Stokesian approximation of the Navier-Stokes equation for the viscous fluid governs the flow in the region outside the two spheres, whereas Brinkman's model describes the flow in the porous region (within the spheres). Since the bipolar coordinate system is the most convenient system to represent separated spheres' geometry, we formulated this problem in the bipolar coordinate system. We then eliminated the pressure term from the equations governing the flow in the region outside the spheres, and they got reduced to a separable equation in terms of the stream function. Further, the flow governing equations inside the porous spheres gave rise to the Helmholtz equation. As the Helmholtz equation is not separable in the bipolar system, we used the spherical coordinate system to describe the fluid flow within each separated spheres and solved the resulting problem in the spherical coordinate system. Because of this, the flow variables on either side of the interface (spheres) are in different coordinate systems. To match the values of the field variables at the boundary, we used the transformation equations between the bipolar and spherical systems and transformed all the variables into the bipolar coordinate system. We then solved the governing equations with appropriate boundary conditions for the arbitrary constants and derived expressions for the stream and pressure functions.We plotted the respective functions for various values of the mathematical model's parameters to understand the flow pattern and pressure distribution in the flow domain and noted our observations. This study revealed an intuiting insight that the pressure distribution inside the porous spheres is independent of the non-dimensional parameter related to the medium's permeability and the fluid's viscosity.

Keywords Bipolar coordinates, Brinkman's model, Helmholtz equation, Coordinate transformation, Separated spheres.

Mathematics Subject Classification 35Q35, 35A09, 35Q30

Nomenclature:
(ξ, η, ϕ) : bipolar coordinates,
$(\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\varphi)$: base vectors in the bipolar coordinate system,
(h_ξ, h_η, h_ϕ) : scale factors in the bipolar coordinate system,
a: parameter that has the dimension of L,
$\tau = \cos\eta$,
(R, θ, z) : Spherical coordinates,
E^2 : the Stokes stream function operator,
S_0 : the region outside the porous spheres,
$S_i, i = 1, 2$: the region within spheres $\xi = \xi_i$,
$\psi^{(i)}$, is the stream function in the region S_i,
$E^2 - \alpha^2$: Helmholtz operator,
U: Uniform velocity of the stream at infinity (LT^{-1}),
$\bar{q}^{(i)}$: Velocity of the fluid in the region $S_i(\mathrm{LT}^{-1})$,
$p^{(i)}$: pressure in the region $S_i(\mathrm{ML}^{-1}\ \mathrm{T}^{-2})$,
$u^{(i)}, v^{(i)}$: velocity components in the region $S_i(\mathrm{LT}^{-1})$,
$k^{(i)} i = 1, 2$: permeability of the porous spheres $\xi = \xi_i(\mathrm{L}^2)$,
μ^* : macroscopically averaged quantity pertaining to the porous medium ($\mathrm{ML}^{-1}\ \mathrm{T}^{-1}$),
μ : viscosity of the fluid ($\mathrm{ML}^{-1}\ \mathrm{T}^{-1}$),
$\lambda_i^2 = \dfrac{\mu}{\mu^* k^{(i)}}$ is a parameter that has the dimension of (L^{-1}),
$\vartheta_{n+1/2}(\tau)$: Gagenbauer function of the first kind,
$P_n(\tau)$: Legendre polynomials,
$K_{n+1/2}(R)$: modified Bessel function of the first kind.

1 Introduction

Fluid flow past separated spheres or flow past two spheres is a classical problem with its origin at the beginning of the 19^{th} century. While considering the flow of viscous fluid past separated spheres, Jeffery found the Laplace equation solution in the bipolar coordinate system [1]. Later, Stimson and Jeffery [2] formulated an equivalent problem of viscous fluid flow past separated spheres in the bipolar coordinate system. They studied the motion set up in the viscous fluid by two solid spheres moving with small constant and equal velocities. Further, they determined the stream function as an infinite series in terms of Legendre polynomials and hyperbolic trigonometric functions and calculated the forces necessary to maintain the two spheres' motion. Dyuro Endo [3] considered the potential flow of fluid past two solid spheres in the bipolar system and derived an expression for the velocity potential as an infinite series in associated Legendre polynomials. Sneddon and Fulton [4] solved the irrotational flow of a perfect fluid past two spheres problem using two sets of spherical polar coordinates (one for each sphere). Payne and Pell [5] studied Stokes flow for a class of axially symmetric bodies with flow past solid separated spheres formulated in the peripolar coordinate system. They derived an expression for the stream function in terms of associated Legendre polynomials. Later, many investigations carried out on fluid flow past two or cluster of solid spheres, wherein analytical or numerical solutions presented as in the references [6–14].

We find very few works related to flow past a pair of separated porous spheres in the literature. Neale *et al.* [15] derived solution to creeping flow relative to an isolated permeable sphere problem and generalised the results to cover flow relative to a swarm of permeable spheres. Wu *et al.* [16] evaluated the hydrodynamic drag force experienced by two highly porous spheres moving along their centerline, wherein the equations governing the flow were solved using CFD software. Recently Radhika *et al.* [17] studied the creeping flow past a pair of porous separated spheres with Darcy's law governing the porous region's flow. They formulated the problem in the bipolar coordinate system and derived an analytical solution to it. Though several researchers studied the problem of flow past porous spheres, to our knowledge, the problem of flow past porous separated spheres was given scant attention, as also observed by Wu *et al.* [16].

Thus, in this paper, we propose to work on the problem of viscous fluid flow past porous spheres. For this, we considered the stokesian approximation of the Navier-Stokes equations to describe the flow in the region outside the two porous spheres and Brinkman equations for the flow in the porous domain. Unlike the work presented by Radhika *et al.* [17], where the bipolar coordinate system suffices to describe the entire fluid flow domain, we had to use two different coordinate systems, one within the porous region (spheres) and the other in the region outside the porous spheres. The reason is, Brinkmann equations used to describe the fluid flow within the porous spheres reduced to Helmholtz form (partial differential equation), which is inseparable in the bipolar coordinate system. While the Navier- Stokes equations taken to describe the flow outside the porous region are separable.

Thus, we derived the flow equations within the porous spheres in the spherical coordinate system and the bipolar system to describe the fluid flow under stokesian approximation outside these spheres.

We assumed that the fluid is incompressible, the flow axisymmetric, then the equations governing the flow in the region S_0 (the region outside the porous spheres) took the form $E^4\psi^{(0)} = 0$, where E^2 is the Stokes stream function operator and $\psi^{(0)}$ is the stream function in the region S_0. This equation is separable in the bipolar coordinate system, and hence, an analytical expression for the stream function is derived using the method of separation of variables. In terms of the stream function, the flow equations in the porous region took the form $E^2\left(E^2 - \alpha^2\right)\psi^{(i)} = 0$, where $E^2 - \alpha^2$ is the Helmholtz operator. We know that this operator is not separable in the bipolar coordinate system. However, as mentioned earlier, we opted to work on the bipolar system in the present problem. Thus we formulated the flow problem within the spheres in the spherical coordinate system wherein the Helmholtz equation is known to be separable. The challenge is now to implement the interface's boundary conditions, wherein two different coordinate systems used, one on either side. Thus, to match the field variables' values at the interface, we used the coordinate transformation between the spherical and bipolar coordinate systems and converted the boundary conditions into a single system, i.e. the bipolar system. The resulting system of equations involving the arbitrary constants is solved using MATHEMATICA software ,and the analytical expressions for the stream and pressure functions are derived. We also plotted these functions for different values of the parameters and discussed them.

2 Mathematical Formulation and the Problem

Consider the flow of a viscous fluid (with a uniform velocity U at infinity) past a pair of separated porous spheres fixed in the flow domain, as shown in Figure 1.

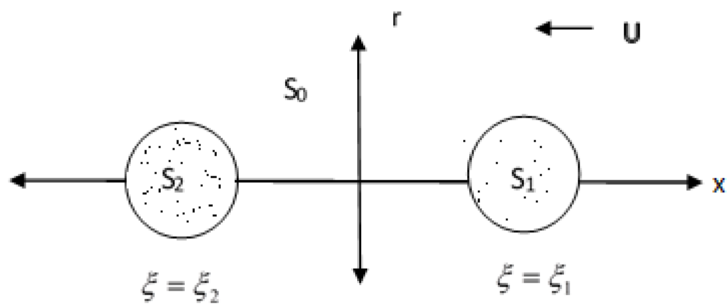


Figure 1: Schematic Diagram of Flow Past Separated Porous Spheres in Bipolar Coordinates

The bipolar system is represented by the coordinates (ξ, η, ϕ), with $(\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\varphi)$ as base vectors and (h_ξ, h_η, h_ϕ) as the corresponding scale factors where

$$x = \frac{a \sinh \xi}{\cosh \xi - \cos \eta} \text{ and } r = \frac{a\, \sin \eta}{\cosh \xi - \cos \eta}. \tag{1}$$

$$h_\xi = \frac{a}{\cosh \xi - \cos \eta};\ h_\eta = \frac{a}{\cosh \xi - \cos \eta};\ h_\phi = \frac{a \sin \eta}{\cosh \xi - \cos \eta} \tag{2}$$

where $-\infty < \xi < \infty, 0 \leq \eta < \pi$.

$\xi = c > 0$, where c is constant, represents spheres on the positive x-axis with center at a distance of $a \coth c$ from the origin (along the x-axis) and with radius equals $a \operatorname{cosech} c$. $\xi = c < 0$ describes spheres on the negative x-axis with their center at a distance of $-a \coth c$ from the origin (along the x-axis) and with radius equals $a \operatorname{cosech} c$.

2.1 Equations Governing the Fluid Flow in the Region S_0

Let $\left(\bar{q}^{(0)}, p^{(0)}\right)$ be the velocity vector and pressure function in the region S_0. Assuming that the flow axisymmetric, we have $\bar{q}^{(0)} = u^{(0)}(\xi, \eta)\, \hat{e}_\xi + v^{(0)}(\xi, \eta)\, \hat{e}_\eta$ and the pressure as $p^{(0)}(\xi, \eta)$. Further, considering the fluid to be incompressible and the flow as steady, the momentum equations under Stokesian approximation take the form:

$$\operatorname{grad} p^{(0)} + \mu \operatorname{curl}(\operatorname{curl} \bar{q}^{(0)}) = 0 \tag{3}$$

Now, introducing the stream function through,

$$h_\eta h_\phi u^{(0)} = -\frac{\partial \psi^{(0)}}{\partial \eta}; \quad h_\xi h_\phi v^{(0)} = \frac{\partial \psi^{(0)}}{\partial \xi} \tag{4}$$

we see that

$$\operatorname{curl} \vec{q}^{(0)} = \left\{ \frac{1}{h_\phi} E^2 \psi^{(0)} \right\} \vec{e}_\varphi \tag{5}$$

$$\operatorname{curl}\operatorname{curl}\vec{q}^{(0)} = \frac{1}{h_\xi h_\eta h_\phi}\left\{h_\xi \frac{\partial}{\partial \eta}\left(E^2\psi^{(0)}\right)\vec{e}_\xi - h_\eta \frac{\partial}{\partial \xi}\left(E^2\psi^{(0)}\right)\vec{e}_\eta\right\} \tag{6}$$

in which the Stokes stream function operatorE^2 is given by

$$E^2 = \frac{h_\phi}{h_\xi h_\eta}\left\{\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi h_\phi}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_\xi}{h_\eta h_\phi}\frac{\partial}{\partial \eta}\right)\right\} \tag{7}$$

Using expressions (5) and (6), equation (3) takes the form

$$\frac{1}{h_\xi}\frac{\partial p^{(0)}}{\partial \xi} + \frac{\mu}{h_\eta h_\phi}\frac{\partial}{\partial \eta}\left(E^2\psi^{(0)}\right) = 0 \tag{8}$$

$$\frac{1}{h_\eta}\frac{\partial p^{(0)}}{\partial \eta} - \frac{\mu}{h_\xi h_\varphi}\frac{\partial}{\partial \xi}\left(E^2\psi^{(0)}\right) = 0 \tag{9}$$

Eliminating $p^{(0)}$ from (8) and (9) gives

$$E^4\psi^{(0)} = 0 \tag{10}$$

which is the equation governing the fluid flow in the region S_0 .

2.2 Equations Governing the Fluid Flow in Regions $S_i, i = 1, 2$

Let $\left(\bar{q}^{(i)}, p^{(i)}\right)$ denote the velocity and pressure in the regions $S_i, i = 1, 2$, where S_1 represents the region inside the porous sphere $\xi = \xi_1$ and S_2 is the region inside the porous sphere $\xi = \xi_2$.

In these regions,we consider Brinkman's law given by

$$\operatorname{div}\bar{q}^{(i)} = 0 \tag{11}$$

$$\operatorname{grad} p^{(i)} + \frac{\mu}{k^{(i)}}\bar{q}^{(i)} + \mu^*\operatorname{curl}\operatorname{curl}\bar{q}^{(i)} = 0 \tag{12}$$

where $k^{(i)}i = 1, 2$ is the permeability of the porous spheres $S_i, i = 1, 2$ respectively, μ^* is a macroscopically averaged quantity pertaining to the porous medium and μ is the viscosity of the fluid [18].

Eliminating $p^{(i)}$ from equations (11) and (12), we get

$$E^2\left(E^2 - \lambda_i^2\right)\psi^{(i)} = 0, \quad i = 1, 2 \tag{13}$$

which is the governing equation in regions $S_i, i = 1, 2$.

Here

$$\lambda_i^2 = \frac{\mu}{\mu^* k^{(i)}}. \tag{14}$$

2.3 Boundary Conditions

The determination of the relevant flow field variables $\psi^{(0)}$ and $p^i, i = 0, 1, 2$ is subject to the following boundary and regularity conditions.

(i) Continuity of the normal velocity component at interfaces:

$$u^{(i)} = u^{(0)} \text{ on } S_i, i = 1, 2. \tag{15}$$

(ii) Tangential velocity component vanish at interfaces:

$$v^{(0)} = 0 \text{ on } S_i, i = 1, 2 \tag{16}$$

(iii) Continuity of pressure at interfaces:

$$p^{(i)} = p^{(0)} \text{ on } S_i, i = 1, 2. \tag{17}$$

(iv) The velocities are regular on the axis, and far away from S_0, the flow is a uniform stream which means, at infinity

$$\psi^{(0)} = -\frac{1}{2}Ur^2, \text{ i.e. } \lim_{\xi\to 0} u^{(0)} = -U, \lim_{\xi\to 0} v^{(0)} = 0. \tag{18}$$

3 Solution to the Equations Governing the Flow in S_0

In view of the linearity property of equation (10) that governs the fluid flow in the region S_0, we find the solution of

$$E^2\psi^{(0)} = f, \tag{19}$$

where f is the solution to

$$E^2 f = 0. \tag{20}$$

Solution to Equation (20):

Equation (20) in the bi-polar coordinate system is

$$\frac{\cosh\xi - \cos\eta}{a^2}\left(a\sin\eta\frac{\partial}{\partial\xi}\left(\frac{\cosh\xi - \cos\eta}{a\sin\eta}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left((\cosh\xi - \cos\eta)\frac{\partial}{\partial\eta}\right)\right) f = 0 \tag{21}$$

Following [2], let us take $\cos\eta = \tau$. Then equation (21) takes the form

$$\frac{\cosh\xi - \tau}{a^2}\left(\frac{\partial}{\partial\xi}\left((\cosh\xi - \tau)\frac{\partial}{\partial\xi}\right) + \left(1-\tau^2\right)\frac{\partial}{\partial\tau}\left((\cosh\xi - \tau)\frac{\partial}{\partial\tau}\right)\right) f = 0 \tag{22}$$

Now, following the method of separation of variables, we assume the solution to (22) as

$$f(\xi,\tau) = (\cosh\xi - \tau)^n\, g(\xi,\tau) \tag{23}$$

Substituting expression shown in (23) in equation (22) and after a straight forward calculation, we get

$$f(\xi,\tau) = (\cosh\xi - \tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi + B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau) \tag{24}$$

where A_n and B_n are arbitrary constants.

3.1 Solution to Equation (19)

Using the expression (24) in equation (19), we get

$$\frac{\cosh\xi-\tau}{a^2}\left(\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)\frac{\partial}{\partial\xi}\right)+\left(1-\tau^2\right)\frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)\frac{\partial}{\partial\tau}\right)\right)\psi^{(0)}$$
$$=(\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau). \tag{25}$$

Again, using the method of separation of variables, we assume the solution of (25) in the form

$$\psi^{(0)}(\xi,\tau)=a^2(\cosh\xi-\tau)^{-3/2}\sum_{n=1}^{\infty}h_n(\xi)\,\vartheta_{n+1}(\tau) \tag{26}$$

Now, substitute the above expression in equation (25). Using the following relations in Gegenbaur functions:

$$\left(1-x^2\right)\vartheta'_{n+1}(x)=-(n+1)\,x\vartheta_{n+1}(x)+(n-1)\,\vartheta_n(x), \tag{27}$$
$$(n+1)\,\vartheta_{n+1}(x)=(2n-1)\,x\vartheta_n(x)-(n-2)\,\vartheta_{n-1}(x), \tag{28}$$

and the orthogonality property of the Legendre polynomials, we get

$$\psi^{(0)}(\xi,\tau)=a^2(\cosh\xi-\tau)^{-3/2}\sum_{n=1}^{\infty}\left(C_n\cosh\left(n-\frac{1}{2}\right)\xi+D_n\sinh\left(n-\frac{1}{2}\right)\xi\right.$$
$$\left.+E_n\cosh\left(n+\frac{3}{2}\right)\xi+F_n\sinh\left(n+\frac{3}{2}\right)\xi\right)\vartheta_{n+1}(\tau), \tag{29}$$

where

$$-(2n-1)\,C_n+(2n+3)\,E_n=A_n \text{ and } -(2n-1)\,D_n+(2n+3)\,F_n=B_n. \tag{30}$$

3.2 Expression for Pressure in the Region S_0

We now derive the expression for the pressure function from equations (8) and (9). For this, let us consider equations (8) and (9) and substitute the expressions for the scale factors from (2) to get

$$\frac{\partial p^{(0)}}{\partial\xi}=-\frac{\mu(\cosh\xi-\tau)}{a}\frac{\partial}{\partial\tau}\left(E^2\psi^{(0)}\right), \tag{31}$$
$$\frac{\partial p^{(0)}}{\partial\tau}=-\frac{\mu(\cosh\xi-\tau)}{(1-\tau^2)}\frac{\partial}{\partial\xi}\left(E^2\psi^{(0)}\right). \tag{32}$$

Eliminating $\psi^{(0)}$ from these equations, we get

$$\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)^{-1}\frac{\partial p^{(0)}}{\partial\xi}\right)+\frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)^{-1}\left(1-\tau^2\right)\frac{\partial p^{(0)}}{\partial\tau}\right)=0. \tag{33}$$

Using the method of separation of variables, we get

$$p^{(0)}(\xi,\tau) = (\cosh\xi-\tau)^{1/2}\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau) \tag{34}$$

where's are Legendre's polynomials.

Substituting the expression for pressure from (34) in equation (31), we have

$$\begin{gathered}\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sinh\xi\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)+\\ (\cosh\xi-\tau)^{1/2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(H_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)=\\ \frac{\mu}{a}\left(\begin{array}{l}\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)+\\ (\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta'_{n+1}(\tau)\end{array}\right)\end{gathered} \tag{35}$$

Multiplying on both sides by $(\cosh\xi-\tau)^{-1}$ and integrating the resulting equation with respect to τ between the limits –1 and 1 gives,

$$\begin{gathered}\frac{1}{2}\sinh\xi\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{P_n(\tau)}{(\cosh\xi-\tau)^{3/2}}d\tau+\\ \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(H_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{P_n(\tau)}{(\cosh\xi-\tau)^{1/2}}d\tau=\\ -\frac{\mu}{2a}\left(\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}(\tau)}{(\cosh\xi-\tau)^{3/2}}d\tau\right)\end{gathered} \tag{36}$$

Using the formulae given in expressions (37) and (38), integrals in the above expression can be evaluated to find the equation involving the constants G_n, H_n, A_n and B_n.

$$\int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi-x)^{1/2}}dx=\frac{2\sqrt{2}}{2n+1}e^{-\left(n+\frac{1}{2}\right)|\xi|}. \tag{37}$$

$$\int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi-x)^{3/2}}dx=\frac{2\sqrt{2}}{\sinh|\xi|}e^{-\left(n+\frac{1}{2}\right)|\xi|}. \tag{38}$$

Substituting the expression for pressure from (34) in equation (32), we have

$$-\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)+$$
$$(\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n'(\tau)=-\frac{\mu}{a(1-\tau^2)}$$
$$\left(\begin{array}{l}-\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sinh\xi\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)+\\(\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\left(A_n\sinh\left(n+\frac{1}{2}\right)\xi+B_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)\end{array}\right) \tag{39}$$

Multiplying on both sides by $(\cosh\xi-\tau)^{-1}$and integrating the resulting equation with respect to τ between the limits -1 and 1 gives,

$$-\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\frac{d}{d\tau}\left(\left(1-\tau^2\right)P_n(\tau)\right)}{(\cosh\xi-\tau)^{1/2}}d\tau+$$
$$\sum_{n=1}^{\infty}\left(H_{n+1}\cosh\left(n+\tfrac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\left(1-\tau^2\right)P_n'(\tau)}{(\cosh\xi-\tau)^{1/2}}d\tau=$$
$$-\frac{\mu}{a}\left(\begin{array}{l}-\frac{1}{2}\sinh\xi\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}(\tau)}{(\cosh\xi-\tau)^{3/2}}d\tau+\\\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\left(A_n\sinh\left(n+\frac{1}{2}\right)\xi+B_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}(\tau)}{(\cosh\xi-\tau)^{1/2}}d\tau\end{array}\right) \tag{40}$$

Using the relations in (37), (38), and the recurrence relations in Legendre polynomials [19], we can derive the expressions for H_n's and G_n's in terms of A_n's and B_n's. These expressions can be written in terms of C_n, D_n, E_n and F_n using the relations in (30) and thus, the pressure function is completely determined.

4 Solution to the Equations Governing the Flow in Regions $S_i, i = 1, 2$

From equations (13), using the superposition principle, we see that each $\psi^{(i)}, i = 1, 2$ can be written as $\psi^{(i)} = \phi_1 + \phi_2, i = 1, 2$ where

$$E^2\varphi_1 = 0, \text{ and} \tag{41}$$
$$\left(E^2-\lambda_i^2\right)\varphi_2 = 0 \tag{42}$$

Since the solutions to equations in (41) and (41) are to be sought in the spherical coordinate system (R, θ, ϕ). We assume two separate spherical coordinate systems to describe the two spheres, with each one at each sphere's center. Following Happel and Brenner [20], the stream function in the region S_1 is

$$\psi^{(1)}(R,\theta)=\sum_{n=1}^{\infty}\Big(L_nR^{n+1}+M_n\sqrt{R}K_{n+1/2}(\lambda_1R)\Big)\vartheta_{n+1}(\varsigma), \tag{43}$$

where $\varsigma=\cos\theta$, and $K_{n+1/2}(\lambda_1R)$ is the modified Bessel function of the first kind and L_n, M_n are arbitrary constants.

Substituting the expression for the stream function $\psi^{(1)}(R,\theta)$ from (43) in (12), we get the pressure function as

$$p^{(1)}(R,\theta)=\lambda_1^2\mu^*\sum_{n=1}^{\infty}\frac{L_nR^n}{n}P_n(\varsigma) \tag{44}$$

Going along the same line, we have the stream function and the pressure in S_2 respectively as

$$\psi^{(2)}(R,\theta)=\sum_{n=1}^{\infty}\big(Q_nR^{n+1}+S_n\sqrt{r}K_{n+1/2}(\lambda_1R)\big)\vartheta_{n+1}(\varsigma), \tag{45}$$

where Q_n and S_n are arbitrary constants, and

$$p^{(2)}(R,\theta)=\lambda_2^2\mu^*\sum_{n=1}^{\infty}\frac{Q_nR^n}{n}P_n(\varsigma) \tag{46}$$

We introduce the non-dimensional quantities defined as to derive the non-dimensional expressions for the pressure and stream function in all three regions.

$$r^*=\frac{r}{a},\ z^*=\frac{z}{a},\ p^{(0)}=\frac{p^{(0)*}\mu U}{a},\ R^*=\frac{R}{a},\psi^{(0)}=Ua^2\psi^{(0)*},$$
$$p^{(i)}=\frac{p^{(i)*}\mu^*U}{a^2},\ i=1,2 \text{ and } \lambda_i^*=a\lambda_i, i=1,2.$$

However, we shall drop '*' from these expressions.

5 Determination of Arbitrary Constants

The eight sets of arbitrary constants $L_n,\ M_n\ Q_n,\ S_n, C_n,\ D_n,\ E_n$ and F_n, in expression (29), (43) and (45), are to be determined using the boundary conditions given in (15)–(18). As mentioned earlier, at the interface, we have the field variables on either side in two different coordinate systems, namely bi-polar and spherical. Thus, we use the following transformation equations (in their non-dimensional form) to transform all the boundary conditions in the bi-polar system.

$$R^2=\left(\frac{\cosh\xi+\tau}{\cosh\xi-\tau}\right),\quad \theta=\tan^{-1}\left(\frac{\sqrt{1-\tau^2}}{\sinh\xi}\right) \tag{47}$$

(i) Continuity of the normal velocity component at interfaces: $u^{(0)} = u^{(1)}$ on $\xi=\xi_1$ gives,

$$-\frac{(\cosh\xi-\tau)^2}{\sin\eta}\frac{\partial\psi^{(0)}}{\partial\tau}=-\frac{1}{R^2\sin\theta}\frac{\partial\psi^{(1)}}{\partial\theta} \text{ on } \xi=\xi_1. \tag{48}$$

Integrating the above expression with respect to τ between the limits -1 and 1 gives,

$$\int_{-1}^{1} \frac{\sin\eta}{(R^2\sin\theta)|_{\xi=\xi_1}(\cosh\xi_1-\tau)^2}\frac{\partial\psi^{(1)}}{\partial\theta}d\tau = 0 \tag{49}$$

Using the coordinate transformations given in (47) and the expression for $\psi^{(1)}(r,\theta)$ from (43), we get an expression involving the constants L_n, M_n. and $u^{(0)} = u^{(2)}$ on $\xi = \xi_2$ gives,

$$\int_{-1}^{1} \frac{\sin\eta}{(R^2\sin\theta)|_{\xi=\xi_2}(\cosh\xi_2-\tau)^2}\frac{\partial\psi^{(2)}}{\partial\theta}d\tau = 0 \tag{50}$$

(ii) Tangential velocity component vanishes at interfaces: $v^{(0)} = 0$ on $\xi = \xi_1$ gives

$$\frac{\partial\psi^{(0)}}{\partial\xi} = 0 \text{ on } \xi = \xi_1. \tag{51}$$

$$\begin{aligned}
&-\frac{3}{2}\frac{\sinh\xi_1}{\sinh\left|\xi_1\right|}* \\
&\sum_{n=1}^{\infty}\Big(C_n\cosh\Big(n-\frac{1}{2}\Big)\xi_1 + D_n\sinh\Big(n-\frac{1}{2}\Big)\xi_1 + E_n\cosh\Big(n+\frac{3}{2}\Big)\xi_1 \\
&\quad + F_n\sinh\Big(n+\frac{3}{2}\Big)\xi_1\Big)\left(\frac{e^{-(n-1/2)|\xi_1|}-e^{-(n+3/2)|\xi_1|}}{2n+1}\right) \\
&\qquad + \sum_{n=1}^{\infty}\frac{1}{2n+1}\Big(\Big(n-\frac{1}{2}\Big)C_n\sinh\Big(n-\frac{1}{2}\Big)\xi_1 + \Big(n-\frac{1}{2}\Big)D_n\cosh\Big(n-\frac{1}{2}\Big)\xi_1 \\
&\qquad\quad + \Big(n+\frac{3}{2}\Big)E_n\sinh\Big(n+\frac{3}{2}\Big)\xi_1 + \Big(n+\frac{3}{2}\Big)F_n\cosh\Big(n+\frac{3}{2}\Big)\xi_1\Big) \\
&\qquad\qquad\left(\frac{e^{-(n-1/2)|\xi_1|}}{2n-1}-\frac{e^{-(n+3/2)|\xi_1|}}{2n+3}\right) = 0.
\end{aligned} \tag{52}$$

Furthermore, $v^{(0)}$ =0 on $\xi = \xi_2$ gives,

$$\frac{\partial\psi^{(0)}}{\partial\xi} = 0 \text{ on } \xi = \xi_2. \tag{53}$$

$$\begin{aligned}
&-\frac{3}{2}\frac{\sinh\xi_2}{\sinh}\left|\xi_2\right| * \sum_{n=1}^{\infty}\Big(C_n\cosh\Big(n-\frac{1}{2}\Big)\xi_2 + D_n\sinh\Big(n-\frac{1}{2}\Big)\xi_2 + E_n\cosh\Big(n+\frac{3}{2}\Big)\xi_2 \\
&\quad + F_n\sinh\Big(n+\frac{3}{2}\Big)\xi_2\Big)\left(\frac{e^{-(n-1/2)|\xi_2|}-e^{-(n+3/2)|\xi_2|}}{2n+1}\right) \\
&\qquad + \sum_{n=1}^{\infty}\frac{1}{2n+1}\Big(\Big(n-\frac{1}{2}\Big)C_n\sinh\Big(n-\frac{1}{2}\Big)\xi_2 + \Big(n-\frac{1}{2}\Big)D_n\cosh\Big(n-\frac{1}{2}\Big)\xi_2 \\
&\qquad\quad + \Big(n+\frac{3}{2}\Big)E_n\sinh\Big(n+\frac{3}{2}\Big)\xi_2 + \Big(n+\frac{3}{2}\Big)F_n\cosh\Big(n+\frac{3}{2}\Big)\xi_2\Big) \\
&\qquad\qquad\left(\frac{e^{-(n-1/2)|\xi_2|}}{2n-1}-\frac{e^{-(n+3/2)|\xi_2|}}{2n+3}\right) = 0.
\end{aligned} \tag{54}$$

(iii) Continuity of pressure at interfaces: $p^{(1)} = p^{(0)}$ on $\xi = \xi_1$ gives

$$\lambda_1^2 \sum_{n=1}^{\infty} \frac{L_n}{n}\Big(R^n P_n(\varsigma)\Big)\Big|_{\xi=\xi_1} = \Big(\cosh \xi_1 - \tau\Big)^{1/2} \sum_{n=0}^{\infty} \Big(H_{n+1} \cosh \Big(n+\frac{1}{2}\Big)\xi_1 + G_{n+1} \sinh \Big(n+\frac{1}{2}\Big)\xi_1\Big) P_n\Big(\tau\Big), \tag{55}$$

and $p^{(2)} = p^{(0)}$ on $\xi = \xi_2$ gives

$$\lambda_2^2 \sum_{n=1}^{\infty} \frac{R_n}{n}\Big(R^n P_n(\varsigma)\Big)\Big|_{\xi=\xi_2} = (\cosh \xi_2 - \tau\Big)^{1/2} \sum_{n=0}^{\infty} \Big(H_{n+1} \cosh \Big(n+\frac{1}{2}\Big)\xi_2 + G_{n+1} \sinh \Big(n+\frac{1}{2}\Big)\xi_2\Big) P_n(\tau\Big), \tag{56}$$

(iv) Regularity condition at infinity:

$$\lim_{\xi\to 0} u^{(0)} = -U$$

$$\Rightarrow \frac{3}{2}(1-\tau)^{-1/2} \sum_{n=1}^{\infty} (C_n + E_n)\vartheta_{n+1}(\tau) - (1-\tau)^{1/2} \sum_{n=1}^{\infty} (C_n + E_n)\vartheta'_{n+1}(\tau) = -1, \tag{57}$$

$$\lim_{\xi\to 0} v^{(0)} = 0 \Rightarrow \sum_{n=1}^{\infty} \left(\left(n-\frac{1}{2}\right) D_n + \left(n+\frac{3}{2}\right) F_n\right)\vartheta_{n+1}(\tau) = 0. \tag{58}$$

We see that the above sets of equations to determine the arbitrary constants are infinite series in infinite sets of constants. Solving these equations for the constants is the most crucial and complex task in this study. We handled the complexity in the following way: For this, we used the definition for equality of two infinite series that states that "Two infinite series are equal if and only if the corresponding partial sums are equal".

Thus, equating the sum to the first one term of the two series (here, we take the right-hand side of the above equations as the zero series), we get eight equations in 8 unknowns $L_1, M_1, R_1, S_1, C_1, D_1, E_1, F_1$ which can be easily solved for these unknowns. Now, equating the sum to the first two terms of the two series, we get eight equations sixteen unknowns, namely $L_1, M_1, R_1, S_1, C_1, D_1, E_1, F_1$ and $L_2, M_2, R_2, S_2, C_2, D_2, E_2, F_2$. Since the values of $L_1, M_1, R_1, S_1, C_1, D_1, E_1, F_1$ are known, we substitute these values in these equations. Then, we again get eight equations in 8 unknowns that can be solved for the unknowns $L_n, M_n, R_n, S_n, C_n, D_n, E_n, F_n$ for $n = 2$. Now, we equate the sum to the first three terms of the two series to get the values of $L_n, M_n, R_n, S_n, C_n, D_n, E_n, F_n$ for $n = 3$. This process is repeated until the difference in the values of the expressions on both the sides of equations (46)–(57) is up to a desired degree of accuracy. Thus, knowing the values of the infinite sets of arbitrary constants, we derive analytical expressions for the non-dimensional stream and the pressure functions in all three regions. In our study, we computed the values of $L_n, M_n, R_n, S_n, C_n, D_n, E_n, F_n$ for $n = 1, 2, 3$, and the difference is in the order 10^{-14} to 10^{-17}.

6 Plot of Streamlines and Pressure Function

Case(I): The two spheres are of equal radius

Figures 2 and 3 respectively present the pressure contours and streamlines in the case of equal spheres with $\xi_1 = 1$ and $\xi_2 = -1$ Figures 4 and 5 depict the same when $\xi_1 = 1.2$ and $\xi_2 = -1.2$. Figures 2 and 4 show that the non-dimensional pressure in the region between the larger spheres is greater than that of smaller spheres. Further, we see from equation (13), and the plots 2(a)–(d) that the pressure function does not depend on the non-dimensional parameter $\lambda_i, i = 1, 2$, and hence the pressure contours for the subsequent cases have been presented only for one particular case when $\lambda_1 = 0.1$, $\lambda_2 = 0.1$.

Figures 3 and 5 depict the streamline for different values of λ_i ,when the two spheres are of equal radius.When the two spheres are of different λ_i, we see that the streamlines within them are dense in the sphere with highλ_i. When the two spheres of with the same λ_i, streamlines are dense when it assumes higher value. This pattern is prevalent irrespective of the radii of the spheres.

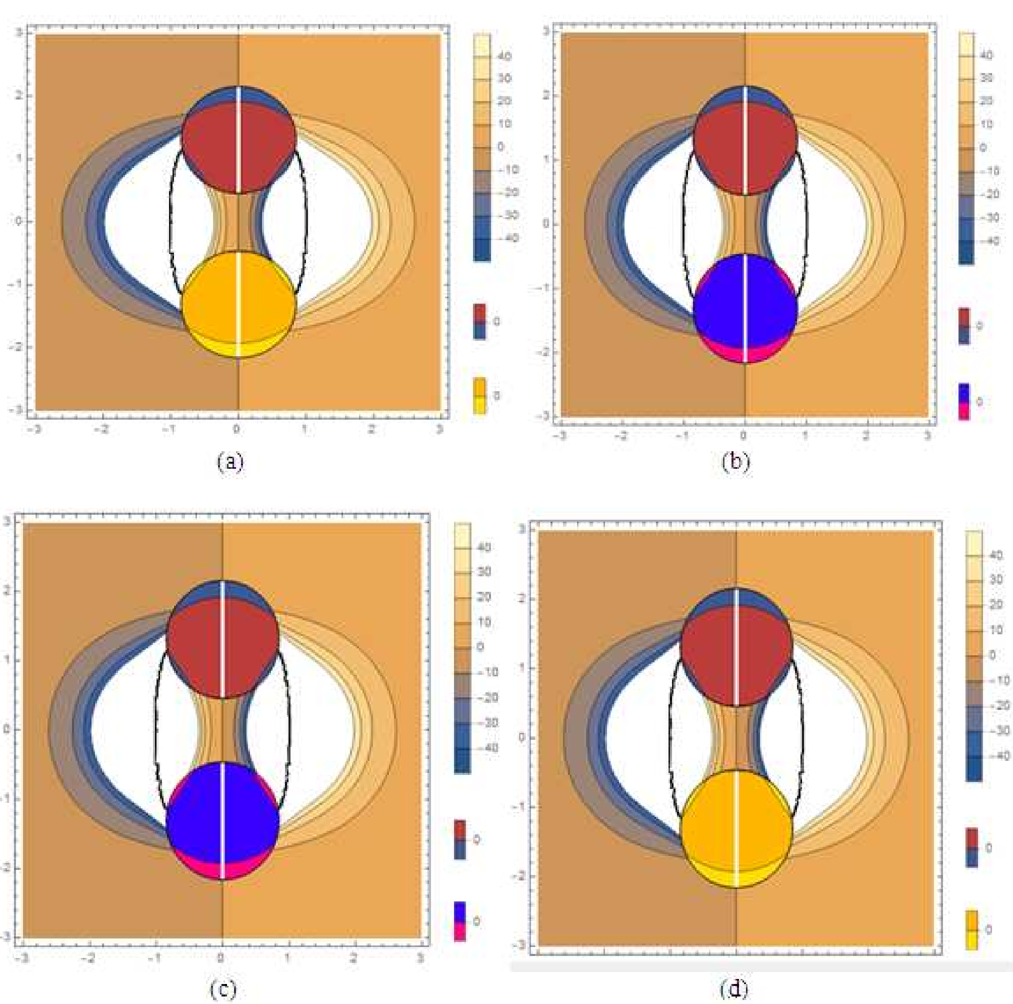


Figure 2: Plot of Pressure Curves in the xy-plane for $\xi_1 = 1.0, \xi_2 = -0.1$, (a) $\lambda_1 = 0.1$, $\lambda_2 = 0.1$ (b) $\lambda_1 = 10$, $\lambda_2 = 10$ (c) $\lambda_1 = 0.1$, $\lambda_2 = 10$ (d) $\lambda_1 = 10$, $\lambda_2 = 0.1$

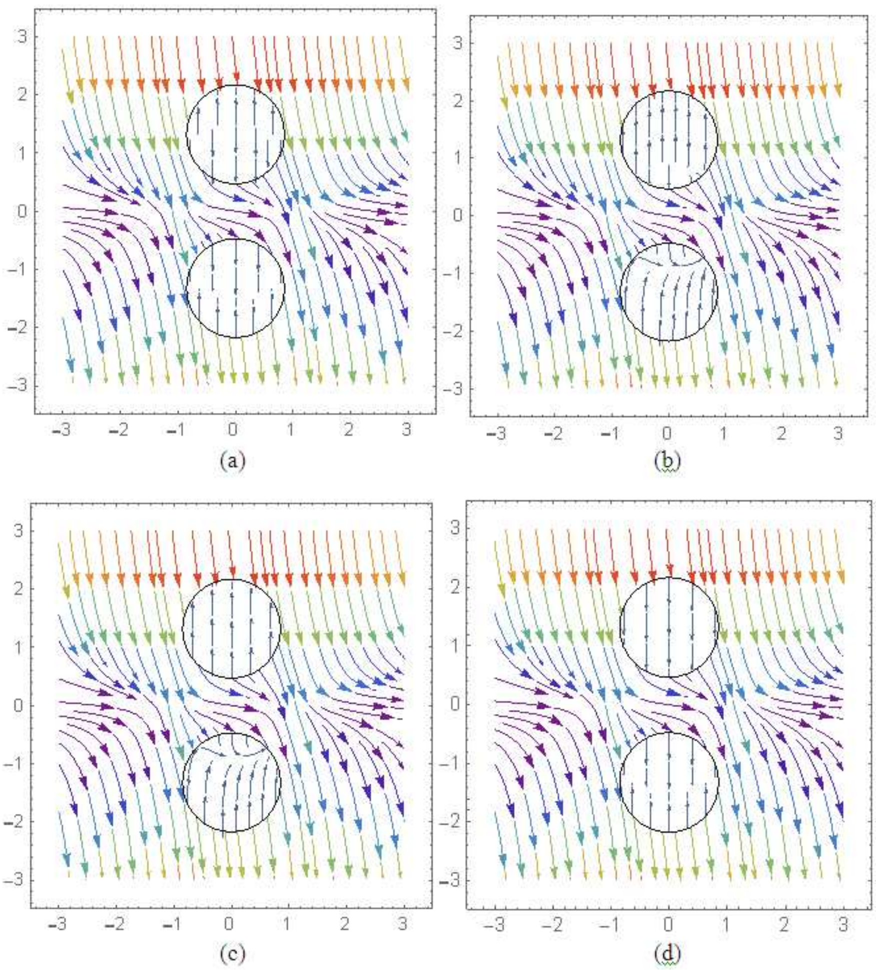


Figure 3: Plot of Streamlines in the xy-plane for $\xi_1 = 1.0$, $\xi_2 = -0.1$, (a) $\lambda_1 = 0.1$, $\lambda_2 = 0.1$ (b) $\lambda_1 = 10$, $\lambda_2 = 10$ (c) $\lambda_1 = 0.1$, $\lambda_2 = 10$ (d) $\lambda_1 = 10$, $\lambda_2 = 0.1$

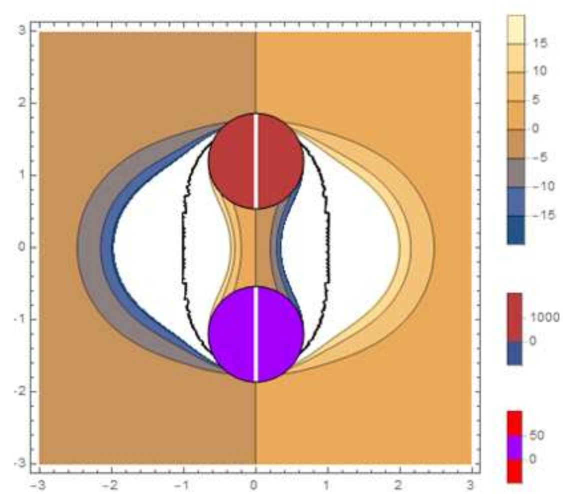


Figure 4: Plot of Pressures Curves in the xy-plane for $\xi_1 = 1.2$, $\xi_2 = -1.2$ and $\lambda_1 = 0.1, \lambda_2 = 0.1$

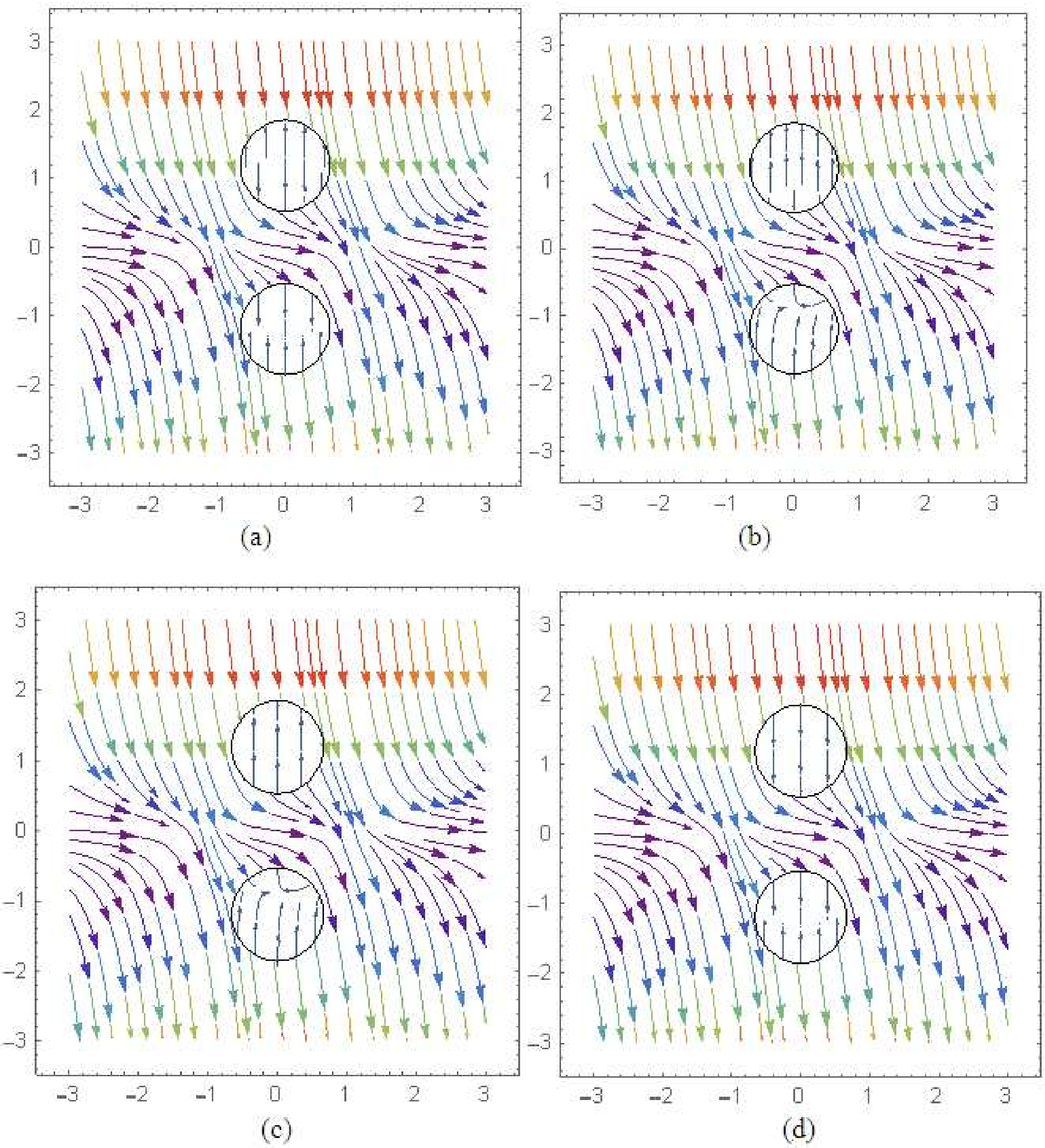


Figure 5: Plot of Streamlines in the xy- plane for $\xi_1 = 1.2$, $\xi_2 = -1.2$, (a) $\lambda_1 = 0.1$, $\lambda_2 = 0.1$ (b) $\lambda_1 = 10$, $\lambda_2 = 10$ (c) $\lambda_1 = 0.1$, $\lambda_2 = 10$ (d) $\lambda_1 = 10$, $\lambda_2 = 0.1$

Case(II): The spheres are of an unequal radius

Figures 6 and 7 respectively depict the pressure contours and the streamlines when the sphere to the right of the origin is larger than the one towards the other side.

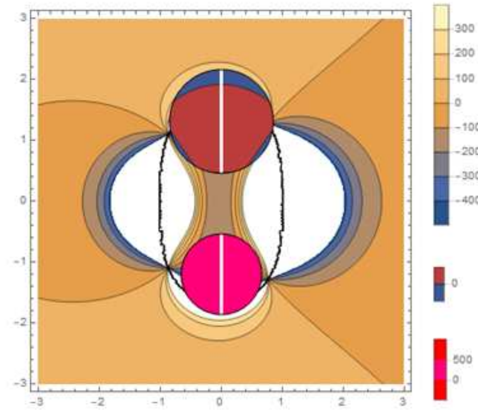


Figure 6: Plot of Pressure Curves in the xy-plane for $\xi_1 = 1.0$, $\xi_2 = -1.2$ and $\lambda_1 = 0.1, \lambda_2 = 0.1$

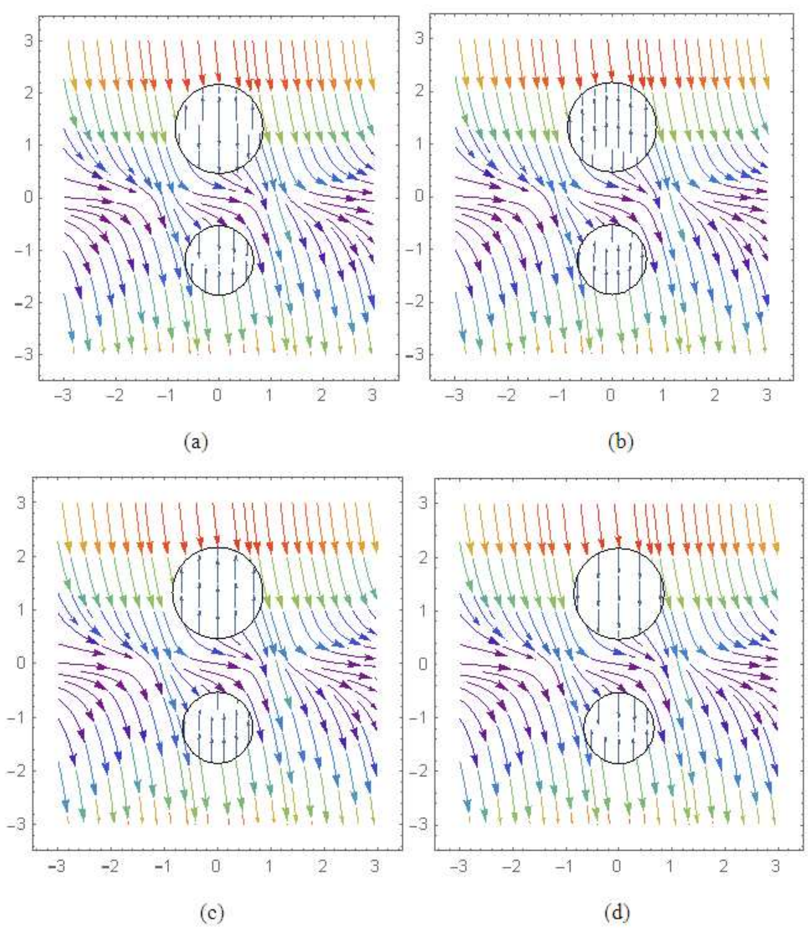


Figure 7: Plot of Streamlines in the xy-plane for $\xi_1 = 1.0$, $\xi_2 = -1.2$, (a) $\lambda_1 = 0.1$, $\lambda_2 = 0.1$ (b) $\lambda_1 = 10$, $\lambda_2 = 10$ (c) $\lambda_1 = 0.1$, $\lambda_2 = 10$ (d) $\lambda_1 = 10$, $\lambda_2 = 0.1$

Further, figures 8 and 9 respectively depict the pressure contours and the streamlines when the sphere to the left of the origin is larger than the one towards the other side.

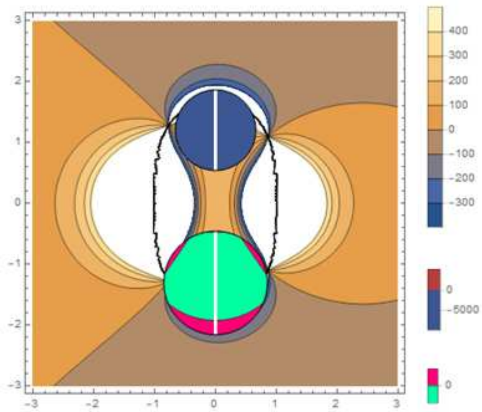
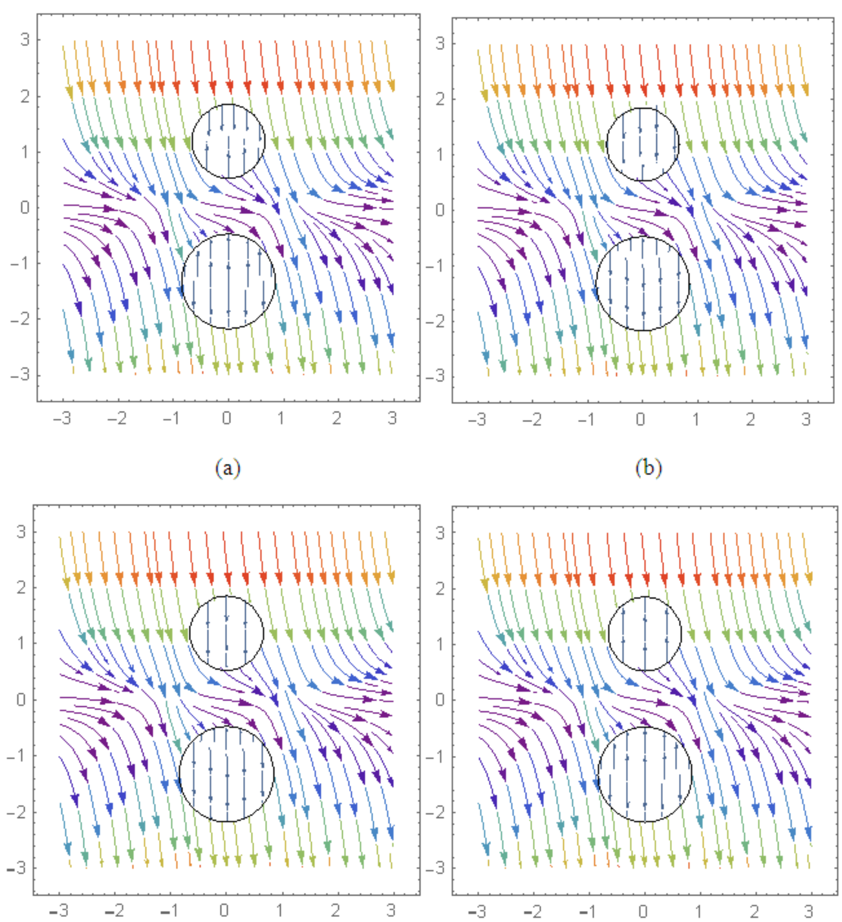


Figure 8: Plot of Pressure Curves in the xy-plane for $\xi_1 = 1.2$, $\xi_2 = -1.0$ and $\lambda_1 = 0.1, \lambda_2 = 0.1$



(a) (b) (c) (d)

Figure 9: Plot of Streamlines in the xy-plane for $\xi_1 = 1.2$, $\xi_2 = -1.0$, (a) $\lambda_1 = 0.1$, $\lambda_2 = 0.1$ (b) $\lambda_1 = 10$, $\lambda_2 = 10$ (c) $\lambda_1 = 0.1$, $\lambda_2 = 10$ (d) $\lambda_1 = 10$, $\lambda_2 = 0.1$

We see from Figures 6 and 8 that the pressure between the two spheres is more when a smaller sphere faces the flow stream.

7 Conclusions

In this work, we presented the analytical solution to Stokes flow of a viscous fluid past a pair of separated porous spheres. Flow within the porous spheres is assumed to follow the Brinkman model. Analytical expressions for the stream and pressure functions are derived, and we presented and discussed the pressure contours and streamlines when the two spheres are of equal radius and unequal radii. An observation made in this work is that there is no variation in the pressure distribution with the change in the non-dimensional parameter related to the porous medium's permeability and the fluid's viscosity.We wish to explore this result further in our future work.

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