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An Extension for the Univariate Exponentially Weighted Moving Average Control Chart

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Abstract The univariate Exponentially Weighted Moving Average (EWMA) control chart, which will be called the EWMA chart hereafter is a good alternative to the Shewhart control chart when one is interested in detecting small shifts quickly. The performance of the EWMA control chart is comparable to that of the cumulative sum (CUSUM) control chart but the former is easier to set up and operate. In this paper, an approach by means of transformation of using the EWMA chart in a multivariate process monitoring will be discussed.

Keywords EWMA; CUSUM; MEWMA; Hotelling; noncentrality parameter; mean vector; covariance matrix; average run length; in-control; out-of-control

1 Introduction

Since the EWMA chart was first introduced by Roberts [1], numerous extensions and variations of the basic EWMA chart have been proposed. Crowder [2 & 3] proposes a simple method for studying the run length distributions of exponentially weighted moving average charts. He also suggests the design of an optimal EWMA control chart based on a desired in-control ARL [3]. Lucas and Saccucci [4] suggest the use of a fast initial response feature to make the EWMA control chart scheme more sensitive to start-up problems. Other approaches of adding the fast initial response feature to the EWMA include the works of Rhoads, Montgomery and Mastrangelo [5]; and Steiner [6]. MacGregor and Harris [7] discuss the use of EWMA based statistics for monitoring the process standard deviation. Borror, Champ and Rigdon [8] describe a procedure for using the EWMA chart for monitoring Poisson counts.

The objective of this paper is to extend the use of an EWMA control chart in the monitoring of a multivariate process. Although a multivariate EWMA (MEWMA) control chart is a logical extension of the EWMA, the use of an EWMA chart to monitor multivariate processes is yet another extension to the literature of EWMA control charts.

2 EWMA Control Chart

The EWMA is defined as

$$Z_i = \beta X_i + (1-\beta)Z_{i-1}, \quad i = 1, 2, \ldots \tag{1}$$

where $0 < \beta \leq 1$ is the smoothing constant and the starting value is the process target value, i.e., $Z_0 = \mu_0$. Here, it is assumed that $X_i, i = 1, 2, \ldots$, are independent and identically (i.i.d.) $N\left(\mu_0, \sigma^2\right)$ observations, where μ_0 and σ^2 represent the process mean and variance of X_i respectively. If μ_0 is unknown, then the average of an in-control preliminary data set is used as the starting value of the EWMA so that $Z_0 = \bar{X}$. The EWMA chart is constructed by plotting Z_i versus the sample number i. The center line and control limits for the EWMA chart are as follow:

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\beta}{2-\beta}\left[1-(1-\beta)^{2i}\right]} \tag{2}$$

$$\text{Center line} = \mu_0 \tag{3}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\beta}{2-\beta}\left[1-(1-\beta)^{2i}\right]} \tag{4}$$

After the EWMA chart has been running for several time periods, the limits will approach steady state values, given for example by Montgomery [9], are

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\beta}{2-\beta}} \tag{5}$$

$$\text{Center line} = \mu_0 \tag{6}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\beta}{2-\beta}} \tag{7}$$

Small values of β are used for a quick detection of small shifts while large values of β are preferable for detecting large shifts.

3 Multivariate EWMA (MEWMA) Control Chart

In the multivariate case, a natural extension is to define vectors of EWMA's,

$$Z_i = RX_i + (I-R)Z_{i-1}, i = 1, 2, \ldots, \tag{8}$$

where $Z_0 = 0$ and $R = \mathrm{diag}(r_1, r_2, ..., r_p)$, $0 < r_j \leq 1, j = 1, 2, \ldots, p$. It is assumed that the multivariate observations, $X_i, i = 1, 2, \ldots$, follow an i.i.d. $N_p\left(\mu_0, \Sigma\right)$ distribution, where μ_0 is the on-target mean vector and Σ is the covariance matrix. The MEWMA chart gives an out-of-control signal as soon as

$$T_i^2 = Z_i' \Sigma_{Z_i}^{-1} Z_i > h \tag{9}$$

where $h(> 0)$ is chosen to achieve a specified in-control ARL and Σ_{Z_i} is the covariance matrix of Z_i. Lowry, Woodall, Champ and Rigdon [10] show that if $r_1 = r_2 = \ldots = r_p = r$, then the MEWMA vectors can be written as

$$Z_i = rX_i + (1-r)Z_{i-1}, i = 1, 2, \ldots, \tag{10}$$

where

$$\Sigma_{Z_i} = \left\{r\left[1-(1-r)^{2i}\right]\Big/(2-r)\right\}\Sigma \tag{11}$$

In the MEWMA control chart design, the asymptotic (as $i \to \infty$) covariance matrix, i.e.,

$$\Sigma_{Z_i} = \{r/(2-r)\}\,\Sigma \tag{12}$$

is usually considered. Analogous to the univariate case, smaller values of r are more effective in detecting small shifts in the mean vector.

4 Using an EWMA Chart in the Monitoring of Multivariate Observation

Assume that $X_f, f = 1, 2, \ldots,$ follow an i.i.d. $N_p(\mu_0, \Sigma)$ distribution. Tracy, Young and Mason [11] show that the statistic

$$T_f^2 = (X_f - \mu_0)'\,\Sigma^{-1}\,(X_f - \mu_0) \tag{13}$$

follows a chi-square distribution with p degrees of freedom. However, if the true population parameters, μ_0 and Σ are both unknown and are estimated, where their estimates are $\bar{X}_m$ and S_m respectively, then the statistic T_f^2 is defined as

$$T_f^2 = \left(X_f - \bar{X}_m\right)' S_m^{-1} \left(X_f - \bar{X}_m\right) \tag{14}$$

where its exact distribution is

$$T_f^2 \sim \frac{p(m+1)(m-1)}{m(m-p)} F_{p,m-p}. \tag{15}$$

In the above discussion, m is the number of observations in a preliminary data set assumed to represent a stable process where both estimates $\bar{X}_m$ and S_m are made. X_f denotes the p dimensional vector of future observations on the p quality characteristics.

Consider the T_f^2 statistic in equation (13), where $T_f^2 \sim \chi_p^2$. It follows that $H_p\left(T_f^2\right)$ has a uniform distribution on the unit interval, where $H_p(\cdot)$ is the chi-square distribution function with p degrees of freedom. If $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal distribution function, then

$$V_f = \Phi^{-1}\left[H_p\left(T_f^2\right)\right], f = 1, 2, \ldots, \tag{16}$$

are i.i.d. $N(0,1)$ random variables. Similarly, for equation (14) where the parameters μ_0 and Σ are both unknown and are estimated,

$$V_f = \Phi^{-1}\left[F_{p,m-p}\left\{\frac{m(m-p)}{p(m+1)(m-1)} T_f^2\right\}\right], f = 1, 2, \ldots, \tag{17}$$

are also i.i.d. $N(0,1)$ variables, where $F_{p,m-p}(\cdot)$ represents the Snedecor F distribution function with $(p, m-p)$ degrees of freedom.

The monitoring of a multivariate process using an EWMA chart can now be done easily since the V_f statistics in equations (16) and (17) are all i.i.d. standard normal variables. The procedure is to monitor the V_f statistics for out-of-control signals since a shift in a multivariate mean vector from the target value, μ_0, will cause the V_f statistics to shift. In a multivariate process monitoring, the performance of the control charts such as the Hotelling or MEWMA charts is determined solely by the distance of the off-target mean vector from the on-target mean vector and not by the particular direction of the shift. Here, the distance of the shift is measured by the square-root of the noncentrality parameter given below:

$$\lambda^2 = (\mu - \mu_0)' \, \Sigma^{-1} \, (\mu - \mu_0) \tag{18}$$

where μ_0 and μ represent the on-target and off-target mean vectors respectively. Due to the directional invariance property of the T_f^2 statistics in equations (13) and (14), the new EWMA chart has only an upper control limit since we are actually monitoring the significance of the magnitude of the shift from μ_0 to μ.

5 The Simulation Study and Results

A simulation study is performed to evaluate the performance of the proposed approach with respect to the MEWMA and Hotelling's control chart based on equations (10) and (13) respectively. A comparison between the performances of these charts are made by means of their ARL profiles computed using computer programs written in SAS version 6.12. Here, it is assumed that the in-control process consists of i.i.d. bivariate observations from a $N_2(\mu_0, \Sigma)$ distribution where μ_0 is the null vector, i.e., $\mu_0 = (0,0)'$ and the covariance matrix is

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Here, ρ is the correlation coefficient between the two quality characteristics. The in-control ARL values (denoted as ARL_0) considered are 200 and 500. Here, ARL is defined as the average number of points that must be plotted on the control chart before an out-of-control signal is observed. Shifts in the process mean that are of the form $\mu = (\delta, 0)'$ for $p = 2$ are considered. Shifts of these forms are investigated for distances of $\lambda = 0.1, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 2, 3, 4$ and 5 from the target mean of $\mu_0 = 0$. The values of δ for $\mu = (\delta, 0)'$ are chosen to give the above distances. Note that shifts in the process mean of other forms such as $\mu = (\delta,\delta)', (0,\delta)', (-\delta,-\delta)', (0,-\delta)'$ or $(-\delta, 0)'$ can also be considered by selecting the appropriate values of to δ give the specified distances.

The UCLs of the Hotelling control chart based on equation (13) are selected to be 10.6 and 12.43 from the chi-square tables with 2 degrees of freedom so that the ARL_0 values are 200 and 500 respectively. Using computer simulation, the limits, i.e., h, of the various MEWMA schemes for $r = 0.05, 0.1, 0.2$ and 0.3 are determined to give these two ARL_0 values. For the EWMA control chart schemes, very small smoothing constants, i.e., β are selected so that shifts of small magnitude can be detected quickly. The values of β that are considered in the simulation study are $\beta = 0.00007, 0.0001, 0.0005, 0.001$ and 0.0015. The EWMA control chart factor, L, in equation (5) which controls the width of the upper control limit is then determined using computer simulation for each of the above values of β to give the desired ARL_0 values of 200 and 500.

Tables 1 and 2 give the ARL results for the various control chart schemes based on ARL_0 values of 200 and 500 respectively. Note that as shown in Tables 1 and 2, the choice of the value of a smoothing constant for the EWMA chart, β, can be made to give approximately similar ARL profiles to a particular MEWMA control chart scheme. It is clearly seen that for both the MEWMA and EWMA control chart schemes, smaller values of r and β will make the two control charts more sensitive to shifts of small magnitude while larger values of these two smoothing constants are desirable for the detection of shifts of big magnitude.

From the results in both the tables, it is obvious that the Hotelling control chart is the least sensitive for small to moderate shifts, i.e., for distances in the range $0.1 \leq \lambda \leq 2$. The EWMA chart with $\beta = 0.00007$ has the best overall performance for all sizes of shifts. However, as noted earlier, the sensitivity of the EWMA chart to small shifts can be further improved by choosing a value of β smaller than 0.00007. On the whole, the results of the EWMA control chart schemes are comparable to that of the MEWMA.

Table 1 ARL profiles for the various control chart schemes based on $ARL_0 = 200$ and $\boldsymbol{\mu} = (\delta, 0)'$.

λ	δ	Hotelling, T_f^2 ($UCL = 10.60$)	MEWMA				EWMA				
			$r = 0.05$, $h = 7.37$	$r = 0.1$, $h = 8.64$	$r = 0.2$, $h = 9.64$	$r = 0.3$, $h = 10.08$	β=0.00007, L=0.0058	β=0.0001, L=0.017	β=0.0005, L=0.065	β=0.001, L=0.133	β=0.0015, L=0.198
0	0	202.02	200.19	200.17	200.36	199.78	200.30	200.21	199.98	199.97	200.16
0.1	0.1	197.50	147.66	159.40	170.48	177.54	131.08	156.74	160.63	167.87	174.54
0.25	0.25	169.89	65.58	76.58	95.57	112.34	36.35	57.17	73.64	92.70	101.90
0.5	0.5	115.24	26.76	28.0	34.65	43.72	9.87	16.13	23.86	31.79	37.33
0.75	0.75	69.70	15.95	15.21	16.75	20.08	4.75	7.42	11.09	14.97	17.51
1	1	41.66	11.30	10.21	10.28	11.36	3.05	4.59	6.58	8.77	10.40
1.25	1.25	25.47	8.74	7.64	7.17	7.49	2.25	3.20	4.57	6.00	7.06
1.5	1.5	15.66	7.16	6.10	5.49	5.48	1.76	2.43	3.40	4.50	5.24
2	2	6.69	5.29	4.42	3.76	3.54	1.31	1.66	2.24	2.94	3.39
3	3	2.13	3.55	2.92	2.41	2.19	1.05	1.14	1.39	1.79	2.08
4	4	1.22	2.75	2.23	1.90	1.65	1.00	1.02	1.09	1.30	1.55
5	5	1.03	2.20	1.97	1.54	1.26	1.00	1.00	1.01	1.07	1.17

Table 2 ARL profiles for the various control chart schemes based on $ARL_0 = 500$ and $\boldsymbol{\mu} = (\delta, 0)'$.

λ	δ	Hotelling, T_f^2 ($UCL = 12.43$)	MEWMA				EWMA				
			$r = 0.05$, $h = 9.58$	$r = 0.1$, $h = 10.78$	$r = 0.2$, $h = 11.68$	$r = 0.3$, $h = 12.068$	β=0.00007, L=0.0258	β=0.0001, L=0.054	β=0.0005, L=0.179	β=0.001, L=0.3363	β=0.0015, L=0.471
0	0	492.92	500.25	500.36	500.08	500.40	500.87	500.99	500.69	500.80	499.79
0.1	0.1	483.18	315.08	365.45	411.24	443.93	297.65	403.34	402.32	428.11	436.96
0.25	0.25	416.68	104.98	139.58	200.53	248.17	91.81	145.67	181.47	220.31	240.04
0.5	0.5	266.53	34.93	39.45	56.62	79.26	25.36	40.22	56.08	72.48	83.11
0.75	0.75	151.80	19.43	19.22	23.30	30.87	11.69	18.41	26.27	34.15	39.04
1	1	84.59	13.39	12.24	12.98	15.63	6.90	10.91	15.44	20.04	22.85
1.25	1.25	48.04	10.25	8.91	8.64	9.53	4.75	7.34	10.33	13.46	15.23
1.5	1.5	28.40	8.31	7.06	6.44	6.65	3.54	5.44	7.61	9.82	11.14
2	2	10.91	6.08	4.98	4.28	4.07	2.34	3.55	4.91	6.28	7.06
3	3	2.79	4.02	3.26	2.65	2.41	1.44	2.17	2.90	3.67	4.11
4	4	1.36	3.10	2.46	2.05	1.81	1.11	1.63	2.16	2.67	3.00
5	5	1.06	2.52	2.06	1.73	1.41	1.01	1.23	1.85	2.13	2.35

6 Conclusions

Computer simulations are used to study the performances of the various EWMA control chart schemes in comparison with the MEWMA schemes and the Hotelling chart. The results show that the performances of the EWMA and MEWMA charts are comparable with one another and are both more sensitive than the Hotelling chart. Thus, the proposed extension is yet another significant contribution to the literature of EWMA control charts. The proposed approach of transforming the multivariate T_f^2 statistic into a univariate skalar which follows a standard normal distribution is deemed as an important contribution as it also enables the monitoring of multivariate observations to be performed using other univariate control charts such as the CUSUM or the moving average charts.

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