

Global Optimization Method For Minimizing Portfolio Selection Risk

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Abstract This study employed the global optimization method called Modified Trusted Region Method (MTRM) to resolve the portfolio selection risk problem. An objective of unconstrained optimization problem was formulated with four sets of fund data. The relationship between the level of acceptable risk and the weighting factor was analyzed numerically. The return of portfolio increased along with the level of acceptable risk since a high return was always accompanied by higher risk. By contrast, the risk of portfolio decreased as the weighting factor increased. The MTRM could resolve the portfolio optimization problem.

Keywords Global optimization; homotopy; portfolio theory; risk; trust region

Mathematics Subject Classification 65K05, 90C26

1 Introduction

Although nonlinear optimization problems are broadly studied in the past 30 years, finding a global optimum for a non-convex and nonlinear optimization problems remains the concern of many researchers [1]. The minimization of the unconstrained optimization problem is defined as

$$\min_x f(x) \quad (1)$$

where $x \in R^n$ is a real vector with components and $f(x)$ is a smooth function [2]. However, solving a general unconstrained nonlinear optimization problem could be very challenging even if the problem is small because the feasible region of the problem is not always convex [2].

When the problem is multi-modal and the algorithm used does not emphasize exploration, then the algorithm will be trapped in one of the local solutions and might cause the solution obtained is not the global optimum. Based on the homotopy technique, many global optimization algorithms have been developed to address this issue. They include the tunneling method, conjugate gradient method, homotopy optimization method (HOM), and

homotopy optimization with perturbation and ensembles (HOPE). Meanwhile, homotopy is a fundamental concept in topology. Homotopy acts as a medium in optimization to transfer solutions successively from one local minimum to another better one [3].

HOM produces a sequence of points approaching a local minimizer of a function. In comparison, HOPE generates a series of ensembles points that are local minimizers of a function. HOPE is a global optimization algorithm extended from HOM [3], yielding a continuous transformation from an easy problem to the given one.

HOPE uses the local minimizers in the previous step as starting points. It perturbs stochastically in various directions with different lengths to find the next minimizers [4]. All minimizers found are stored in an ensemble. The points in the ensemble will be perturbed again to find the next solution. These steps are repeated until its stopping criterion is achieved. The performance of HOPE will increase along with the computation efforts. However, HOPE becomes expensive when the number of ensembles increases.

Subsequently, an algorithm known as homotopy with two-step predictor-corrector method (HSPM) was developed by Kerk to overcome the computational burden of HOPE [5]. Although HSPM performed better than HOPE in time of complexity, it was designed to solve the one variable optimization problem only. Hence, it is not comparable with HOPE. HSPM was improved further with better time complexity when the inner loop of its algorithm was altered slightly, giving rise to the algorithm called Improved Homotopy with 2-Step Predictor-corrector Method (I-HSPM) [6]. When extended, it resolves multi-variable optimization problems, and this gradient-based algorithm is now known as Modified Trusted Region Method (MTRM) [7].

In general, MTRM generates a series of the trusted region (TR) using the Poincare-Miranda theorem (PMT) for identifying a TR that confers the convexity. This attribute allows MTRM to determine the convex parts within the given domain of a function. In other words, MTRM could separate a big non-convex problem into several small but convex ones [7].

This study used MTRM to resolve the risk problems of portfolio selection. In general, portfolio selection describes how investors should allocate their wealth among several assets in financial mathematics, risk management, and economics [8,9]. It also answers the trade-off risk and returns while selecting assets under certain conditions [10].

2 Modified Trusted Region Method (MTRM)

MTRM is a global optimization algorithm that resolves multi-variable problems. It is a gradient-based algorithm. It finds the area of the zeroes on the gradient of an objective function. Such areas are called the trusted region (TR). In the algorithm of MTRM, a TR must have negative derivative function values followed by positive ones. With this condition, MTRM can filter out the maximizers and also the saddle points.

Unlike Hansens method [11], MTRM does not eliminate any area with no minimizer. Instead, it identifies areas with an existing minimizer, and such area is the TR. MTRM is a unique algorithm because it identifies convex regions within a non-convex region.

When initialized with the homotopy function, the first loop in the MTRM is called the identification part of TR. This part contains the preparation steps for identifying and gathering the TRs. If other TRs intercept the same region, only one will be kept while others will be eliminated in the next step. In the subsequent local search, the endpoint values will be

calculated and stored in an ensemble with all the minimizers found in the local search step. Then, the global solution will be determined.

The algorithm of MTRM for cases with two variables is given below.

Algorithm 1: Algorithm MTRM

1. Construct a homotopy function, $H(x, y, \lambda)$ by combination of target function and auxiliary function.
 2. Set a closed interval, $[x_1, x_2]$, $[y_1, y_2]$ for the homotopy function $H(x, y, \lambda)$.
 3. Choose step sizes for st , sx and sy , where st is a step size for λ , step size sx and sy are used to divide $[x_1, x_2]$, $[y_1, y_2]$ into several subintervals respectively.
 4. For $\lambda = l, \dots, 1$
 5. For $j = 0, y_1 + syj \leq y_2, j + +$
 6. For $i = 0, x_1 + sxi \leq x_2, i + +$
 7. Compute $f_x[x_i, y_i]$.
 8. end.
 9. end.
 10. Determine subinterval $_m = [\{x_i, y_j\}, \{x_{i+1}, y_j\}]$ where $\{x_i, y_j\}$ gives a negative function value while $\{x_{i+1}, y_j\}$ gives a positive function value and $m = 1, 2, \dots, M$.
 11. For $i = 0, x_1 + sxi \leq x_2, i + +$
 12. For $j = 0, y_1 + syj \leq y_2, j + +$
 13. Compute $f_y[x_i, y_i]$.
 14. end.
 15. end.
 16. Determine subinterval $_n = [\{x_i, y_j\}, \{x_i, y_{j+1}\}]$ where $\{x_i, y_j\}$ gives a negative function value while $\{x_i, y_{j+1}\}$ gives a positive function value and $n = 1, 2, \dots, N$.
 17. end
 18. Filtration: subinterval $_k =$ delete duplicate subintervals among subinterval $_m$ and subinterval $_n$ where $k = 1, 2, \dots, K$.
 19. For $k = 1, 2, \dots, K$
 20. let $\{x_i, y_j\}$ be the initial point.
 21. run a local search method on the target function and store the solution found in an ensemble.
 22. end
 23. Calculate the function value of endpoints and add into the ensemble.
 24. Select the lowest function value for the function value $f(x, y)$.
 25. Stop.
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MTRM has proven to hold the descent property and global convergence. It has also shown its robustness capability of locating the global minimizers [7].

3 Portfolio Selection

The portfolio theory deals with the selection of portfolios that maximize the expected returns consistent with an individually acceptable level of risk [12, 13]. Investors are risk-averse [14],

i.e., when faced with two investments with different risks but the same expected return, they prefer the one with the lowest risk. Usually, investors would keep a risky investment if, and only if, the expected return is sufficiently high enough to offset the assuming risk.

The portfolio theory quantifies risk into the variance of an assets expected return [15]. The variance of a random variable is a measure of the dispersion of the possible outcomes around the expected value. The larger the variance or standard deviation, the greater the investment risk. In this respect, the risk could be reduced by constructing a portfolio for diversification, which is the central concept of portfolio theory. In diversification, money is put in different asset classes and securities to spread out the investment [13].

In general, when someone wants to invest, a bank manager will offer some potential assets to the customer, for example, two funds in different sectors, one in information technology (IT), one in manufacturing. The question that arises here is, how much should one invest in each fund? Is 50: 50 or 70: 30? The question could be answered using a model with the fund data from the bank to compute the appropriate allocation for each fund using MTRM in minimizing the overall risk of the portfolio while maximizing the return.

4 Problem Description and Methodology

The portfolio optimization is unique with various mathematical models for different types of portfolios and even for portfolios intended for the same purpose. However, the model used is not uniquely defined [16]. The portfolio optimization is usually modeled in a linear programming structure with many constraints. But, it can also be represented by an unconstrained minimization problem by reformulating the problem [17]. This paper used the method introduced in another study [17] to formulate the portfolio model for computing the best combination of expected return and risk. Other authors [18–21] also used the same method to resolve portfolio problems.

The expected portfolio return given by the investment fraction y_1, y_2, \dots, y_n could be represented by Equation (2) [17]

$$R = \sum_{i=1}^n \bar{r}_i y_i, \quad (2)$$

where

$$\bar{r}_i = \frac{\sum_{j=1}^m r_{ij}}{m} \quad (3)$$

is the mean of return of each asset.

The variance and covariance could then be calculated from the past performance, i.e., the return rate r_{ij} . The variance of asset i could be expressed by Equation (4),

$$\sigma_{ii} = \frac{\sum_{j=1}^m (r_{ij} - \bar{r}_i)^2}{m} \quad (4)$$

and the covariance of assets i and k could be represented by Equation (5).

$$\sigma_{ik} = \frac{\sum_{j=1}^m (r_{ij} - \bar{r}_i)(r_{kj} - \bar{r}_k)}{m}. \quad (5)$$

Then, the portfolio risk could be defined by Equation (6).

$$V_{min} = \sum_{i=1}^n \sigma_{ii} y_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sigma_{ij} y_i y_j. \quad (6)$$

By using matrix-vector notation, portfolio optimization problems can be formulated as

$$\text{minimize } F = -r^T y + \frac{\rho}{v_a^2} (y^T Q y - v_a)^2 \quad (7)$$

subject to

$$\sum_{i=1}^n y_i = 1, \quad (8)$$

where r is the mean of return and ρ is a weighting factor controlling the balance between return and risk, v_a represents acceptable level of risk, and Q is a variance-covariance matrix given below.

$$Q = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}.$$

It contains variances and covariances which computed based on past performance history of fund's data.

5 Results and Discussion

Table 1 shows the return of PB Cash Management Fund (PBCMF), PB Islamic Bond Fund (PBIBF), PB Islamic Equity Fund (PBIEF), and PB Asean Dividend Fund (PBADF) for the past ten years. The data were downloaded from the official website of Public Mutual: <https://www.publicmutual.com.my/Our-Products/Fund-Overview>. y_1 and y_2 are very low risk funds while the indicator for y_3 and y_4 are very high risk [22].

Table 2 shows the mean return for each asset generated from the fund data where \bar{r}_1 represents the mean return for y_1 , \bar{r}_2 represents the mean return for y_2 , \bar{r}_3 represents the mean return for y_3 and \bar{r}_4 represents the mean return for y_4 . Meanwhile, the matrix of variance-covariance was computed using Equations (4) and (5).

$$Q = \begin{bmatrix} 0.1546 & -0.0417 & -1.7301 & 0.0089 \\ -0.0417 & 2.1245 & -1.6930 & -4.0224 \\ -1.7301 & -1.6930 & 67.6209 & 18.8094 \\ 0.0089 & -4.0224 & 18.8096 & 64.0090 \end{bmatrix}.$$

Then, the objective function was formed with Equation (7), yielding the minimize F in Equation (9).

$$\begin{aligned}
\text{minimize } F = & -3.088y_1 - 4.920y_2 - 5.481y_3 - 5.878y_4 + \frac{\rho}{v_a}(0.1546y_1^2 \\
& + 2.1245y_2^2 + 67.6209y_3^2 + 64.009y_4^2 - 0.0834y_1y_2 - 3.4602y_1y_3 \\
& + 0.0178y_1y_4 - 3.386y_2y_3 - 8.0448y_2y_4 + 37.619y_3y_4). \tag{9}
\end{aligned}$$

Table 1: Returns of Fund Data

Year	PBCMF(y_1)	PBIBF(y_2)	PBIEF(y_3)	PBADF(y_4)
2011	2.75	9.35	-1.89	0.77
2012	2.91	6.14	15.96	21.15
2013	2.87	3.38	14.10	2.30
2014	3.06	3.52	1.08	11.87
2015	3.46	4.05	8.68	-0.23
2016	3.31	5.43	-4.59	9.90
2017	3.29	4.89	8.34	16.03
2018	3.60	5.96	-8.64	-4.92
2019	3.40	7.46	6.37	3.24
2020	2.22	5.14	15.40	-1.33

Table 2: Mean of Each Asset

\bar{r}_i	mean
\bar{r}_1	3.088
\bar{r}_2	4.920
\bar{r}_3	5.481
\bar{r}_4	5.878

With MTRM, the weight for y_1, y_2, y_3 and y_4 was computed. These weights were appreciable, i.e., with different values of ρ and v_a . The sum of the invested proportions equals to 1. The numerical results are shown in Table 3.

Based on Table 3, the global solution is attained when $\rho = 1$ and $v_a = 1$. The global minimum $F = -4.7538$ and the weight for y_1, y_2, y_3 and y_4 are 0.0837, 0.8123, 0.0281 and 0.0759 respectively. With Equation (2), the expected return could be computed using Equation (10).

$$R = \bar{r}_1y_1 + \bar{r}_2y_2 + \bar{r}_3y_3 + \bar{r}_4y_4 = 4.8551. \tag{10}$$

Meanwhile the V_{min} value was computed using Equation (6) with outputs given in Equation (11) below.

$$\begin{aligned}
V_{min} = & 0.1546y_1^2 + 2.1245y_2^2 + 67.6209y_3^2 + 64.009y_4^2 - 0.0834y_1y_2 - 3.4602y_1y_3 + \\
& 0.0178y_1y_4 - 3.386y_2y_3 - 8.0448y_2y_4 + 37.6188y_3y_4 = 1.3183. \tag{11}
\end{aligned}$$

The numerical results showed that when the level of acceptable risk was high, the portion for the high-risk fund, like y_3 and y_4 would increase. For example, when $\rho = 1$, the weight for y_4 increased from 0.0252, 0.0426, 0.0555, and finally reaching 0.0759 as v_a varied from 0.2 to 1.0.

Table 3: Numerical Results

ρ	v_a	y_1	y_2	y_3	y_4	R	V_{min}	F
1	0.2	06154	0.3322	0.0272	0.0252	3.832	0.2368	-3.7980
	0.4	0.4331	0.4968	0.0275	0.0426	4.1828	0.4894	-4.1329
	0.6	0.2982	0.6186	0.0277	0.0555	4.4424	0.7557	-4.3751
	0.8	0.1845	0.7213	0.0279	0.0633	4.6612	1.0326	-4.5766
	1.0	0.0837	0.8123	0.0281	0.0759	4.8551	1.3183	-4.7538
10	0.2	0.6485	0.3023	0.0271	0.0221	3.7683	0.2042	-3.7639
	0.4	0.4817	0.4529	0.0274	0.0380	4.0893	0.4100	-4.0831
	0.6	0.3693	0.5593	0.0276	0.0438	4.3009	0.6039	-4.3074
	0.8	0.2673	0.6465	0.0278	0.0584	4.5018	0.8262	-4.4911
	1.0	0.1831	0.7225	0.0278	0.0666	4.6440	1.0365	-4.6506
100	0.2	0.6526	0.2986	0.0271	0.0217	3.7604	0.2005	-3.7600
	0.4	0.4875	0.4476	0.0274	0.0375	4.0782	0.4012	-4.0774
	0.6	0.3719	0.5520	0.0276	0.0485	4.3006	0.6019	-4.2997
	0.8	0.2774	0.6373	0.0278	0.0575	4.4825	0.8029	-4.4812
	1.0	0.1955	0.7113	0.0279	0.0653	4.6401	1.0038	-4.6386
1000	0.2	0.6531	0.2982	0.0265	0.0222	3.7597	0.2001	-3.7591
	0.4	0.5033	0.4277	0.0180	0.0510	4.0569	0.4003	-4.0566
	0.6	0.3751	0.5463	0.0259	0.0527	4.2978	0.6004	-4.2975
	0.8	0.2981	0.5010	0.0269	0.0740	4.4599	0.8006	-4.4591
	1.0	0.2040	0.6945	0.0270	0.0745	4.6328	1.0010	-4.6321

Figure 1 shows a positive relationship between the return R and v_a , and Figure 2 shows a negative relationship between the risk V_{min} and ρ . The return of the portfolio increased along with v_a since a high return was always accompanied by a higher risk (Figure 1) [9]. By contrast, the risk of portfolio decreased as ρ increased (Figure 2).

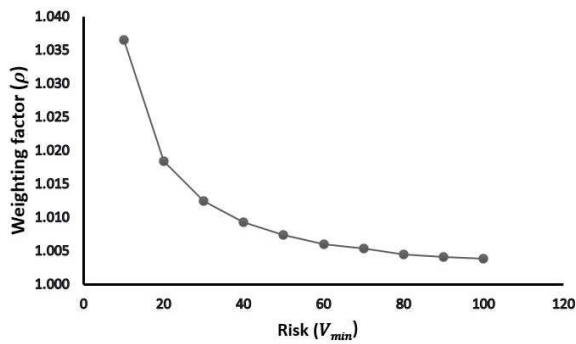


Figure 1: R versus v_a .

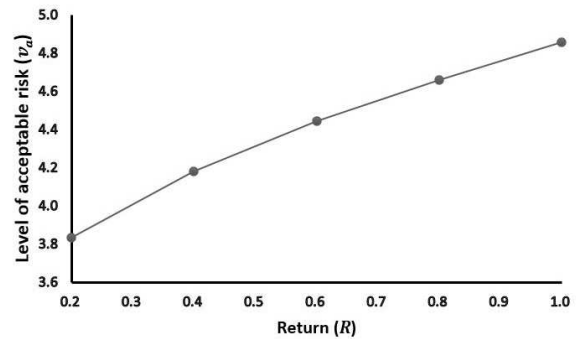


Figure 2: V_{min} versus ρ .

6 Conclusion

This study used the MTRM to resolve the portfolio selection risk problem. A portfolio model was formulated with real-life data. The interrelation between the level of acceptable risk (v_a) and the weightage of each fund (ρ) was analyzed with five experiments implemented at each ρ as v_a varied from 0.2 to 1.0. In general, a risk-averse investor would invest a larger portion of funds on low-risk assets compared to the high-risk assets. Besides, the lower the risk, the lower the potential return, and vice versa. In conclusion, the MTRM played its role well in resolving the portfolio optimization problem.

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