MATEMATIKA, MJIM, Volume 39, Number 1, 33–42

Besse Extended Cubic B-spline Collocation Method for Solving Benjamin-Bona-Mahony Equation

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Article history
Received: 4 November 2022
Received in revised form: 22 February 2023
Accepted: 19 March 2023
Published online: 15 April 2023

Abstract Extended cubic B-spline collocation method is formulated to solve the Benjamin-Bona-Mahony equation without linearization. The Besse relaxation scheme is applied on the nonlinear terms and therefore transforms the equation into a system of two linear equations. The time derivative is discretized using Forward Difference Approximation whereas the spatial dimension is approximated using extended cubic B-spline function. Applying the von-Neumann stability analysis, the proposed technique are shown unconditionally stable. Two numerical examples are presented and the results are compared with the exact solutions and recent methods.

Keywords Benjamin-Bona-Mahony equation; extended cubic B-spline; Besse relaxation scheme; nonlinear term; collocation method; discretization.

Mathematics Subject Classification 65N99

1 Introduction

The Benjamin-Bona-Mahony equation (BBME), also known as the regularized long-wave equation, of the form

$$u_t + u_x + \epsilon u u_x - \mu u_{xxt} = 0, \tag{1}$$

with initial condition

$$u(x,\ 0) = f(x), \quad a \leq x \leq b, \tag{2}$$

and the boundary conditions

$$u(a,\ t) = u(b,\ t) = 0, \quad 0 \leq t \leq T, \tag{3}$$

where t is time, x is the space, u is the wave amplitude, $f(x)$ is a localized disturbance inside the interval $[a,\ b]$, and T, ϵ, and μ are positive constants.

Peregrine [1-2] first derived the BBME for modelling the propagation of unidirectional weakly nonlinear and weakly dispersive water waves. Benjamin *et al.* [3] then introduced the BBME as a regularized version of the Korteweg-de Vries equation for shallow water waves. This equation models the nonlinear dispersive waves that arise in the surface waves of long wavelengths in liquids, magnetohydrodynamic waves in plasma, longitudinal dispersive waves in elastic rods, rotating flow down a tube, and pressure waves in liquid-gas bubble mixtures [3].

The BBME has been solved numerically by various methods based on finite difference, finite element, Petrov-Galerkin finite element, Fourier pseudo-spectral, collocation methods, and many other methods [4-5]. Lately, the B-spline collocation method with quadratic B-spline [4], cubic B-spline (CB) [6-8], cubic trigonometric B-spline (CTB) [9-11], and extended cubic B-spline (ECB) [12-16] have been formulated successfully on some differential equations with accurate results and high efficiency. There are several techniques used to handle the nonlinear term in the BBME namely Taylor series expansion [17], quasilinearization [18] and Adomian polynomials [19]. This article aims to solve the BBME using the Besse ECB collocation method without linearizing the algebraic system. This approach was chosen to minimize the cost of evaluating the nonlinear term.

BBME is one type of nonlinear partial differential equations. In order to solve this equation, the BBME is separated into two equations. The Besse relaxation scheme replaced the nonlinear term of the equation by its average value. Forward difference approximation is used to discretize the time derivative and ECB collocation method replace the spatial terms. The stability of the proposed scheme was analyzed using von-Neumann method. Lastly, the results were compared with the exact solution and other methods.

2 Extended Cubic B-spline

The ECB basis function is a piecewise polynomial function of degree 4 with C^2 continuity, as shown in Equation (4). This function is extended from the CB basis function. Figure 1 shows the plot of the basis function where x is variable and $\lambda \in \mathbb{R}$. ECB possesses the same properties as the CB basis function for $-8 \le \lambda \le 1$ [12].

$$E_{4,\,i}(x,\ \lambda) = \frac{1}{24}h^4 \begin{cases} 4h(1-\lambda)(x-x_i)^3 + 3\lambda(x-x_i)^4 & , x \in [x_i, x_{i+1}) \\ (4-\lambda)h^4 + 12h^3(x-x_{i+1}) + 6h^2(2+\lambda)(x-x_{i+1})^2 & \\ \qquad -12h(x-x_{i+1})^3 - 3\lambda(x-x_{i+1})^4 & , x \in [x_{i+1}, x_{i+2}) \\ (4-\lambda)h^4 + 12h^3(x_{i+3}-x) + 6h^2(2+\lambda)(x_{i+3}-x)^2 & \\ \qquad -12h(x_{i+3}-x)^3 - 3\lambda(x_{i+3}-x)^4 & , x \in [x_{i+2}, x_{i+3}) \\ 4h(1-\lambda)(x_{i+4}-x)^3 + 3\lambda(x_{i+4}-x)^4 & , x \in [x_{i+3}, x_{i+4}) \\ 0 & , \text{otherwise.} \end{cases} \tag{4}$$

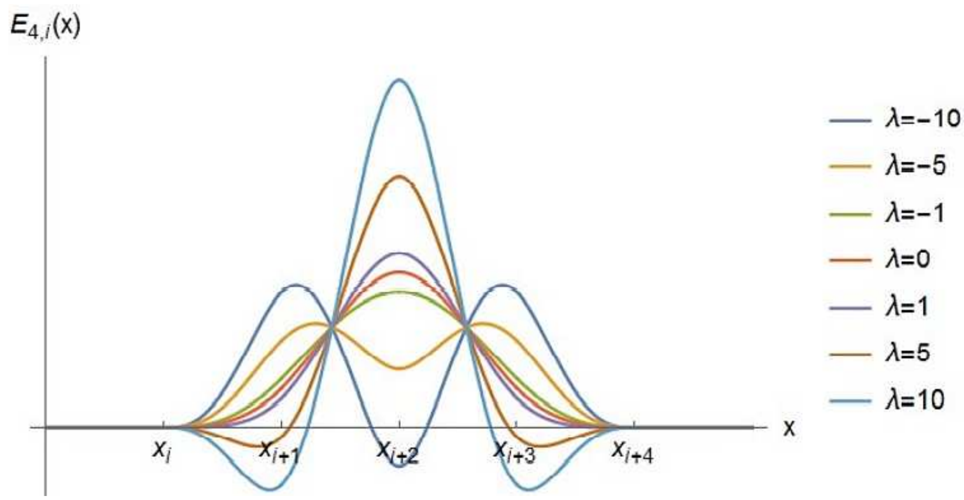

Figure 1: Extended Cubic B-spline Basis Function of Degree 4

The ECB function can be arbitrarily generated by taking a linear combination of the ECB basis,

$$U(x,\ t) = \sum_{i=-3}^{n-1} C_i(t)\, E_{4,i}(x), \quad x \in [x_0, x_n], \tag{5}$$

where $C_i(t)$ are time dependent unknowns. The ECB function, $U(x,\ t)$, and its derivatives with respect to x can be simplified as follows [12] for $i = 0,\ 1,\ 2,\ \ldots,\ n$:

$$\begin{aligned} U_i^j &= \left(\frac{4-\lambda}{24}\right)C_{i-3}^j + \left(\frac{8+\lambda}{12}\right)C_{i-2}^j + \left(\frac{4-\lambda}{24}\right)C_{i-1}^j, \\ (U_x)_i^j &= -\frac{1}{2h}C_{i-3}^j + \frac{1}{2h}C_{i-1}^j, \\ (U_{xx})_i^j &= \left(\frac{2+\lambda}{2h^2}\right)C_{i-3}^j - \frac{2}{h^2}C_{i-2}^j + \left(\frac{2+\lambda}{2h^2}\right)C_{i-1}^j. \end{aligned} \tag{6}$$

3 Besse Extended Cubic B-spline Collocation Method

Let (x_i, t_j) be the grid points discretizing region $\Delta = [a,b] \times [0,T]$ where $x_i = a + ih,\ i = 0,\ 1,\ \ldots,\ n,\ n \in \mathbb{Z}^+$, and $t_j = jk,\ j = 0,\ 1,\ \ldots,\ N,\ N \in \mathbb{Z}^+$. The terms h and k denote the space and time step size, respectively. The Besse relaxation scheme [20] is applied on the temporal dimension to avoid linearizing the nonlinear systems. Firstly, the BBME is replaced with a system of two equations [21]

$$V = U,$$
$$U_t + U_x + \epsilon V U_x = 0.$$

The time derivative term is discretized by the forward difference approach resulting in the following coupled system:

$$\begin{cases} \dfrac{V_i^{j+\frac{1}{2}} + V_i^{j-\frac{1}{2}}}{2} = U_i^j, \\ \dfrac{U_i^{j+1} - U_i^j}{k} - \mu \dfrac{(U_{xx})_i^{j+1} - (U_{xx})_i^j}{k} + \dfrac{(U_x)_i^{j+1} + (U_x)_i^j}{2} + \dfrac{(\epsilon V U_x)_i^{j+1} + (\epsilon V U_x)_i^j}{2} = 0 \end{cases} \tag{7}$$

where $t_{i+\frac{1}{2}} = t_j + \frac{k}{2}$ and $V_i^{j+\frac{1}{2}}$ is an approximation of V at the point $\left(x_i,\ t_{j+\frac{1}{2}}\right)$. The ECB function from Equation (5) is then presumed to be the solutions for the system in Equation (7). That is, the terms U_i^j in Equation (7) are substituted with Equation (5) and Equation (6) and simplified into

$$\begin{cases} V_i^{j+\frac{1}{2}} = \left(\dfrac{4-\lambda}{12}\right)\left(C_{i-3}^j + C_{i-1}^j\right) + \left(\dfrac{8+\lambda}{6}\right) C_{i-2}^j - \left(\dfrac{4-\lambda}{24}\right)\left(C_{i-3}^{j-\frac{1}{2}} + C_{i-1}^{j-\frac{1}{2}}\right) \\ \qquad\qquad - \left(\dfrac{8+\lambda}{12}\right) C_{i-2}^{j-\frac{1}{2}}, \\ P C_{i-3}^{j+1} + Q C_{i-2}^{j+1} + R C_{i-1}^{j+1} = R C_{i-3}^j + Q C_{i-2}^j + P C_{i-1}^j, \end{cases} \tag{8}$$

where

$$P = \frac{4-\lambda}{24} - \frac{\mu(2+\lambda)}{2h^2} - \frac{k}{4h}\left(1 + \epsilon V_i^{j+\frac{1}{2}}\right),\ Q = \frac{8+\lambda}{12} + \frac{\mu(2+\lambda)}{h^2} \text{ and } R = \frac{4-\lambda}{24} - \frac{\mu(2+\lambda)}{2h^2} + \frac{k}{4h}\left(1 + \epsilon V_i^{j+\frac{1}{2}}\right).$$

Equation (8) is a system of linear equations of order $(n+1)$ with $(n+3)$ unknowns. The two boundary conditions from Equation (3) are added into the system to make it square ensuring a unique solution. The system is solved recursively for $C_i^j,\ j = 0,\ 1,\ 2,\ \ldots,\ m-1$ and C_i^j are substituted back in Equation (5) as the approximate solution for the BBME.

4 Von-Neumann Stability Analysis

In this section, we present the stability of the Besse ECB approximation using the von-Neumann method. The growth of error in a single Fourier mode is considered as

$$C_i^j = \zeta^j e^{\iota \eta i h} \tag{9}$$

where ι is the complex number and η is the mode number. Equation (9) is substituted into Equation (8) to obtain

$$\begin{aligned} &\left[\frac{4-\lambda}{24} - \mu\left(\frac{2+\lambda}{2h^2}\right) - \frac{k}{4h}\left(1 + \epsilon V_i^{j+\frac{1}{2}}\right)\right] \zeta^{j+1} e^{\iota\eta(i-3)h} \\ &\quad + \left[\frac{8+\lambda}{12} + \mu\left(\frac{2+\lambda}{h^2}\right)\right] \zeta^{j+1} e^{\iota\eta(i-2)h} \\ &\quad + \left[\frac{4-\lambda}{24} - \mu\left(\frac{2+\lambda}{2h^2}\right) + \frac{k}{4h}\left(1 + \epsilon V_i^{j+\frac{1}{2}}\right)\right] \zeta^{j+1} e^{\iota\eta(i-1)h} \\ &= \left[\frac{4-\lambda}{24} - \mu\left(\frac{2+\lambda}{2h^2}\right) + \frac{k}{4h}\left(1 + \epsilon V_i^{j+\frac{1}{2}}\right)\right] \zeta^j e^{\iota\eta(i-3)h} \\ &\quad + \left[\frac{8+\lambda}{12} + \mu\left(\frac{2+\lambda}{h^2}\right)\right] \zeta^j e^{\iota\eta(i-2)h} \\ &\quad + \left[\frac{4-\lambda}{24} - \mu\left(\frac{2+\lambda}{2h^2}\right) - \frac{k}{4h}\left(1 + \epsilon V_i^{j+\frac{1}{2}}\right)\right] \zeta^j e^{\iota\eta(i-1)h} \end{aligned} \tag{10}$$

Dividing both sides of Equation (10) by $\zeta^j e^{\iota\eta(i-2)h}$ and simplifying the terms lead to

$$\begin{aligned}\left[\frac{4-\lambda}{24}-\mu\left(\frac{2+\lambda}{2h^2}\right)\right]\left[e^{-\iota\eta h}+e^{\iota\eta h}\right]\zeta+\frac{k}{4h}\left(1+\epsilon V_i^{j+\frac{1}{2}}\right)\left[-e^{-\iota\eta h}+e^{\iota\eta h}\right]\zeta\\+\left[\frac{8+\lambda}{12}+\mu\left(\frac{2+\lambda}{h^2}\right)\right]\zeta=\left[\frac{4-\lambda}{24}-\mu\left(\frac{2+\lambda}{2h^2}\right)\right]\left[e^{-\iota\eta h}+e^{\iota\eta h}\right]\\+\frac{k}{4h}\left(1+\epsilon V_i^{j+\frac{1}{2}}\right)\left[-e^{-\iota\eta h}+e^{\iota\eta h}\right]+\left[\frac{8+\lambda}{12}+\mu\left(\frac{2+\lambda}{h^2}\right)\right]\end{aligned}\tag{11}$$

By using Euler's equation, Equation (11) can be written as

$$\zeta=\frac{P-\iota Q}{P+\iota Q}\tag{12}$$

where

$$P=\left[\frac{8+\lambda}{12}+\mu\left(\frac{2+\lambda}{h^2}\right)\right]\cos(\eta h)\ +\frac{8+\lambda}{12}+\mu\left(\frac{2+\lambda}{h^2}\right)\ \text{and}\ Q=\frac{k}{4h}\left(1+\epsilon V_i^{j+\frac{1}{2}}\right)\sin(\eta h).$$

It is stable if and only if $|\zeta|\leq 1$. Since $|\zeta|=\left|\frac{P-\iota Q}{P+\iota Q}\right|=1$, the scheme is unconditionally stable.

5 Initial State

In order to kick-start the calculation, the unknowns at $j=0$, $\boldsymbol{C}^0$, needs to be predetermined. Hence, the initial condition from Equation (2) is treated similarly as the previous section where the term u is substituted by the ECB function in Equation (5):

$$U_i^0=\left(\frac{4-\lambda}{24}\right)C_{-3}^0+\left(\frac{8+\lambda}{12}\right)C_{-2}^0+\left(\frac{4-\lambda}{24}\right)C_{-1}^0=f(x_i).$$

This generates a system of linear equations of order $(n+1)\times(n+3)$,

$$\boldsymbol{P}\boldsymbol{C}^0=\boldsymbol{Q},$$

where

$$\boldsymbol{P}=\begin{bmatrix}\frac{4-\lambda}{24}&\frac{8+\lambda}{12}&\frac{4-\lambda}{24}&0&\cdots&0\\0&\frac{4-\lambda}{24}&\frac{8+\lambda}{12}&\frac{4-\lambda}{24}&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\ddots&\ddots\\0&\cdots&0&\frac{4-\lambda}{24}&\frac{8+\lambda}{12}&\frac{4-\lambda}{24}\end{bmatrix},\ \boldsymbol{C}^0=\begin{bmatrix}C_{-3}^0\\C_{-2}^0\\\vdots\\C_{n-1}^0\end{bmatrix},\boldsymbol{Q}=\begin{bmatrix}f(x_0)\\f(x_1)\\\vdots\\f(x_n)\end{bmatrix}.$$

This system is solved for $\boldsymbol{C}^0$ using least square method.

6 Numerical Experiments

Two problems were solved by the Besse ECB collocation method. The accuracy of the method would be measured by using the absolute error, infinite-norm, and two-norm, whose formulas are detailed as follows, respectively:

$$\text{Absolute Error}=|U_i-u_i|,$$

$$L_\infty=\|U_i-u_i\|_\infty=max_{i=1}^{n-1}\ |U_i-u_i|,$$

$$L_2=\|U_i-u_i\|_2=\sqrt{h\sum_{i=1}^{n-1}|U_i-u_i|^2}.$$

The terms u and U denote the exact and approximate solutions, respectively. The numerical experiments were carried out using *Mathematica* 11.3. The analytical solution of the single solitary wave BBME is given by

$$u(x,t)=3c\text{sech}^2\left[\sqrt{\frac{\epsilon c}{4\mu(1+\epsilon c)}}(x-(1+\epsilon c)t-x_o)\right],$$

representing a single solitary wave of amplitude $3c$ where $(1+\epsilon c)$ is the wave velocity and $\sqrt{\dfrac{\epsilon c}{4\mu(1+\epsilon c)}}$ is the width of the wave pulse. The following initial condition was used on both problems:

$$U(x,0) = 3c\,\mathrm{sech}^2\left[\sqrt{\frac{\epsilon c}{4\mu(1+\epsilon c)}}\,(x-x_o)\right].$$

6.1 Problem 1

Equation (1) was solved by the ECB collocation method using the following values: $x \in [-40,\ 60]$, $t \in [0,\ 20]$, $\epsilon = 2$, $\mu = 1$, $x_0 = 0$, $c = 0.1$, $\lambda = -0.003879$, $h = 0.125$, and $k = 0.1$ [21]. The exact solution is displayed in Figure 2 and the approximate solutions at $t = 20$ are plotted together with the respective exact solution in Figure 3. Table 1 compares the analytical and approximate solutions, while Table 2 shows the comparison of norms between the results generated by the proposed Besse ECB collocation method with those of CB, CTB, and Besse CB collocation methods [9, 21] at $t = 20$. From the tables, the approximate solution agrees with the exact solution with reasonable errors and Besse ECB collocation method generated more accurate results than the other methods. Table 3 displays the comparison of norms error with different space and time step sizes at $t = 20$. When $k = 0.125$ the accuracy of the errors increases as value of h decreases. But the accuracy of the errors decreases as value of h and k decreases. We can conclude that, the accuracy will be affected when we use too small value of k.

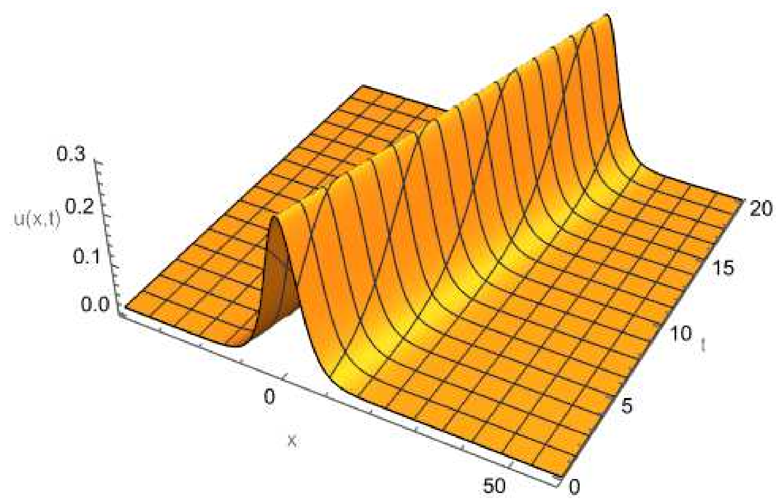

Figure 2: The Exact Solution for Problem 1

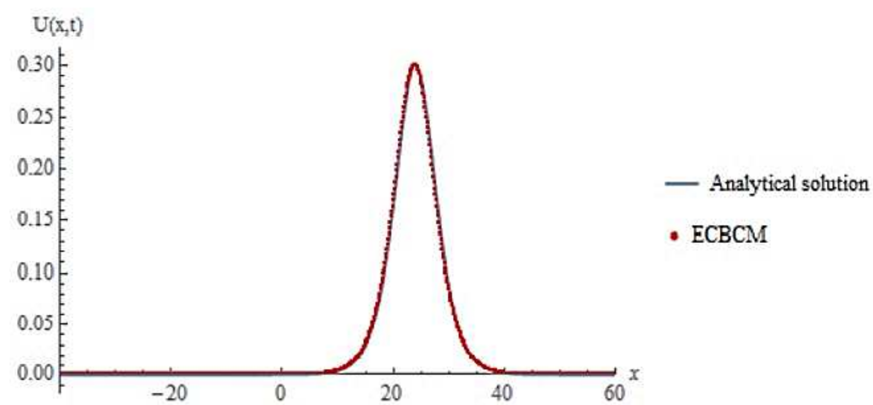

Figure 3: The Exact and Approximate Solutions for Problem 1 at $t = 20$ with $h = 0.125$, $k = 0.1$, and $\lambda = -0.003879$

Table 1: Comparison between the Exact and Approximate Solutions for Problem 1 at $t = 20$ with $h = 0.125,\ k = 0.1$, and $\lambda = -0.003879$

x	$U(x,t)$	$u(x,t)$	Absolute Error
−30	3.198765×10^{-10}	1.108007×10^{-7}	1.104808×10^{-7}
−20	1.896628×10^{-8}	4.075516×10^{-8}	2.178888×10^{-8}
−10	1.124556×10^{-6}	1.163362×10^{-6}	3.880578×10^{-8}
0	6.667046×10^{-5}	6.381200×10^{-5}	2.858461×10^{-6}
10	3.927576×10^{-3}	3.944548×10^{-3}	1.697250×10^{-5}
20	1.640573×10^{-1}	1.640329×10^{-1}	2.435410×10^{-5}
30	8.779135×10^{-2}	8.776815×10^{-2}	2.319719×10^{-5}
40	1.742279×10^{-3}	1.739641×10^{-3}	2.637974×10^{-6}
50	2.946867×10^{-5}	2.940728×10^{-5}	6.138126×10^{-8}
		L_∞-norm	2.435746×10^{-5}
		L_2-norm	8.830497×10^{-5}

Table 2: Comparison of Norms of Results Generated by Besse ECB, CB [9], CTB [9] and Besse CB [21] Collocation Methods for Problem 1 at $t = 20$ with $h = 0.125,\ k = 0.1$, and $\lambda = -0.003879$

Collocation Method	L_∞-norm	L_2-norm
Besse ECB	2.435746×10^{-5}	8.830497×10^{-5}
Besse CBS [21]	2.632789×10^{-4}	6.537858×10^{-4}
CB [9]	2.567974×10^{-4}	6.202924×10^{-4}
CTB [9]	6.529259×10^{-4}	1.890132×10^{-3}

Table 3: Comparison of Norms Error with Different Space and Time Step Sizes for Problem 1 at $t = 20$ and $\lambda = -0.003879$

k	h	L_∞-norm	L_2-norm
0.125	0.125	9.754177×10^{-5}	2.810605×10^{-4}
	0.1	9.118558×10^{-5}	2.509043×10^{-4}
	0.05	8.509371×10^{-5}	2.171473×10^{-4}
0.1	0.125	2.435746×10^{-5}	8.830497×10^{-5}
	0.1	4.224126×10^{-5}	1.194515×10^{-4}
	0.05	6.608743×10^{-5}	1.687267×10^{-4}
0.05	0.125	1.844198×10^{-4}	4.583457×10^{-4}
	0.1	2.023971×10^{-4}	4.966866×10^{-4}
	0.05	2.262652×10^{-4}	5.487074×10^{-4}

6.2 Problem 2

Equation (1) was solved by the ECB collocation method using the following values: $x \in [-40, 60]$, $t \in [0, 20]$, $\epsilon = 1$, $\mu = 1$, $x_0 = 0$, $c = 0.1$, $\lambda = -0.002624$, $h = 0.125$ and $k = 0.1$ [21]. The exact solution is displayed in Figure 4 and the approximate solutions at $t = 20$ are plotted together with the respective exact solution in Figure 5. Table 4 compares the analytical and approximate solutions, while Table 5 shows the comparison of norms between the results generated by the proposed Besse ECB collocation method with those of CB, CTB, and Besse CB collocation methods [9, 21] at $t = 20$. From the tables, the approximate solution agrees with the exact solution with reasonable errors and Besse CBS collocation method generated more accurate results than the other methods. Similar to Problem 1, Table 6 shows the comparison of norms error with different space and time step sizes at $t = 20$. Using the same argument, the accuracy will be affected when we use too small value of k.

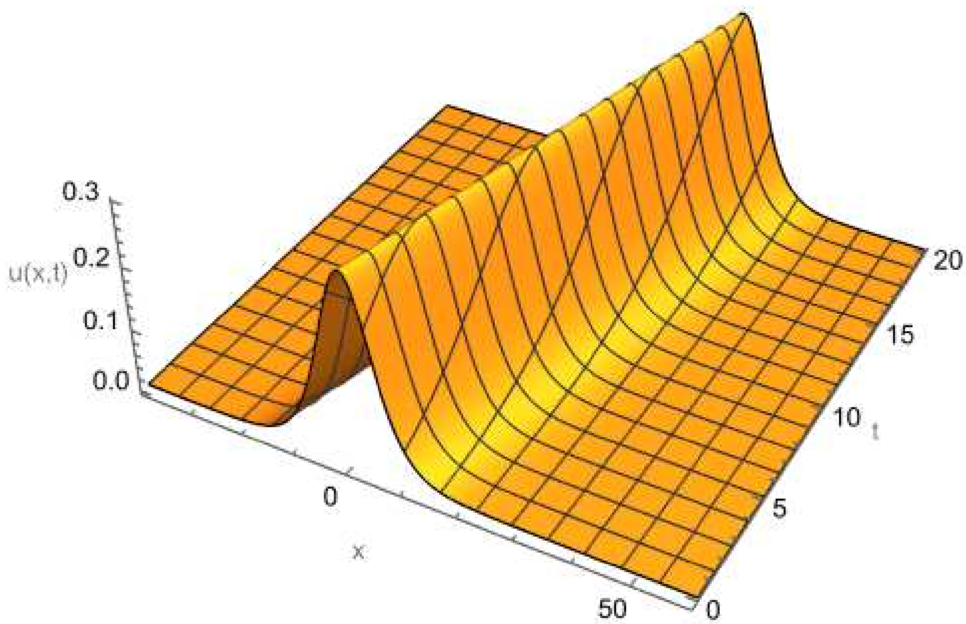

Figure 4: The Exact Solution for Problem 2

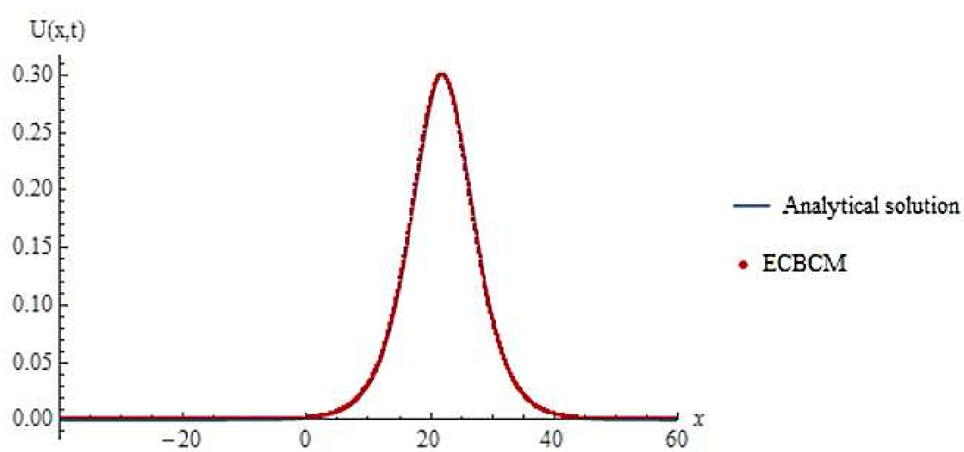

Figure 5: The Exact and Approximate Solutions for Problem 2 at $t = 20$ with $h = 0.125$, $k = 0.1$, and $\lambda = -0.002624$

Table 4: Comparison between Exact and Approximate Solutions for at $t = 20$ with $h = 0.125$, $k = 0.1$, and $\lambda = -0.002624$

x	$U(x,t)$	$u(x,t)$	Absolute Error
−30	1.862327×10^{-7}	7.625434×10^{-6}	7.439201×10^{-6}
−20	3.797524×10^{-6}	5.102140×10^{-6}	1.304616×10^{-6}
−10	7.742735×10^{-5}	7.748605×10^{-5}	5.870608×10^{-8}
0	1.574909×10^{-3}	1.574547×10^{-3}	3.613120×10^{-7}
10	3.053834×10^{-2}	3.053744×10^{-2}	8.967256×10^{-7}
20	2.742989×10^{-1}	2.742965×10^{-1}	2.376183×10^{-6}
30	9.058729×10^{-2}	9.058379×10^{-2}	3.500554×10^{-6}
40	5.228378×10^{-3}	5.227063×10^{-3}	1.315754×10^{-6}
50	2.585485×10^{-4}	2.584637×10^{-4}	8.480421×10^{-8}
		L_∞-norm	1.266780×10^{-5}
		L_2-norm	3.313883×10^{-5}

Table 5: Comparison of Norms of Results Generated by Besse ECB, CB [9], CTB [9], and Besse CB [21] Collocation Methods for Problem 2 at $t = 20$ with $h = 0.125$, $k = 0.1$, and $\lambda = -0.002624$

Collocation Method	L_∞-norm	L_2-norm
Besse ECB	1.266780×10^{-5}	3.313883×10^{-5}
Besse CBS [21]	9.610112×10^{-5}	2.521466×10^{-4}
CB [9]	9.607819×10^{-5}	2.472435×10^{-4}
CTB [9]	4.546975×10^{-4}	1.578019×10^{-3}

Table 6: Comparison of Norms Error with Different Space and Time Step Sizes for Problem 2 at $t = 20$ and $\lambda = -0.002624$

k	h	L_∞-norm	L_2**-norm**
0.125	0.125	4.373310×10^{-5}	1.199783×10^{-4}
	0.1	3.728655×10^{-5}	1.068436×10^{-4}
	0.05	2.973689×10^{-5}	9.134967×10^{-5}
0.1	0.125	1.266780×10^{-5}	3.313883×10^{-5}
	0.1	1.268449×10^{-5}	3.794381×10^{-5}
	0.05	1.524244×10^{-5}	4.472656×10^{-5}
0.05	0.125	5.876552×10^{-5}	1.617571×10^{-4}
	0.1	6.523481×10^{-5}	1.752378×10^{-4}
	0.05	7.384749×10^{-5}	1.940618×10^{-4}

7 Conclusion

The solitary wave solution of the BBME has been numerically approximated for two problems using the proposed Besse ECB collocation method. For both problems, the approximate solution agrees with the exact solution with reasonable errors and Besse ECB collocation method generated more accurate results than CB [9], CTB [9], and Besse CB [21] collocation methods. This approach improves the accuracy of Besse CB collocation method by incorporating ECB in place of CB. The proposed scheme was proved to be unconditionally stable. Since the errors do not overgrow for both problems, we conclude that Besse ECB collocation method can be used to solve the BBME. However, the error trends were affected when the time steps are too small. More investigation could be carried out to clarify this issue.

Acknowledgments

We acknowledge the financial support from the School of Mathematical Sciences, Universiti Sains Malaysia.

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