MATEMATIKA, MJIM, Volume 38, Number 1, 33–51

Fermatean Fuzzy Weighted Geometric Aggregation Operator in Multiple Attribute Group Decision Making Problems

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Article history
Received: 16 July 2021
Received in revised form: 16 March 2022
Accepted: 24 March 2022
Published on line: 30 April 2022

Abstract Fermatean fuzzy set, an extension of Intuitionistic fuzzy set and Pythagorean fuzzy set, plays a remarkable role in dealing with ambiguity owing to its vast space compared to Intuitionistic fuzzy set and Pythagorean fuzzy set. Aggregation operators have been helpful in multi-attribute group decision making (MAGDM) problems, because of their importance and efficacy in coping with uncertainty. The purpose of this paper is to prove important theorems in the domain of the Fermatean fuzzy weighted geometric operator (FFWG) and discuss its essential properties. Most importantly, how to utilize the Fermatean fuzzy weighted geometric operator in the MAGDM problem. An algorithm of the proposed method has been established. The proposed operator is applied to decision making problems to show the validity, practicality and effectiveness of the new approach. The main advantage of using the FFWG method is that this method gives more accurate results as compared to the existing methods.

Keywords FFS; FFWG operator; MAGDM problem

Mathematics Subject Classification 00A69; 03B52.

1 Introduction

In many complex real-life problems, uncertainty makes things worse for the decision-makers since it is challenging to cope with problems in which uncertainty is involved. Due to the strange nature of ambiguity, the decision-makers tend to make mistakes that lead to devastating results. So the concept of Fuzzy set (FS) theory was brought into light in 1965 by L. A. Zadeh [1] to deal with fuzziness. Atanassov proposed the concept of Intuitionistic fuzzy set (IFS), which is the generalization of FS [2]. The IFS has got the attention of the researchers since its publication. So many papers have been published related to IFS [3–8]. They [9] presented the definition of intuitionistic fuzzy generators. Xu [10] introduced the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic fuzzy hybrid aggregation operator. They [11] develop some new geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator. Wei [12] proposed two new aggregation operators: induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IIFOWG) operator. The definition of Pythagorean fuzzy set (PFS) was proposed by Yager [13, 14]. In the year 2015, they [15] developed a Pythagorean fuzzy superiority and inferiority ranking method to solve uncertainty multiple attribute group decision making problem. Recently in 2020, they [16] proposed an agricultural product supplier selection algorithm based on the Pythagorean fuzzy power Bonferroni mean operator under Pythagorean fuzzy environment. They [17] introduced the notion of the Fermatean fuzzy set (FFS) to face situations that could not be handled well by IFS and PFS. It is the extension of FS, IFS, and PFS; it is a more powerful tool compared to IFS and PFS since it covers more space than them. In 2020, they [18] proposed Fermatean fuzzy Yager weighted average (FFYWA), Fermatean fuzzy Yager ordered weighted average (FFYOWA), Fermatean fuzzy Yager hybrid weighted average (FFYHWA), Fermatean fuzzy Yager weighted geometric (FFYWG), Fermatean fuzzy Yager ordered weighted geometric (FFYOWG), and Fermatean fuzzy Yager hybrid weighted geometric (FFYHWG) operator.

Because FFSs are more powerful than IFSs and PFSs to model the uncertainty in the practical MCDM or MCGDM problems, in this paper we will propose a new group decision method to handle effectively MCGDM problems with Fermatean fuzzy information. Motivated by the mentioned operators and their applications in diverse areas of life including engineering, medical, and business, in our study, we have explored the essential properties of FFWG operator along with solved examples, and have applied the FFWG operator to the MAGDM problem. This paper has five sections. Section 2 is devoted to some basic definitions. Section 3 is devoted to some operational laws and relations. In section 4, we present the algorithm of the FFWG method, and construct the MAGDM problem. In section 5, we conclude our paper.

2 Preliminaries

Definition 2.1. [2] *Let a set X be a universe of discourse, The IFS A is an object having the form:*

$$A = \{\langle x, \sigma_A(x), \varrho_A(x)\rangle : x \in X\},$$

where $\sigma_A(x)$, and $\varrho_A(x)$ are two functions from X to [0,1], satisfying the conditions that $0 \leq \sigma_A(x) + \varrho_A(x) \leq 1$ for all $x \in X$.

Definition 2.2. [13] *Let a set X be a universe of discourse, The PFS P is an object having the form:*
$P = \{\langle x, \sigma_P(x), \varrho_P(x)\rangle : x \in X\}$, where $\sigma_P(x)$, and $\varrho_P(x)$ are two mappings from X to [0,1], satisfying the conditions that $0 \leq \sigma_P^2(x) + \varrho_P^2(x) \leq 1$.

Definition 2.3. [17] *Let a set X be a universe set, The FFS β is an object having the form:*

$$\beta = \{\langle x, \sigma_\beta(x), \varrho_\beta(x)\rangle : x \in X\},$$

where $\sigma_\beta(x)$ *represents membership function, and* $\varrho_\beta(x)$ *represents non-membership function, both are mappings from* X *to [0,1], satisfying the conditions that* $0 \leq \sigma_\beta^3(x) + \varrho_\beta^3(x) \leq 1$ *for all* $x \in X$*, and* $\pi_\beta = \sqrt[3]{1 - (\sigma_\beta(x))^3 - (\varrho_\beta(x))^3}$ *is called the degree of indeterminacy of* x *to* β*. For simplicity, we consider the Fermatean fuzzy numbers (FFNs) be the components of the FFS.*

For understanding the FFS better, we give an instance to illuminate the understandability of the FFS: the point when someone needs will plan as much craving for the level for an alternative x_i on a criterion C_j, he might provide for the degree on which that alternative x_i fulfills those criteria C_j likewise 0.9, what's more correspondingly the elective x_i dissatisfies the criterion C_j similarly as 0.6. We can definitely get $0.9 + 0.6 > 1$, and, therefore, it does not follow the condition of intuitionistic fuzzy sets. Also, we can get $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$, which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 \leq 1$, which is good enough to apply the FFS to control it.

From Figure 1, it is clear that FFS covers more space, and it is the best tool in dealing with ambiquity compared to "Intuitionistic fuzzy set" and "Pythagorean fuzzy set".

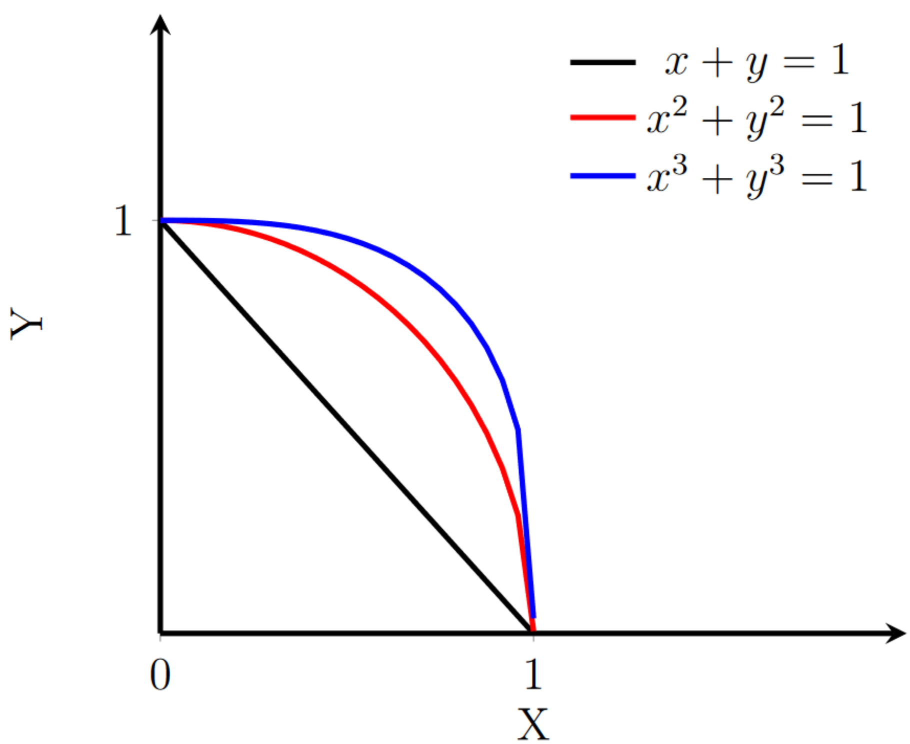

Figure 1: Comparison of IFS, PFS, and FFS

Definition 2.4. [17] *Let* $\beta = (\sigma_\beta, \varrho_\beta)$, $\beta_1 = (\sigma_{\beta_1}, \varrho_{\beta_1})$ *and* $\beta_2 = (\sigma_{\beta_2}, \varrho_{\beta_2})$ *be three FFNs, then their operations are defined as follows*:

(1) $\beta^c = (\varrho_\beta, \sigma_\beta)$,

(2) $\beta_1 \boxplus \beta_2 = \left(\sqrt[3]{(\sigma_{\beta_1})^3 + (\sigma_{\beta_2})^3 - (\sigma_{\beta_1})^3(\sigma_{\beta_2})^3}, \varrho_{\beta_1}\varrho_{\beta_2} \right)$,

(3) $\beta_1 \boxtimes \beta_2 = \left(\sigma_{\beta_1}\sigma_{\beta_2}, \sqrt[3]{(\varrho_{\beta_1})^3 + (\varrho_{\beta_2})^3 - (\varrho_{\beta_1})^3(\varrho_{\beta_2})^3} \right)$,

(4) $\alpha\beta = \left(\sqrt[3]{1 - (1 - (\sigma_\beta)^3)^\alpha}, (\varrho_\beta)^\alpha \right)$,

(5)$\beta^\alpha = \left((\sigma_\beta)^\alpha, \sqrt[3]{1 - (1 - (\varrho_\beta)^3)^\alpha} \right)$.

In the following, we will discuss some special cases.

(1) If $\beta = (\sigma_\beta, \varrho_\beta) = (1, 1)$ i,e. $\sigma_\beta = 1, \varrho_\beta = 1$,
then $\beta^\gamma = (1, 1)$.

$$\begin{aligned}
\beta^\gamma &= \left(\sigma_\beta^\gamma, \sqrt[3]{1 - \left(1 - \varrho_\beta^3\right)^\gamma} \right)\\
&= \left(1, \sqrt[3]{1 - (1 - 1)^\gamma} \right)\\
&= \left(1, \sqrt[3]{1 - (0)} \right)\\
&= \left(1, \sqrt[3]{1} \right)\\
&= (1, 1)
\end{aligned}$$

(2) If $\beta = (\sigma_\beta, \varrho_\beta) = (0, 0)$ i,e. $\sigma_\beta = 0, \varrho_\beta = 0$,
then $\beta^\gamma = (0, 0)$.

$$\begin{aligned}
\beta^\gamma &= \left(\sigma_\beta^\gamma, \sqrt[3]{1 - \left(1 - \varrho_\beta^3\right)^\gamma} \right)\\
&= \left(0, \sqrt[3]{1 - (1 - 0)^\gamma} \right)\\
&= \left(0, \sqrt[3]{1 - (1)^\gamma} \right)\\
&= \left(0, \sqrt[3]{1 - 1} \right)\\
&= (0, 0)
\end{aligned}$$

(3) If $\beta = (\sigma_\beta, \varrho_\beta) = (0, 1)$ i,e. $\sigma_\beta = 0, \varrho_\beta = 1$,
then $\beta^\gamma = (0, 1)$.

$$\begin{aligned}
\beta^\gamma &= \left(\sigma_\beta^\gamma, \sqrt[3]{1 - \left(1 - \varrho_\beta^3\right)^\gamma} \right)\\
&= \left(0, \sqrt[3]{1 - (1 - 1)^\gamma} \right)\\
&= \left(0, \sqrt[3]{1 - (0)} \right)\\
&= (0, 1)
\end{aligned}$$

(4) If $\gamma \to 0$ and $0 \leq \sigma_\beta, \varrho_\beta \leq 1$, then
$\beta^\gamma = (\sigma_\beta, \varrho_\beta) \to (1, 0)$ i.e. $\beta^\gamma \to (1, 0)(\gamma \to 0)$

$$\begin{aligned}
\beta^\gamma &= \left(\sigma_\beta^\gamma, \sqrt[3]{1 - \left(1 - \varrho_\beta^3\right)^\gamma} \right)\\
&= \left(1, \sqrt[3]{1 - (1 - 1)^\gamma} \right)\\
&= \left(1, \sqrt[3]{1 - (1 - 1)^0} \right)
\end{aligned}$$

$= \left(1, \sqrt[3]{1-1}\right)$
$= (1, 0)$

(5) If $\gamma \to +\infty$ and $0 \le \sigma_\beta, \varrho_\beta \le 1$, then
$\beta^\gamma = (\sigma_\beta, \varrho_\beta) \to (0, 1)$ i.e. $\beta^\gamma \to (0,1)(\gamma \to +\infty)$
$\beta^\gamma = \left(\sigma_\beta^\gamma, \sqrt[3]{1-\left(1-\varrho_\beta^3\right)^\gamma}\right)$
$= \left(0, \sqrt[3]{1-(1-1)^\gamma}\right)$
$= \left(0, \sqrt[3]{1-(0)^\gamma}\right)$
$= \left(0, \sqrt[3]{1}\right)$
$= (0, 1)$

(6) If $\gamma = 1$, then $\beta^\gamma = (\sigma_\beta, \varrho_\beta)$. i.e,
$\beta^\gamma \to \beta(\gamma = 1)$
$\beta^\gamma = \left(\sigma_\beta^\gamma, \sqrt[3]{1-\left(1-\varrho_\beta^3\right)^\gamma}\right)$
$= \left(\sigma_\beta^1, \sqrt[3]{1-\left(1-\varrho_\beta^3\right)^1}\right)$
$= \left(\sigma_\beta, \sqrt[3]{1-\left(1-\varrho_\beta^3\right)}\right)$
$= \left(\sigma_\beta, \sqrt[3]{1-1+\varrho_\beta^3}\right)$
$= \left(\sigma_\beta, \sqrt[3]{\varrho_\beta^3}\right)$
$= (\sigma_\beta, \varrho_\beta)$

Definition 2.5. **[17]** *Let $\beta_1 = (\sigma_{\beta_1}, \varrho_{\beta_1})$ and $\beta_2 = (\sigma_{\beta_2}, \varrho_{\beta_2})$ be two FFNs (Fermatean fuzzy numbers), a nature quasi-ordering on the FFNs is defined as follows*:

$\beta_1 \ge \beta_2$ if and only if $\sigma_{\beta_1} \ge \sigma_{\beta_2}$ and $\varrho_{\beta_1} \le \varrho_{\beta_2}$.

Definition 2.6. **[17]** *Let $\beta = (\sigma_\beta, \varrho_\beta)$ be a FFN, then the score function of β can be defined as follows*:

$SC(\beta) = (\sigma_\beta)^3 - (\varrho_\beta)^3,$

where $SC \in [-1, 1]$.

Definition 2.7. **[17]** *Let $\beta_1 = (\sigma_{\beta_1}, \varrho_{\beta_1})$ and $\beta_2 = (\sigma_{\beta_2}, \varrho_{\beta_2})$ be two FFNs, $SC(\beta_1)$ and $SC(\beta_2)$ be the score function of β_1 and β_2 respectively, then*

(1) If $SC(\beta_1) < SC(\beta_2)$, then $\beta_1 < \beta_2$;
(2) If $SC(\beta_1) > SC(\beta_2)$, then $\beta_1 > \beta_2$;
(3) If $SC(\beta_1) = SC(\beta_2)$, then $\beta_1 \sim \beta_2$.

Definition 2.8. **[17]** *Let $\beta = (\sigma_\beta, \varrho_\beta)$ be an FFN, then the accuracy function of β can* *be defined as follows*:

$acc(\beta) = (\sigma_\beta)^3 + (\varrho_\beta)^3.$

Definition 2.9. [**17**] *Let* $\beta_1 = (\sigma_{\beta_1}, \varrho_{\beta_1})$ *and* $\beta_2 = (\sigma_{\beta_2}, \varrho_{\beta_2})$ *be two FFNs,* $SC(\beta_j)$ *and* $acc(\beta_j)(j = 1, 2)$ *are the score values and accuracy values of* β_1 *and* β_2 *respectively, then*

(1) *If* $SC(\beta_1) < SC(\beta_2)$*, then* $\beta_1 < \beta_2$;
(2) *If* $SC(\beta_1) > SC(\beta_2)$*, then* $\beta_1 > \beta_2$;
(3) *If* $SC(\beta_1) = SC(\beta_2)$*, then*

(i) *If* $acc(\beta_1) < acc(\beta_2)$*, then* $\beta_1 < \beta_2$;
(ii) *If* $acc(\beta_1) > acc(\beta_2)$*, then* $\beta_1 > \beta_2$;
(iii) *If* $acc(\beta_1) = acc(\beta_2)$*, then* $\beta_1 = \beta_2$.

3 Operational Laws and Relations

This section is devoted to some operational laws and relations.

Theorem 3.1 *let* $\beta_1 = (\sigma_{\beta_1}, \varrho_{\beta_1})$ *and* $\beta_2 = (\sigma_{\beta_2}, \varrho_{\beta_2})$ *be two FFNs,* $\sigma^3_{\beta_1} \in (0,1), \varrho^3_{\beta_1} \in (0,1), \sigma^3_{\beta_2} \in (0,1), \varrho^3_{\beta_2} \in (0,1), \sigma^3_{\beta_1} + \varrho^3_{\beta_1} \leq 1,\ \sigma^3_{\beta_2} + \varrho^3_{\beta_2} \leq 1$*. Let* $\alpha_1 = \beta_1 \boxtimes \beta_2$ *and* $\alpha_2 = \beta^\gamma (\gamma > 0)$*. Then* α_1 *and* α_2 *are also FFNs.*

Proof Let $\beta_1 = (\sigma_{\beta_1}, \varrho_{\beta_1})$ and $\beta_2 = (\sigma_{\beta_2}, \varrho_{\beta_2})$ be two FFNs which means that $\sigma^3_{\beta_1} + \varrho^3_{\beta_1} \leq 1$ and $\sigma^3_{\beta_2} + \varrho^3_{\beta_2} \leq 1$. Therefore,$1 - \varrho^3_{\beta_1} \geq \sigma^3_{\beta_1} \geq 0, 1 - \varrho^3_{\beta_2} \geq \sigma^3_{\beta_2} \geq 0$.

$$
\begin{aligned}
&(\sigma_{\beta_1}\sigma_{\beta_2})^3 + \left(\sqrt[3]{\varrho^3_{\beta_1} + \varrho^3_{\beta_2} - \varrho^3_{\beta_1}\varrho^3_{\beta_2}}\right)^3 \\
&\leq (1 - \varrho^3_{\beta_1})(1 - \varrho^3_{\beta_2}) + \left(\sqrt[3]{\varrho^3_{\beta_1} + \varrho^3_{\beta_2} - \varrho^3_{\beta_1}\varrho^3_{\beta_2}}\right)^3 \\
&= (1 - \varrho^3_{\beta_1})(1 - \varrho^3_{\beta_2}) + \varrho^3_{\beta_1} + \varrho^3_{\beta_2} - \varrho^3_{\beta_1}\varrho^3_{\beta_2} \\
&= 1 - \varrho^3_{\beta_1} - \varrho^3_{\beta_2} + \varrho^3_{\beta_1}\varrho^3_{\beta_2} + \varrho^3_{\beta_1} + \varrho^3_{\beta_2} - \varrho^3_{\beta_1}\varrho^3_{\beta_2} \\
&= 1
\end{aligned}
$$

Thus α_1 is a FFN. Now let $\sigma^\gamma_\beta \geq 0$ and $\varrho^\gamma_\beta \geq 0, 1 - \varrho^3_\beta \geq \sigma^3_\beta \geq 0$ and hence
$(1 - \varrho^3_\beta)^\gamma \geq (\sigma^3_\beta)^\gamma \geq 0$.
Since

$$
\begin{aligned}
&\left(\sigma^\gamma_\beta\right)^3 + \left(\sqrt[3]{1 - \left(1 - \varrho^3_\beta\right)^\gamma}\right)^3 \\
&\leq \left(1 - \varrho^3_\beta\right)^\gamma + \left(\sqrt[3]{1 - \left(1 - \varrho^3_\beta\right)^\gamma}\right)^3 \\
&= \left(1 - \varrho^3_\beta\right)^\gamma + 1 - \left(1 - \varrho^3_\beta\right)^\gamma \\
&= 1
\end{aligned}
$$

Thus α_2 is also a FFN. $\square$

Definition 3.2 [19] *Let* $\beta_j = (\sigma_{\beta_j}, \varrho_{\beta_j})(j = 1, \ldots, n)$ *be FFNs then the Fermatean fuzzy* *weighted geometric aggregation operator is a mapping* $FFWG : \Omega^n \rightarrow \Omega$ *and can be defined as:*

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = \beta_1^{w_1} \boxtimes \beta_2^{w_2} \boxtimes \ldots \boxtimes \beta_n^{w_n} \tag{1}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ *is the weighted vector of* $\beta_j (j = 1, 2, 3, \ldots, n)$ *with condition* $w_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{n} w_j = 1$.

If $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, *then the FFWG operator is converted to FFG operator which is defined as:*

$$FFG(\beta_1, \beta_2, \ldots, \beta_n) = (\beta_1 \boxtimes \beta_2 \boxtimes \ldots \boxtimes \beta_n)^{\frac{1}{n}} \tag{2}$$

Example 3.3 Let
$\beta_1 = (0.3, 0.5), \beta_2 = (0.4, 0.7)$
$\beta_3 = (0.7, 0.5), \beta_2 = (0.6, 0.8)$
and $w = (0.5, 0.2, 0.2, 0.1)^T$.

Thus

$$FFWG_w(\beta_1, \beta_2, \beta_3, \beta_4)$$
$$= \left(\prod_{j=1}^{4} \sigma_{\beta_j}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{4} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}}\right)$$
$$= \left((0.3)^{0.5} \bullet (0.4)^{0.2} \bullet (0.7)^{0.2} \bullet (0.6)^{0.1}, \sqrt[3]{1 - (1 - 0.5^3)^{0.5}(1 - 0.7^3)^{0.2}(1 - 0.5^3)^{0.2}(1 - 0.8^3)^{0.1}}\right)$$
$$= (0.4035, 0.6042).$$

Theorem 3.4 *Let* $\beta_j = \left(\sigma_{\beta_j}, \varrho_{\beta_j}\right) (j = 1, 2, \ldots, n)$ *are FFNs, then their aggregated value obtained by applying FFWG operator is also a FFN, and*

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = \boxtimes_{j=1}^{n} (\beta_j)^{w_j}$$
$$= \left(\prod_{j=1}^{n} \sigma_{\beta_j}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{n} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}}\right) \tag{3}$$

the weighted vector of $\beta_j (j = 1, 2, \ldots, n)$ *is* $w = (w_1, w_2, \ldots, w_n)^T$ *with some conditions* $w_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{n} w_j = 1$.

Proof By mathematical induction we can prove that equation (3) holds for all n. First we show that equation (3) holds for $n = 2$, since

$$\beta_1^{w_1} = \left(\sigma_{\beta_1}^{w_1}, \sqrt[3]{1 - \left(1 - \varrho_{\beta_1}^3\right)^{w_1}}\right)$$

So

$$\beta_2^{w_2} = \left(\sigma_{\beta_2}^{w_2}, \sqrt[3]{1-\left(1-\varrho_{\beta_2}^3\right)^{w_2}}\right)$$

$$\beta_1^{w_1} \boxtimes \beta_2^{w_2}$$

$$= \left(\sigma_{\beta_1}^{w_1}, \sqrt[3]{1-\left(1-\varrho_{\beta_1}^3\right)^{w_1}}\right) \boxtimes \left(\sigma_{\beta_2}^{w_2}, \sqrt[3]{1-\left(1-\varrho_{\beta_2}^3\right)^{w_2}}\right)$$

$$= \Bigg(\sigma_{\beta_1}^{w_1}\sigma_{\beta_2}^{w_2},$$

$$\sqrt[3]{\left(\sqrt[3]{1-\left(1-\varrho_{\beta_1}^3\right)^{w_1}}\right)^3 + \left(\sqrt[3]{1-\left(1-\varrho_{\beta_2}^3\right)^{w_2}}\right)^3 - \left(\sqrt[3]{1-\left(1-\varrho_{\beta_1}^3\right)^{w_1}}\right)^3\left(\sqrt[3]{1-\left(1-\varrho_{\beta_2}^3\right)^{w_2}}\right)^3}\ \Bigg)$$

$$= \Bigg(\sigma_{\beta_1}^{w_1}\sigma_{\beta_2}^{w_2},$$

$$\sqrt[3]{1-\left(1-\varrho_{\beta_1}^3\right)^{w_1} + 1-\left(1-\varrho_{\beta_2}^3\right)^{w_2} - \left(1-\left(1-\varrho_{\beta_1}^3\right)^{w_1}\right)\left(1-\left(1-\varrho_{\beta_2}^3\right)^{w_2}\right)}\ \Bigg)$$

$$= \left(\sigma_{\beta_1}^{w_1}\sigma_{\beta_2}^{w_2}, \sqrt[3]{1-\left(1-\varrho_{\beta_1}^3\right)^{w_1}\left(1-\varrho_{\beta_2}^3\right)^{w_2}}\right)$$

$$= \left(\prod_{j=1}^{2}\sigma_{\beta_j}^{w_j}, \sqrt[3]{1-\prod_{j=1}^{2}\left(1-\varrho_{\beta_j}^3\right)^{w_j}}\right)$$

Thus equation (3) holds for $n = 2$. Let us suppose that equation (3) is true for $n = k$. Then we have

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_k) = \left(\prod_{j=1}^{k}\sigma_{\beta_j}^{w_j}, \sqrt[3]{1-\prod_{j=1}^{k}\left(1-\varrho_{\beta_j}^3\right)^{w_j}}\right)$$

Now we show that equation (11) is true for $n = k+1$

$FFWG_w(\beta_1, \beta_2, \ldots, \beta_{k+1})$

$= \beta_1^{w_1} \boxtimes \beta_2^{w_2} \boxtimes \ldots \boxtimes \beta_{k+1}^{w_{k+1}}$

$$= \left(\prod_{j=1}^{k}\sigma_{\beta_j}^{w_j}, \sqrt[3]{1-\prod_{j=1}^{k}\left(1-\varrho_{\beta_j}^3\right)^{w_j}}\right) \boxtimes \left(\left(\sigma_{\beta_{k+1}}\right)^{w_{k+1}}, \sqrt[3]{1-\left(1-\varrho_{\beta_{k+1}}^3\right)^{w_{k+1}}}\right)$$

$$= \Bigg(\prod_{j=1}^{k}\sigma_{\beta_j}^{w_j}\left(\sigma_{\beta_{k+1}}\right)^{w_{k+1}},$$

$$\sqrt[3]{\left(\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\varrho_{\beta_j}^3\right)^{w_j}}\right)^3 + \left(\sqrt[3]{1-\left(1-\varrho_{\beta_{k+1}}^3\right)^{w_{k+1}}}\right)^3 - \left(\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\varrho_{\beta_j}^3\right)^{w_j}}\right)^3\left(\sqrt[3]{1-\left(1-\varrho_{\beta_{k+1}}^3\right)^{w_{k+1}}}\right)^3}\ \Bigg)$$

$$= \left(\prod_{j=1}^{k} \sigma_{\beta_j}^{w_j} \left(\sigma_{\beta_{k+1}}\right)^{w_{k+1}}, \right.$$

$$\left. \sqrt[3]{1 - \prod_{j=1}^{k} \left(1 - \varrho_{\beta_j}^3\right)^{w_j} + 1 - \left(1 - \varrho_{\beta_{k+1}}^3\right)^{w_{k+1}} - \left(1 - \prod_{j=1}^{k} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}\right) \left(1 - \left(1 - \varrho_{\beta_{k+1}}^3\right)^{w_{k+1}}\right)} \right)$$

$$= \left(\prod_{j=1}^{k+1} \sigma_{\beta_j}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{k+1} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}} \right)$$

Hence equation (3) holds for $n = k + 1$. Thus equation (3) holds for all n. □

Example 3.5 Let $\beta_1 = (0.3, 0.7), \beta_2 = (0.4, 0.6), \beta_3 = (0.5, 0.6), \beta_4 = (0.6, 0.3)$, be four FFNs, and their weighted vector is $w = (0.4, 0.3, 0.2, 0.1)^T$, then if we apply the FFWG operator we get the Fermatean fuzzy number.

Thus

$$FFWG_w \left(\beta_1, \beta_2, \beta_3, \beta_4\right)$$

$$= \left(\prod_{j=1}^{4} \sigma_{\beta_j}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{4} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}} \right)$$

$$= \left((0.3)^{0.4} \bullet (0.4)^{0.3} \bullet (0.5)^{0.2} \bullet (0.6)^{0.1}, \sqrt[3]{1 - (1 - 0.7^3)^{0.4} (1 - 0.6^3)^{0.3} (1 - 0.6^3)^{0.2} (1 - 0.3^3)^{0.1}} \right)$$

$$=(0.3882,\ 0.6329)$$

Theorem 3.6 *Let $\beta_j = (\sigma_{\beta_j}, \varrho_{\beta_j})(j = 1, 2, 3, \ldots, n)$ be the FFNs and the weighted vector of $\beta_j(j = 1, 2, 3, \ldots, n)$ is $w = (w_1, w_2, \ldots, w_n)^T$ with some conditions $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. If $\beta_j(j = 1, 2, 3, \ldots, n)$ are mathematically equal. Then*

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = \beta$$

Proof As we know that

$$FFWG_w \left(\beta_1, \beta_2, \ldots, \beta_n\right) = \beta_1^{w_1} \boxtimes \beta_2^{w_2} \boxtimes \ldots \boxtimes \beta_n^{w_n}$$

Let $\beta_j(j = 1, 2, 3, \ldots, n) = \beta$, then

$$\begin{aligned} FFWG_w \left(\beta_1, \beta_2, \ldots, \beta_n\right) &= \beta_1^{w_1} \boxtimes \beta_2^{w_2} \boxtimes \ldots \boxtimes \beta_n^{w_n} \\ &= (\beta)^{\sum_{j=1}^{n} w_j} \end{aligned}$$

$$= \beta. \ \square$$

Example 3.7 Let $\beta_1 = (0.4, 0.5), \beta_2 = (0.4, 0.5), \beta_3 = (0.4, 0.5), \beta_4 = (0.4, 0.5)$, be four FFNs, and their weighted vector is $w = (0.4, 0.3, 0.2, 0.1)^T$, then if we apply the FFWG operator we get the Fermatean fuzzy number.

Thus

$$FFWG_w(\beta_1, \beta_2, \beta_3, \beta_4)$$

$$= \left(\prod_{j=1}^{4} \sigma_{\beta_j}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{4} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}} \right)$$

$$= \left((0.4)^{0.4} \bullet (0.4)^{0.3} \bullet (0.4)^{0.2} \bullet (0.4)^{0.1}, \sqrt[3]{1 - (1 - 0.5^3)^{0.4}(1 - 0.5^3)^{0.3}(1 - 0.5^3)^{0.2}(1 - 0.5^3)^{0.1}} \right)$$

$$=(0.4,\ 0.5)$$

Theorem 3.8 *Let* $\beta_j = (\sigma_{\beta_j}, \varrho_{\beta_j})(j = 1, 2, 3, \ldots, n)$ *be the FFNs and the weighted vector of* $\beta_j(j = 1, 2, 3, \ldots, n)$ *is* $w = (w_1, w_2, \ldots, w_n)^T$ *with some conditions* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *If* $\beta_j(j = 1, 2, 3, \ldots, n)$ *are mathematically equal. If*

$$\beta^- = \left(\min_j \left(\sigma_{\beta_j}\right), \max_j \left(\varrho_{\beta_j}\right) \right),$$

$$\beta^+ = \left(\max_j \left(\sigma_{\beta_j}\right), \min_j \left(\varrho_{\beta_j}\right) \right),$$

then

$$\beta^- \leq FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) \leq \beta^+ \tag{4}$$

for all w.

Proof As we know that

$$\min_j \left(\sigma_{\beta_j}\right) \leq \sigma_{\beta_j} \leq \max_j \left(\sigma_{\beta_j}\right) \tag{5}$$

$$\min_j \left(\varrho_{\beta_j}\right) \leq \varrho_{\beta_j} \leq \max_j \left(\varrho_{\beta_j}\right). \tag{6}$$

From equation (5), we have

$$\Leftrightarrow \min_j \left(\sigma_{\beta_j}\right) \leq \sigma_{\beta_j} \leq \max_j \left(\sigma_{\beta_j}\right)$$

$$\Leftrightarrow \min_j \left(\sigma_{\beta_j}\right)^{w_j} \leq \sigma_{\beta_j}^{w_j} \leq \max_j \left(\sigma_{\beta_j}\right)^{w_j}$$

$$\Leftrightarrow \prod_{j=1}^{n} \min_j \left(\sigma_{\beta_j}\right)^{w_j} \leq \prod_{j=1}^{n} \sigma_{\beta_j}^{w_j} \leq \prod_{j=1}^{n} \max_j \left(\sigma_{\beta_j}\right)^{w_j}$$

$$\Leftrightarrow \min_j \left(\sigma_{\beta_j}\right) \leq \prod_{j=1}^{n} \sigma_{\beta_j}^{w_j} \leq \max_j \left(\sigma_{\beta_j}\right).$$

Now from equation (6), we have

$$\Leftrightarrow \sqrt[3]{1-\max_j \left(\varrho_{\beta_j}\right)^3} \leq \sqrt[3]{1-\varrho_{\beta_j}^3} \leq \sqrt[3]{1-\min_j \left(\varrho_{\beta_j}\right)^3}$$
$$\Leftrightarrow \sqrt[3]{\prod_{j=1}^{n}\left(1-\max_j \left(\varrho_{\beta_j}\right)^3\right)^{w_j}} \leq \sqrt[3]{\prod_{j=1}^{n}\left(1-\varrho_{\beta_j}^3\right)^{w_j}} \leq \sqrt[3]{\prod_{j=1}^{n}\left(1-\min_j \left(\varrho_{\beta_j}\right)^3\right)^{w_j}}$$
$$\Leftrightarrow \sqrt[3]{1-\max_j \left(\varrho_{\beta_j}\right)^3} \leq \sqrt[3]{\prod_{j=1}^{n}\left(1-\varrho_{\beta_j}^3\right)^{w_j}} \leq \sqrt[3]{\prod_{j=1}^{n}\left(1-\min_j \left(\varrho_{\beta_j}\right)^3\right)}$$

$$\Leftrightarrow \min_j \left(\varrho_{\beta_j}\right) \leq \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\varrho_{\beta_j}^3\right)^{w_j}} \leq \max_j \left(\varrho_{\beta_j}\right). \tag{7}$$

Let $FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = \beta = (\sigma_\beta, \varrho_\beta)$, then

$$R(\beta) = \sigma_\beta^3 - \varrho_\beta^3 \leq \max_j (\sigma_\beta)^3 - \min_j (\varrho_\beta)^3$$
$$= R(\beta^+)$$

Thus $R(\beta) \leq R(\beta^+)$. Again

$$R(\beta) = \sigma_\beta^3 - \varrho_\beta^3 \geq \min_j (\sigma_\beta)^3 - \max_j (\varrho_\beta)^3$$
$$= R(\beta^-).$$

Thus $R(\beta) \geq R(\beta^-)$. If $R(\beta) < R(\beta^+)$, $R(\beta) > R(\beta^-)$, then

$$\beta^- < FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) < \beta^+ \tag{8}$$

If $R(\beta) = R(\beta^+)$, then

$$\Leftrightarrow \sigma_\beta^3 - \varrho_\beta^3 = \max_j (\sigma_\beta)^3 - \min_j (\varrho_\beta)^3$$
$$\Leftrightarrow \sigma_\beta^3 = \max_j (\sigma_\beta)^3, \varrho_\beta^3 = \min_j (\varrho_\beta)^3$$
$$\Leftrightarrow \sigma_\beta = \max_j (\sigma_\beta), \varrho_\beta = \min_j (\varrho_\beta)$$

Since

$$A(\beta) = \sigma_\beta^3 + \varrho_\beta^3 = \max_j (\sigma_\beta)^3 + \min_j (\varrho_\beta)^3 = A(\beta^+),$$

thus

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = \beta^+. \tag{9}$$

If $R(\beta) = R(\beta^-)$, then

$$\Leftrightarrow \sigma_\beta^3 - \varrho_\beta^3 = \min_j (\sigma_\beta)^3 - \max_j (\varrho_\beta)^3$$

$$\Leftrightarrow \sigma_\beta^3 = \min_j (\sigma_\beta)^3, \varrho_\beta^3 = \max_j (\varrho_\beta)^3$$

$$\Leftrightarrow \sigma_\beta = \min_j (\sigma_\beta), \varrho_\beta = \max_j (\varrho_\beta).$$

Since

$$A(\beta) = \sigma_\beta^3 + \varrho_\beta^3 = \min_j (\sigma_\beta)^3 + \max_j (\varrho_\beta)^3 = A(\beta^-),$$

thus

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = \beta^- \tag{10}$$

Thus from equation (8) to (10), we have

$$\beta^- \leq FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) \leq \beta^+, \text{ for all } w. \ \square$$

Theorem 3.9 *Let $\beta_j = \left(\sigma_{\beta_j}, \varrho_{\beta_j}\right)(j = 1, 2, 3, \ldots, n)$, and $\beta_j^* = \left(\sigma_{\beta_j^*}, \varrho_{\beta_j^*}\right)(j = 1, 2, 3, \ldots, n)$ be the two collection of FFNs with the weighted vector $w = (w_1, w_2, \ldots, w_n)^T$ satisfying the conditions $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. If $\sigma_{\beta_j} \leq \sigma_{\beta_j^*}$ and $\varrho_{\beta_j} \geq \varrho_{\beta_j^*}$ then*

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) \leq FFWG_w(\beta_1^*, \beta_2^*, \ldots, \beta_n^*) \tag{11}$$

Proof Since, $\sigma_{\beta_j} \leq \sigma_{\beta_j^*}$ and $\varrho_{\beta_j} \geq \varrho_{\beta_j^*}$ then

$$\Leftrightarrow \sigma_{\beta_j}^{w_j} \leq \sigma_{\beta_j^*}^{w_j}$$

$$\Leftrightarrow \prod_{j=1}^{n} \sigma_{\beta_j}^{w_j} \leq \prod_{j=1}^{n} \sigma_{\beta_j^*}^{w_j}$$

Now using the non-membership function we have

$$\Leftrightarrow 1 - \varrho_{\beta_j}^3 \leq 1 - \varrho_{\beta_j^*}^3$$

$$\Leftrightarrow \sqrt[3]{\prod_{j=1}^{n} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}} \leq \sqrt[3]{\prod_{j=1}^{n} \left(1 - \varrho_{\beta_j^*}^3\right)^{w_j}}$$

$$\Leftrightarrow \sqrt[3]{1 - \prod_{j=1}^{n} \left(1 - \varrho_{\beta_j^*}^3\right)^{w_j}} \leq \sqrt[3]{1 - \prod_{j=1}^{n} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}}$$

Let

$$\beta = FFWG_w(\beta_1, \beta_2, \ldots, \beta_n)$$

and

$\beta^* = FFWG_w(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$

Then from equation (11) we have, $R(\beta) \leq R(\beta^*)$.

If $R(\beta) < R(\beta^*)$, then

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) < FFWG_w(\beta_1^*, \beta_2^*, \ldots, \beta_n^*) \tag{12}$$

If $R(\beta) = R(\beta^*)$, then

$\Leftrightarrow \sigma_\beta^3 - \varrho_\beta^3 = \sigma_{\beta^*}^3 - \varrho_{\beta^*}^3$

$\Leftrightarrow \sigma_\beta^3 = \sigma_{\beta^*}^3, \varrho_\beta^3 = \varrho_{\beta^*}^3$

$\Leftrightarrow \sigma_\beta = \sigma_{\beta^*}, \varrho_\beta = \varrho_{\beta^*}$

Since $A(\beta) = \sigma_\beta^3 + \varrho_\beta^3 = \sigma_{\beta^*}^3 + \varrho_{\beta^*}^3 = A(\beta^+)$, thus

$$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) = FFWG_w(\beta_1^*, \beta_2^*, \ldots, \beta_n^*) \tag{13}$$

Thus from equation (12) and (13), we have

$FFWG_w(\beta_1, \beta_2, \ldots, \beta_n) \leq FFWG_w(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$. $\square$

Example 3.10 Let $\beta_1 = (0.3, 0.5), \beta_2 = (0.4, 0.6), \beta_3 = (0.2, 0.7)$,
$\beta_4 = (0.1, 0.8)$, $\beta_1^* = (0.6, 0.4), \beta_2^* = (0.7, 0.2), \beta_3^* = (0.5, 0.4)$,
$\beta_4^* = (0.4, 0.4)$, and $w = (0.4, 0.3, 0.2, 0.1)^T$,

Now using the FFWG operator we get the following result.
$FFWG_w(\beta_1, \beta_2, \beta_3, \beta_4)$

$$= \left(\prod_{j=1}^{4} \sigma_{\beta_j}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{4} \left(1 - \varrho_{\beta_j}^3\right)^{w_j}} \right)$$

$$= \left((0.3)^{0.4} \bullet (0.4)^{0.3} \bullet (0.2)^{0.2} \bullet (0.1)^{0.1}, \sqrt[3]{1 - (1 - 0.5^3)^{0.4}(1 - 0.6^3)^{0.3}(1 - 0.7^3)^{0.2}(1 - 0.8^3)^{0.1}}\right)$$

$=(0.2702,\ 0.6265)$

$FFWG_w(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*)$

$$= \left(\prod_{j=1}^{4} \sigma_{\beta_j^*}^{w_j}, \sqrt[3]{1 - \prod_{j=1}^{4} \left(1 - \varrho_{\beta_j^*}^{3}\right)^{w_j}} \right)$$

$$= \left((0.6)^{0.4} \bullet (0.7)^{0.3} \bullet (0.5)^{0.2} \bullet (0.4)^{0.1}, \sqrt[3]{1-(1-0.4^3)^{0.4}(1-0.2^3)^{0.3}(1-0.4^3)^{0.2}(1-0.4^3)^{0.1}} \right)$$

$$=(0.5818, 0.3623)$$

Hence

$$FFWG_w(\beta_1, \beta_2, \beta_3, \beta_4) \leq FFWG_w(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*).$$

4 An Application of the FFWG Operator to MAGDM Problem

In this section, we discuss an application of the $FFWG$ operator to MADM. Now we are utilizing FFNs to develop the MADM.

Algorithm Let $V = \{V_1, V_2, \ldots, V_n\}$ be a finite set of n alternatives, and suppose $C = \{C_1, C_1, \ldots, C_m\}$ is a finite set of m attributes, and $E = \{e_1, e_2, \ldots, e_k\}$ be the set of k experts.

Let $w = (w_1, w_2, \ldots, w_m)^T$ be the weighted vector of the attributes $C_j\,(j = 1, 2, \ldots, m)$, also $w_j \in [0, 1]$ and $\sum\limits_{j=1}^{m} w_j = 1$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T$ be the weighted vector of the $E^s\,(s = 1, 2, \ldots, k)$, also $\alpha_s \in [0, 1]$ and $\sum\limits_{s=1}^{k} \alpha_s = 1$.

This method has the following steps.

Step 1: Construct the Fermatean fuzzy decision matrices $K^s = \left[d_{ij}^{(s)}\right]_{n\times m} (s = 1, 2, \ldots, k)$ for decision. If the criteria have two types, one is benefit criteria and the other is cost criteria, then the decision-maker transforms the Fermatean fuzzy decision matrix, $K^s = \left[d_{ij}^{(s)}\right]_{n\times m}$, into the normalized Fermatean fuzzy decision matrix,

$T^s = \left[t_{ij}^{(s)}\right]_{n\times m}$, where

$$t_{ij}^{(s)} = \begin{cases} d_{ij}, & \text{for benefit criteria } C_j \ (j = 1, 2, \ldots, m), \\ d_{ij}^c, & \text{for cost criteria } C_j \ (j = 1, 2, \ldots, m) \end{cases}$$

where d_{ij}^c represents the complement of d_{ij}. The normalization is not required when all the criteria have the same type.

Step 2: Utilize the FFWG operator to combine the entire individual FFDMS. This step is utilized to get the ideal opinion of all experts.

$T^s = \left[t_{ij}^{(s)}\right]_{n\times m}$ $(s = 1, 2, \ldots, k)$ into the collective FFDM $T = [t_{ij}]_{n\times m}$, with condition $t_{ij} = (\sigma_{ij}, \xi_{ij})\,(i = 1, 2, \ldots, n)\,(j = 1, 2, \ldots, m)$

Step 3: Aggregate all the preference values $t_{ij} = (\sigma_{ij}, \xi_{ij})\,(i = 1, 2, \ldots, n)\,(j = 1, 2, \ldots, m)$ by utilizing the FFWG operator to achieve the overall preference value $t_i\,(i = 1, 2, \ldots, n)$ analogous to the alternatives $V_i\,(i = 1, 2, \ldots, n)$.

Step 4: The scores of $t_i\,(i = 1, 2, \ldots, n)$ are determined in this step. In case there is a difference between two or more score functions then the accuracy degrees must be calculated.

Step 5: To find the best alternative, arrange the score values in descending order, and pick the first alternative in descending order as the best alternative.

Example 4.1 Consider a multiattribute decision problem for mounting a global positioning system (GPS). Suppose four alternatives $V_i(i = 1, 2, 3, 4)$ are considered according to four criteria: (C_1) accuracy, (C_2) reliability, (C_3) service, and (C_4) functionality. C_1, C_2, and C_3 are the benefit criteria while C_4 is the cost criteria. Three experts $E^s(s = 1, 2, 3)$ are called for assessments. Let $\alpha = (0.6, 0.3, 0.1)$ be the weight vector of the experts $E^s(s = 1, 2, 3)^T$, and $w = (0.4, 0.4, 0.1, 0.1)^T$ be the weight vector of criteria $C_i(i = 1, 2, 3, 4)$.

The assessment values provided by the experts are in the following tables.

Table 1: Fermatean fuzzy decision matrix E_1

	C_1	C_2	C_3	C_4
V_1	(0.7,0.4)	(0.7,0.6)	(0.6,0.5)	(0.5,0.6)
V_2	(0.6,0.4)	(0.7,0.5)	(0.5,0.6)	(0.6,0.4)
V_3	(0.4,0.5)	(0.1,0.4)	(0.6,0.5)	(0.7,0.4)
V_4	(0.6,0.7)	(0.6,0.4)	(0.7,0.4)	(0.7,0.5)

Table 2: Fermatean fuzzy decision matrix E_2

	C_1	C_2	C_3	C_4
V_1	(0.1,0.9)	(0.6,0.5)	(0.3,0.4)	(0.5,0.6)
V_2	(0.1,0.7)	(0.7,0.5)	(0.4,0.7)	(0.6,0.4)
V_3	(0.4,0.7)	(0.6,0.4)	(0.9,0.3)	(0.7,0.6)
V_4	(0.4,0.6)	(0.5,0.7)	(0.3,0.8)	(0.7,0.6)

From the Figure 2 and Table 9, we observe that the ranking order of all alternatives utilizing different operators is different. The fluctuations reflected in the graphs illustrate

Table 3: Fermatean fuzzy decision matrix E_3

	C_1	C_2	C_3	C_4
V_1	(0.3,0.8)	(0.5,0.8)	(0.3,0.5)	(0.5,0.6)
V_2	(0.6,0.4)	(0.7,0.5)	(0.4,0.8)	(0.2,0.9)
V_3	(0.6,0.7)	(0.5,0.7)	(0.8,0.4)	(0.6,0.4)
V_4	(0.3,0.6)	(0.6,0.4)	(0.7,0.5)	(0.2,0.7)

Table 4: Normalize FFDM E_1

	C_1	C_2	C_3	C_4
V_1	(0.7,0.4)	(0.7,0.6)	(0.6,0.5)	(0.6,0.5)
V_2	(0.6,0.4)	(0.7,0.5)	(0.5,0.6)	(0.4,0.6)
V_3	(0.4,0.5)	(0.1,0.4)	(0.6,0.5)	(0.4,0.7)
V_4	(0.6,0.7)	(0.6,0.4)	(0.7,0.4)	(0.5,0.7)

Table 5: Normalize FFDM E_2

	C_1	C_2	C_3	C_4
V_1	(0.1,0.9)	(0.6,0.5)	(0.3,0.4)	(0.6,0.5)
V_2	(0.1,0.7)	(0.7,0.5)	(0.4,0.7)	(0.4,0.6)
V_3	(0.4,0.7)	(0.6,0.4)	(0.9,0.3)	(0.6,0.7)
V_4	(0.4,0.6)	(0.5,0.7)	(0.3,0.8)	(0.6,0.7)

Table 6: Normalize FFDM E_3

	C_1	C_2	C_3	C_4
V_1	(0.3,0.8)	(0.5,0.8)	(0.3,0.5)	(0.6,0.5)
V_2	(0.6,0.4)	(0.7,0.5)	(0.4,0.8)	(0.9,0.2)
V_3	(0.6,0.7)	(0.5,0.7)	(0.8,0.4)	(0.4,0.6)
V_4	(0.3,0.6)	(0.6,0.4)	(0.7,0.5)	(0.7,0.2)

Table 7: The ideal opinion of all experts

	C_1	C_2	C_3	C_4
V_1	(0.3587,0.7339)	(0.6463, 0.6102)	(0.4547,0.4749)	(0.6 ,0.5)
V_2	(0.3505 , 0.5409)	(0.7,0.5)	(0.4573,0.6626)	(0.4338 , 0.5822)
V_3	(0.4166 , 0.6035)	(0.2011 , 0.4588)	(0.6974 ,0.4491)	(0.4517 ,0.6919)
V_4	(0.4957, 0.6656)	(0.5681 , 0.5409)	(0.5429 ,0.6174)	(0.5462, 0.6807)

Table 8: The collective preference values, scores and ranking of alternatives

Alternatives	Collective preference	Scores	Ranking
V_1	(0.4725, 0.6462)	-0.1643	4
V_2	(0.4647 , 0.5657)	-0.0807	1
V_3	(0.3742 , 0.5563)	-0.1198	3
V_4	(0.5310 , 0.6270)	-0.0968	2

Figure 2: Ranking order comparison using using PFWG and FFWG operators

Table 9: Comparative analysis

Method	Score values				Order of alternatives
	V_1	V_2	V_3	V_4	
PFWG [20]	-0.2334	0.0425	-0.2669	-0.1191	$V_2 > V_4 > V_1 > V_3$
FFWG	-0.1643	-0.0807	-0.1198	-0.0968	$V_2 > V_4 > V_3 > V_1$

the stability between the alternatives V_1 to V_4. The Pythagorean fuzzy weighted geometric operator (PFWG) shows less stability as compared to the Fermatean fuzzy weighted geometric operator . Since the FFWG operator is more reliable and stable in decision-making problems.

4.1 Decision Process

Step 1 is utilized to get the FFDMs, and then we normalize them. Step 2 is used to get the ideal opinions of all experts. In steps 3 and 4, we achieve collective preference values utilizing the FFWG operator, and score values, respectively. In step 5, we rank the alternatives. From Table 8, it is clear that $V_2 > V_4 > V_3 > V_1$. So the best option is V_2.

5 Conclusion

Aggregation operators are useful to associate a unique representative value for each alternative, when there are various attributes that apply to any given case. FFSs, a remarkable extension of PFSs, permit modeling of situations with higher more generality than PFSs, because they still apply in cases where the membership m and non-membership n values sum up to more than 1 but they satisfy $m^3 + n^3 \leq 1$. Fermatean fuzzy sets can handle uncertain information more easily in the process of decision-making. So we have established the essential properties of FFWG operator in the domain of FFSs to overcome the deficiencies resulted in decision-making problems in IFSs and PFSs. We have also constructed a MCDM problem, and the FFWG operator has been applied to it. The outcomes of the problem showed that the FFWG operator is more reliable and accurate in decision-making problems as compared to the PFWG operator. In the future, we will extend Pythagorean fuzzy soft Yager hybrid weighted operators to Fermatean fuzzy soft Yager hybrid weighted operators. Also we will extend the proposed method to address the MAGDM problems in which the decision information takes the form of interval-valued FFNs. Other applications of this approach will be considered, especially in business decision-making and statistics.

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