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Constructing Bootstrap Confidence Intervals of Process Capability Indices for a Three Parameter Weibull Distribution

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Abstract Statistical Quality Control (SQC) technique is used in investigating the quality improvement features in a manufacturing process. One of the important tools in SQC is the process capability indices (PCIs), for measuring and comparing the characteristics of a production process to engineering specifications. This article evaluates the performance of the PCIs for a three parameter Weibull distribution which is commonly employed in almost all fields of studies, such as reliability, stability and breakage data. In this article, three different techniques of constructing bootstrap confidence intervals (BCIs) of PCIs are investigated using simulations for the three parameter Weibull distribution. The three different techniques considered are percentile bootstrap, bias-corrected percentile bootstrap and normal bootstrap techniques. By using simulation, the authors investigate the average widths of the BCI of each of the three different techniques. It is found that the average widths of the bias-corrected percentile bootstrap technique are shorter than that of the other two techniques.

Keywords BCI; PCI; three parameter Weibull distribution

Mathematics Subject Classification 62F40, 62F25, 62G99

1 Introduction

PCIs play a major role in the process capability assessment of a production process. PCIs are useful in manufacturing as they provide useful information about the capability of a process in meeting the specification limits. PCIs of different processes can be easily compared as they are just single numbers. Anis and Tahir [1] noted that PCIs also have limitations, as they require the assumptions of statistical properties to work well in deciding about the capability of a process. The main objective in process capability assessment is to use the PCI measure to achieve the goal of producing high quality products at low costs.

Numerous PCIs are available in the literature. For example, Juran and Gryna [2] introduced the first index, C_p to study whether a process is capable of meeting the two-sided specification limits. Kane [3] presented several PCIs that can be used with bilateral and unilateral tolerances, with or without target values. Chen and Hsu [4] adopted the PCIs with distribution free tolerance intervals to estimate the standard deviation, while Clements [5] presented empirical non-normal percentiles to evaluate both the indices, C_p and C_{pk} and identified four key objectives of PCIs, which include, among others, the ability to compare different processes and classify the nearness to target (i.e. the Taguchi loss function concept). Clements [5] proposed the structure of C_p and C_{pk} based on the lognormal distribution and discussed about the use of process capability to regulate the specification limits. Hsu *et al.* [6] discussed the PCIs that are used to quantify process improvement, as well as being effective measures for quality assurance. Senvar and Sennaroglu [7] compared three different techniques, namely the Clements' Approach, Box-Cox Transformation and Johnson Transformation, for investigating the effect of tail behavior on the performance index of the process using Weibull-distributed data. Dey *et al.* [8] studied the BCIs for generalized PCIs using simulations, based on the Lindley, as well as power Lindley distributions. Dey and Saha [9] investigated the BCIs for S_{pmk} under five different life time distributions, i.e. log-logistic, generalized exponential, lognormal, Weibull and gamma distributions.

The index C_{pk} with a modified adjustment under various sample sizes was discussed by Wu [10], where the modified index is more reliable and it does not cause overestimation, hence, it is a better process capability measure in real-life applications. Yang *et al.* [11] developed a new key performance indicator that handles prototype production, where a Bayesian approach of estimation for the index C_{pk} was used and a real-life application from a process producing lithium-ion batteries was provided. By using simulation studies, robust point and interval estimations of the PCIs were investigated by Wang *et al.* [12] for non-normal distributions. Meng *et al.* [13] proposed a hypothesis testing technique for C_{pk} using the generalized p-value, for four common distributions. Three types of improved BCIs were developed by Park *et al.* [14]. The percentile bootstrap using the *R* software was developed by Rousselet *et al.* [15]. The three parameter Weibull distribution is widely used in monitoring and reliability analyses. Wais [16] explained that the two parameter Weibull distribution is inadequate in wind power analysis, instead, the three parameter Weibull distribution is needed. Moreover, the inter-failure time analysis for robust estimation was conducted by Sürücü and Sazak [17] by considering the three parameter Weibull distribution.

2 Parametric Schemes for Process Capability Indices

This section discusses the generalized form of the PCIs presented in Franklin and Wasserman [18]. The first PCI is

$$C_p(\lambda, \nu) = \frac{d - \lambda|\mu - M|}{3\sqrt{\sigma^2 + \nu(\mu - T)^2}}, \tag{1}$$

where $d = \frac{USL-LSL}{2}$, $M = \frac{USL+LSL}{2}$, λ and ν are two non-negative parameters, while T, μ and σ are the process target, mean and standard deviation values, respectively.

$C_p(\lambda, \nu)$ in Equation (1) reduces to the index

$$C_p = \frac{d}{3\sigma} = \frac{USL - LSL}{6\sigma}, \tag{2}$$

when both λ and ν are zeros. Here, USL is the upper specification limit, while LSL is the lower specification limit. When $\lambda = 1$ and $\nu = 0$, the index $C_p(\lambda, \nu)$ in Equation (1) becomes the index C_{pk} defined as follows:

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma} = \frac{\min(USL - \mu,\ \mu - LSL)}{3\sigma}. \tag{3}$$

Sometimes the Taguchi loss based index, C_{pm} is employed to measure the performance of a process. Chan *et al.* [19] established C_{pm}, based on the quadratic loss by substituting $\lambda = 0$ and $\nu = 1$ into Equation (1), which gives

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{4}$$

Pearn *et al.* [20] presented the capability index known as C_{pmk}. By choosing λ and ν both equal to one, the $C_p(\lambda, \nu)$ index in Equation (1) reduces to the index C_{pmk} as follows:

$$C_{pmk} = min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}\ ,\frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}. \tag{5}$$

The index C_{pmk} gives a sustainable quality assurance with respect to the process yield. Among the PCIs mentioned in this section, the widely used ones are the indices, C_{pm} and C_{pmk}.

3 Weibull Distribution

The Weibull distribution has received much interest in real-life applications in reliability, such as to model the breaking strength of materials. The Weibull distribution has the density function given as (Dodson [21])

$$f(z) = \frac{\alpha'}{\beta}\left(\frac{z - u}{\beta}\right)^{\alpha' - 1} e^{\left(\frac{x-\mu}{\beta}\right)^{\alpha'}} \tag{6}$$

where $z > \alpha', \mu > 0$ and $\beta > 0$. The parameters α', β and μ are shape, scale and location, respectively. The mean, variance and skewness of the Weibull distribution with three parameters are given as follows:

$$E(Z) = \mu + \beta\Gamma\left(1 + \frac{1}{\alpha'}\right) \tag{7}$$

$$\mathrm{Var}(Z) = \beta^2\left[\Gamma\left(1 + \frac{2}{}\right) - \Gamma^2\left(1 + \frac{1}{}\right)\right] \tag{8}$$

$$\mathrm{sk}(Z) = \frac{\Gamma\left(1 + \frac{1}{}\right)^3 - \Gamma\left(1 + \frac{1}{}\right)\Gamma\left(1 + \frac{2}{}\right) + \Gamma\left(1 + \frac{3}{}\right)}{\left(\Gamma\left(1 + \frac{2}{}\right) - \Gamma\left(1 + \frac{3}{}\right)\right)^{\frac{3}{2}}} \tag{9}$$

In Equations (7) - (9), $\Gamma(\,.\,)$ represents the gamma function.

4 Methods of Bootstrap Confidence Interval

The bootstrap technique originated from Efron [22]. Efron [23] and Hall *et al.* [24] provide theoretical details about the bootstrap technique. This technique can be used to construct confidence intervals for parameters when the usual interval estimation approach is not feasible. BCIs are commonly applied in constructing the confidence intervals for various PCIs. Suppose that $\varsigma_1,\ \varsigma_2,\ ...,\ \varsigma_n$ constitute a random sample with n observations taken from a distribution of interest, say ϕ, i.e. $\varsigma_1,\ \varsigma_2,\ \ldots,\ \varsigma_n \sim \phi$. Let $\hat{\theta}$ represent an estimator of an arbitrary PCI in Section 2, say C_{pm}. Then the bootstrap technique is implemented as follows:

i. A bootstrap sample with n observations (with replacement) is taken from the original sample by using $\frac{1}{n}$ as the mass at each point, where this bootstrap sample is denoted as $\varsigma_1^*, \varsigma_2^*, \ldots, \varsigma_n^*$.
ii. From the k^{th} bootstrap sample, for $1 \leq k \leq n$, the k^{th} bootstrap estimator of θ (an arbitrary PCI) can be denoted as $\hat{\theta}^* = \hat{\theta}\left(\varsigma_1^*, \varsigma_2^*, \ldots\ldots, \varsigma_n^*\right)$.
iii. If the number of resamples in the bootstrap technique is B, then a total of B estimates of $\hat{\theta}^*$ can be obtained. Arranging the whole collection from the smallest to the largest value constitutes an empirical bootstrap distribution of $\hat{\theta}$ (Kashif *et al.* [22]). $B = 1000$ bootstrap resamples is considered in this article. The confidence intervals of $\hat{\theta}$ can be constructed using any of the following three bootstrap techniques.

4.1 Method 1: Standard Bootstrap (SB) Confidence Interval

The sample average and sample standard deviation are computed as follows using the 1000 bootstrap estimates of $\bar{\theta}^*$:

$$\bar{\theta}^* = (1000)^{-1} \sum_{i=1}^{1000} \hat{\theta}^* \tag{10}$$

$$s_{\bar{\theta}^*} = \sqrt{\frac{1}{999} \sum_{i=1}^{1000} \left(\hat{\theta}^*(i) - \bar{\theta}^*\right)^2} \tag{11}$$

Consequently, the $100\,(1-\alpha)\,\%$ SB confidence interval is obtained as

$$\mathrm{CI_{SB}} = \bar{\theta}^* \pm z_{\left(1-\frac{\alpha}{2}\right)}\ s_{\hat{\theta}^*}, \tag{12}$$

where $z_{\left(1-\frac{\alpha}{2}\right)}$ is the $\left(1-\frac{1}{\alpha}\right)^{\text{th}}$ quantile of the standard normal variable.

4.2 Method 2: Percentile Bootstrap (PB) Confidence Interval

Since there is a total of B resamples of $\hat{\theta}^*$, these resamples will produce B estimates of $\hat{\theta}^*$. An arrangement of these estimates from the smallest value to the largest value will form an empirical distribution of $\hat{\theta}^*$. From the ordered empirical distribution of $\hat{\theta}^*$, choose the $100\left(\frac{\alpha}{2}\right)$ and $100\left(1-\frac{\alpha}{2}\right)$ percentiles as the end points of the interval, which results in the $100\,(1-\alpha)\,\%$ PB confidence interval for $\hat{\theta}^*$ given as

$$\mathrm{CI_{PB}} = \left(\hat{\theta}^*_{1000\left(\frac{\alpha}{2}\right)}, \hat{\theta}^*_{1000\left(1-\frac{\alpha}{2}\right)}\right). \tag{13}$$

For example, the 95% confidence interval with 1000 bootstrap estimates is

$$\mathrm{CI}_{\mathrm{PB}} = \left(\hat{\theta}^*_{(25)}, \hat{\theta}^*_{(975)}\right), \tag{14}$$

where $\hat{\theta}^*_{(25)}$ and $\hat{\theta}^*_{(975)}$ represent the 25^{th} and 975^{th} ordered collection of the bootstrap estimates of $\hat{\theta}^*$.

4.3 Bias-Corrected Percentile Bootstrap (BCPB) Confidence Interval

This technique was established to address the potential bias that could occur as the bootstrap distribution is based on a sample from the complete bootstrap distribution, which may be shifted higher or lower than would be expected. The following steps explain the implementation of this technique:

1. By means of the (ordered) distribution of $\hat{\theta}^*$, calculate

$$l_0 = \Pr\left(\hat{\theta}^* \leq \hat{\theta}\right) \tag{15}$$

2. By letting ρ^{-1} as the inverse distribution function of the standard normal variable, calculate

$$q_0 = \rho^{-1}(l_0). \tag{16}$$

3. The lower percentile and upper percentile of the ordered distribution of $\widehat{\theta}^*$ are

$$P_L = \rho\left(2q_0 + z_{\left(\frac{\alpha}{2}\right)}\right) \tag{17}$$

and

$$P_U = \rho\left(2q_0 + z_{\left(1-\frac{\alpha}{2}\right)}\right), \tag{18}$$

respectively, where ρ, $z_{\left(\frac{\alpha}{2}\right)}$ and $z_{\left(1-\frac{\alpha}{2}\right)}$ are the distribution function, $\left(\frac{\alpha}{2}\right)^{\text{th}}$ quantile and $\left(1-\frac{\alpha}{2}\right)^{\text{th}}$ quantile, respectively, of the standard normal distribution. Consequently, the $100\,(1-\alpha)\,\%$ BCPB confidence interval is constructed as

$$\mathrm{CI}_{\mathrm{BCPB}} = \left(\hat{\theta}^*_{1000(P_L)}, \hat{\theta}^*_{1000(P_U)}\right). \tag{19}$$

The average width (AW) is considered to compare the three different types of BCIs. The AW of the BCI is computed using a total of M trials. Next, the estimated AW is computed as

$$\mathrm{AW} = \frac{\sum_{i=1}^{M}\left(U_{p_i} - L_{w_i}\right)}{M}, \tag{20}$$

where L_{W_i} and U_{W_i} are the estimated lower confidence limit and upper confidence limit of the $100(1-\alpha)\%$ confidence interval for any of the three types of BCIs discussed in Sections 4.1, 4.2 and 4.3, based on the i^{th} replicate.

5 Results and Discussion

An evaluation of the performances of PCIs, based on the C_{pm} and C_{pmk} indices, is conducted by means of simulation. From the skewness value, sk (Z), the behavior of the Weibull distribution is categorized as highly positive skewed, moderately skewed and negatively skewed, based on the various values of the shape ($\alpha\prime$), scale (β) and location (μ) parameters considered. The value of sk (Z) is calculated using Equation (9).

In this simulation study, the data are generated using sample sizes, $n \in \{25, 50, 75, 100, 300, 500\}$ from the Weibull distribution, for each of the three skewness levels mentioned in Table 1. Note that the parameter values of $\alpha\prime, \beta$ and μ as shown in Table 1 are specified in order to obtain the three levels of skewness. Additionally, LSL = 0 and USL = 6 are used.

Table 1: Skewness of the Weibull distribution

Shape parameter, $\alpha\prime$	Scale parameter, β	Location, μ parameter, μ	Skewness, sk (Z)	Behavior of distribution
0.5	1.2	1.6	6.52	Highly positive skewed
1.0	1.2	1.6	1.00	Moderately skewed
1.5	1.2	1.6	−2.12	Negatively skewed

The estimated 95% BCIs of C_{pm} and C_{pmk} for the three parameter Weibull distribution, presented in Tables 2 and 3 for the three types of BCIs (SB, PB and BCPB), are obtained based on the R language. 1000 bootstrap resamples are considered. In studying the performances of the three types of BCIs, the AWs of these confidence intervals are computed as the difference between the average upper confidence limit and average lower confidence limit. The analysis of results is based on a comparison between the AWs of the estimated 95% BCIs of C_{pm} and that of C_{pmk}, shown in Table 4. In the case of interval estimation, the widths of all three types of BCIs are affected by both the shape parameter and sample size. The three BCIs (SB, PB and BCPB) and their respective AWs are estimated using the R language based on 1000 bootstrap resamples. For evaluating the performances of the three BCIs, the AW of each of the confidence interval is computed by subtracting the lower confidence limit from the upper confidence limit.

In Table 4, the results show that, with the exception of $n = 25$ for the highly positive and moderately skewed processes, the AWs of the three types of BCIs generally become smaller when the sample size becomes larger, for the indices C_{pm} and C_{pmk}. For example, for the negatively skewed Weibull distribution in Table 4, the AWs of the 95% SB confidence intervals of C_{pm} are {0.6502, 0.5159, 0.3528, 0.3492, 0.1892, 0.1501} for $n \in \{25, 50, 75, 100, 300, 500\}$, where it is obvious that the AWs decrease as n increases. The results in Table 4 show that generally the AWs of the 95% confidence intervals given by the BCPB technique are smaller than that of the other two techniques. Hence, it can be concluded that the performance of the BCPB technique is better than that of the other two techniques. Additionally, the AWs of the BCIs for the PB technique are generally smaller than that of the SB technique. Thus, in terms of performance, the BCPB technique performs best, while the PB technique comes in second and the SB technique comes in last.

Table 2: Estimated 95% BCI of C_{pm} for the Weibull distribution

Weibull distribution	Sample, size, n	SB	PB	BCPB
Highly positive skewed	25	(0.7141, 0.9868)	(0.7487, 1.0202)	(0.7370, 0.9997)
	50	(0.2179, 0.6433)	(0.3350, 0.7370)	(0.2903, 0.6615)
	75	(0.1252, 0.4302)	(0.2182, 0.5347)	(0.1959, 0.4501)
	100	(0.1292, 0.3799)	(0.1963, 0.4454)	(0.1745, 0.3881)
	300	(0.1525, 0.3341)	(0.1880, 0.3607)	(0.1638, 0.3332)
	500	(0.1635, 0.2625)	(0.1788, 0.2783)	(0.1731, 0.2667)
Moderately skewed	25	(0.7589, 1.2506)	(0.8414, 1.3295)	(0.8123, 1.2725)
	50	(0.5730, 1.1054)	(0.6630, 1.1709)	(0.5859, 1.1024)
	75	(0.6805, 1.1895)	(0.7514, 1.2087)	(0.6584, 1.1627)
	100	(0.8020, 1.0826)	(0.8257, 1.1051)	(0.8009, 1.0767)
	300	(0.7673, 0.9799)	(0.7825, 0.9943)	(0.7648, 0.9770)
	500	(0.7864, 0.9378)	(0.7946, 0.9465)	(0.7854, 0.9385)
Negatively skewed	25	(0.7671, 1.4173)	(0.8844, 1.5214)	(0.8216, 1.4285)
	50	(1.0203, 1.5362)	(1.0801, 1.5950)	(1.0270, 1.5339)
	75	(1.1775, 1.5303)	(1.2145, 1.5633)	(1.2033, 1.5472)
	100	(1.0576, 1.4068)	(1.0909, 1.4311)	(1.0565, 1.3967)
	300	(1.1723, 1.3615)	(1.1796, 1.3696)	(1.1614, 1.3546)
	500	(1.1584, 1.3085)	(1.1616, 1.3107)	(1.1536, 1.3037)

6 Real Life Example

In this section, a real life example is presented to demonstrate the application of the proposed methodology for all PCIs. For this purpose, the data in Chang and Lu [25], which represent the thickness of an oil seal process are adopted. There are 65 observations in the data which are reported in Table 5.

The summary statistics for the thickness of oil seal process data are reported in Table 6.

Chang and Lu [25] mentioned that the thickness of oil seal process dataset is not normal. Kao [26] concluded that adopting the three parameter Weibull distribution is appropriate for this dataset. By fitting the three parameter Weibull distribution, the maximum likelihood estimators for the shape, scale and location parameters are $\widehat{\alpha'}$= 5.504809, $\widehat{\beta}$ = 2.650830 and $\hat{\ }$= 1.348899, respectively. The objective of this example is to use this dataset to study the performance of the third generation PCIs (C_{pm} and C_{pmk}) and to construct their BCIs. For this purpose, the USL = 2.5 and LSL = 1.5 are used to estimate the PCIs and construct their BCIs. The point and interval estimates of the third generation

Table 3: Estimated 95% BCI of C_{pmk} for the Weibull distribution

Weibull 3	Sample size, n	SB	PB	BCPB
Highly positive skewed	25	(-0.2415, 0.3788)	(-0.0085, 0.5664)	(-0.0480, 0.5081)
	50	(-0.1123, 0.2982)	(0.0264, 0.4234)	(0.0091, 0.3493)
	75	(0.0145, 0.3030)	(0.0752, 0.3668)	(0.0592, 0.3268)
	100	(-0.1693, 0.3490)	(0.0467, 0.4984)	(0.0017, 0.4072)
	300	(-0.0209, 0.1262)	(0.0153, 0.1591)	(0.0015, 0.1362)
	500	(0.0405, 0.1651)	(0.0623, 0.1865)	(0.0549, 0.1707)
Moderately skewed	25	(0.4411, 0.8915)	(0.4998, 0.9614)	(0.4741, 0.9307)
	50	(0.4949, 0.9570)	(0.4949, 0.9625)	(0.4463, 0.9345)
	75	(0.5787, 0.8713)	(0.6117, 0.9207)	(0.5771, 0.8838)
	100	(0.7023, 0.9256)	(0.7278, 0.9512)	(0.7052, 0.9276)
	300	(0.6708, 0.8390)	(0.6871, 0.8519)	(0.6637, 0.8406)
	500	(0.7163, 0.8596)	(0.7266, 0.8733)	(0.7185, 0.8576)
Negatively skewed	25	(0.8450, 1.2960)	(0.9040, 1.3730)	(0.9000, 1.3500)
	50	(0.8610, 1.2670)	(0.8990, 1.3090)	(0.8740, 1.2760)
	75	(0.9330, 1.2460)	(0.9640, 1.2780)	(0.9610, 1.2670)
	100	(0.9080, 1.1930)	(0.9460, 1.2350)	(0.9290, 1.2100)
	300	(1.0390, 1.2070)	(1.0520, 1.2180)	(1.0430, 1.2060)
	500	(1.0670, 1.1950)	(1.0760, 1.2100)	(1.0730, 1.2040)

Table 4: AWs of the 95% BCIs of C_{pm} and C_{pmk} for the Weibull distribution

			C_{pm}			C_{pmk}		
Effect of skewness	N	$\alpha\prime$	AW			AW		
			SB	PB	BCPB	SB	PB	BCPB
Highly positive skewed	25	0.5	0.2727	0.2715	0.2627	0.6203	0.5749	0.5561
	50	0.5	0.4254	0.4020	0.3712	0.4105	0.3970	0.3402
	75	0.5	0.3050	0.3165	0.2542	0.2885	0.2916	0.2676
	100	0.5	0.2507	0.2491	0.2136	0.5183	0.4517	0.4055
	300	0.5	0.1816	0.1727	0.1694	0.1471	0.1438	0.1347
	500	0.5	0.0990	0.0995	0.0936	0.1246	0.1242	0.1158
Moderately skewed	25	1.0	0.4917	0.4881	0.4602	0.4504	0.4616	0.4566
	50	1.0	0.5324	0.5079	0.5165	0.4621	0.4676	0.4882
	75	1.0	0.5090	0.4573	0.5043	0.2926	0.3090	0.3067
	100	1.0	0.2806	0.2794	0.2758	0.2233	0.2234	0.2224
	300	1.0	0.2126	0.2118	0.2122	0.1682	0.1648	0.1769
	500	1.0	0.1514	0.1519	0.1531	0.1433	0.1467	0.1391
Negatively skewed	25	1.5	0.6502	0.6370	0.6069	0.4510	0.4690	0.4500
	50	1.5	0.5159	0.5149	0.5069	0.40600	0.4100	0.4020
	75	1.5	0.3528	0.3488	0.3439	0.3130	0.3140	0.3060
	100	1.5	0.3492	0.3402	0.3402	0.2850	0.2890	0.2810
	300	1.5	0.1892	0.1900	0.1932	0.1680	0.1660	0.1630
	500	1.5	0.1501	0.1491	0.1501	0.1280	0.1340	0.1310

Table 5: Thickness of an oil seal process data

2.4	2.3	2.0	2.2	2.2	2.2	2.2	2.4	2.1	2.0	2.2	2.0	2.0
1.8	2.3	2.0	2.4	2.4	1.9	1.8	2.1	1.8	2.0	2.3	1.8	1.9
2.1	1.7	1.6	2.2	1.9	1.6	1.9	2.4	1.9	2.1	2.0	2.1	2.1
2.0	1.9	2.5	1.8	1.8	1.8	2.0	2.0	1.9	2.0	2.1	1.8	2.1
1.7	2.0	1.6	1.6	2.1	1.9	1.8	1.9	2.2	2.0	2.2	2.1	2.3

Table 6: Summary statistics of the data

Statistic	Value	Statistic	Value
N	65	Shape	3.43807
Min.	1.6	Scale	0.7489
Max.	2.5	Location	1.3488
Mean	2.0215	Mean (Weibull)	2.0222
S.d	0.2190	Variance (Weibull)	0.04686
Q(1)	1.9	Median (Weibull)	2.0221
Q(2)	2	USL	2.500
Q(3)	2.2	LSL	1.500
Skewness	0.0567	Target value	2.00
Kurtosis	2.5307	USL (p)	3.00
		LSL (P)	1.00

PCIs together with the first and second generation PCIs (C_p and C_{pk}) are reported in Table 7. The values in parentheses are the lower and upper confidence limits of each BCI, whereas the value below each BCI is the width of the BCI.

A comparison of the three BCIs (SB, PB and BCPB) for four different PCIs (C_p , C_{pk}, C_{pm} and C_{pmk}) shows that the BCPB confidence interval gives the shortest width. Therefore, the BCPB confidence interval is recommended in this situation.

7 Conclusion

In continuous quality improvement for measuring the performance of a production process, the PCIs are commonly used. The implementation of the PCIs requires a process to follow a normal distribution but in many real life problems, the normality assumption cannot be fulfilled. The objective of this study is to evaluate the performance of third generation PCIs, i.e. C_{pm} and C_{pmk} and to construct their BCIs when the process follows a three parameter Weibull distribution. The performance of the C_{pm} and C_{pmk} indices are evaluated at different skewness levels, i.e., highly positive skewed, moderately skewed and negatively skewed, using different sample sizes for the three parameter Weibull distribution. At each skewness level, three BCIs, namely, SB, PB and BCPB are constructed using 1000 bootstrap resamples. The BCI which shows a smaller width is recommended. The results show that the BCPB confidence interval performs better than the other two confidence intervals, in terms of having a smaller width, both for the simulated data and real life data. In future, the performance of third generation PCIs and construction of BCIs, based on other three parameter distributions, such as the three parameter exponential and three parameter lognormal distributions can be explored.

Table 7: Summary statistics of the data

PCIs	Point Estimates	BCIs		
		SB	PB	BCPB
C_p	0.7698	(0.6726, 0.8908) 0.2182	(0.6857, 0.9024) 0.2167	(0.6380, 0.8548) 0.2168
C_{pk}	0.7357	(0.6157, 0.8585) 0.2428	(0.6252, 0.8664) 0.2412	(0.6275, 0.8564) 0.2289
C_{pm}	0.7658	(0.6634, 0.8782) 0.2148	(0.6767, 0.8905) 0.2138	(0.6474, 0.8574) 0.2100
C_{pmk}	0.7319	(0.5979, 0.8575) 0.2596	(0.6016, 0.8618) 0.2602	(0.6056, 0.8548) 0.2492

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