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Computationally Efficient Laplace Transform with Modified Variational Iteration Method for Solving Fourth-Order Fractional Integro-Differential Equations

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Abstract In this paper, linear and nonlinear fourth-order Fractional Integro Differential Equations (FIDEs) with boundary value problems are solved by Laplace Transform with Modified Variational Iteration Method (LT-MVIM). A new technique based on the VIM is introduced to remove the random choice of initial guess by setting a specific rule depends on unknown parameters. These parameters contributed to the increase in the number of terms of the polynomial approximation and its degree, which, in turn, accelerates the convergence and increases the accuracy from one iteration compared to the standard method, where the initial approximation is still randomly chosen. Moreover, the standard method requires an infinite number of iterations, which need massive calculations in each iteration. Some examples are given in order to show the accuracy of the solutions obtained by the proposed method. Furthermore, comparisons are made between the solutions obtained by the proposed method and Laplace Transform Variational Iteration Method (LT-VIM) based on the exact solutions, revealing that the LT-MVIM contributes to accelerating the convergence of approximate solution to the exact solution by reducing the computational work to obtain the approximate solution using one iteration. Whereas, LT-VIM needs more iterations to obtain a suitable approximate solution, which results in an increase in the computational workload.

Keywords Variational iteration method; Laplace transform; nonlinear boundary value problems; Fourth-order fractional integro differential equations (FIDEs).

Mathematics Subject Classification 46N60, 92B99.

1 Introduction

In recent years, several studies have focused on fractional differential equations (FDEs) due to their versatile applications in many fields. One basic example is the isochronous problem, where fractional calculus has been used, and it shows its usefulness in solving some types of integral equations [1]. Furthermore, the hydrogeology application known as the fractional advection-dispersion equation which shows the fundamental connection between fractional derivatives and the stable distributions [2]. Moreover, most of FDEs have no exact solutions because they are difficult to be modelled and what is more to be solved. Therefore, analytical and numerical methods are used to obtain approximate solutions to these equations, such as Homotopy Perturbation Method (HPM) [3], Variational Iteration Method (VIM) [4], and Adomian decomposition method (ADM) [5] are used to obtain approximate solutions to these equations.

FIDEs are one of the important classes of FDEs. Physical and chemical processes such as elasticity, electric drives, circuits systems and heat transfer are modeled by these classes [6, 7]. Analytical methods were used to obtain approximate solutions for this type of equation such as HPM [8], VIM [9], and homotopy analysis method (HAM) [10]. Among the above, VIM is chosen for this study because of its versatility in solving fractional problems.

Different from derivative of integer order, there are various definitions associated with the fractional derivatives. These definitions are commonly not equivalent to each other. The two most applied are Riemann-Liouville and Caputo derivative. One of the main advantages of Caputo fractional derivative is that it allows integer order initial and boundary conditions to be included in the formulation of the problems, which have a clear physical interpretation.

In this paper, we will find the approximate solution of fourth-order FIDEs as following:

$$D_*^{\alpha}u(t) = f(t) + \gamma u(t) + \int_0^t [g(x)u(x) + h(x)F(u(x))]dx, 0 \leq t \leq b, 3 < \alpha \leq 4, \tag{1}$$

with boundary conditions:

$$u(0) = \gamma_0, \qquad u''(0) = \gamma_2, \tag{2}$$

$$u(b) = \beta_0, \qquad u''(b) = \beta_2, \tag{3}$$

where $D_*^{\alpha}u(t)$ is fractional derivative in the caputo sense, whereby γ_0, γ_2, β_0, β_2 and γ are real constant, and $F(u(x))$ is any nonlinear continuous function, and g and h can be determined by Taylor polynomials. Many mathematical of real-life problems usually formulation of physical phenomena involving this equations such as diffusion processes, rheology and damping laws. Furthermore, The activity of interacting inhibitory and excitatory neurons can be described by a system of integro-differential equations.

The VIM was first introduced by He in 1998 [11], and was further developed in 1999 [12] and 2007 [13]. Many authors were also interested in the VIM in order to deal with linear and non-linear equations generated in solving engineering and science problems [14]. The free choice of initial approximation is one of the advantages of VIM [15]. Furthermore, the iterations are convergent to the exact solution very rapidly [16]. In addition, it can be applied on nonlinear terms directly without any restrictive assumption. The main points of VIM are the identification of the Lagrange multiplier, the construction of correct function and selection of the initial approximation. Recently, the VIM was extended to the FDE, but the outcome was not very successful because of the difficulty to identify Lagrange Multiplier, which has later been identified as one of the drawbacks of applying this method. Therefore, in order to avoid this disadvantage, the authors use Laplace transform to identify the Lagrange Multiplier. Another disadvantage of the VIM is that it repeats computations, which consumes time and effort that most authors clearly want to avoid [17]. For this reason, with the objective to improvise the existing method, the Laplace transform (LT) is combined with the VIM to solve boundary value problems for fourth-order fractional integro differential equations so that the identification of the Lagrange multiplier can be easily selected. Then the LT-VIM is modified into LT-MVIM by setting a specific rule for choosing the initial approximation, which removes the random selection of the initial approximation and reduces the size of computational work to obtain the approximate solution using one iteration. Furthermore, LT-MVIM greatly accelerates the convergence of the solution.

2 Basic Definitions

In this section, important definitions are given that help in this paper.

Definition 1 *For $Re(r) > 0$, the gamma function is:*

$$\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du. \tag{4}$$

Definition 2 *For $\alpha > 0$, the Caputo fractional derivatives can be written as:*

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{5}$$

for $n-1 < \alpha \leq n, n \in N, x > 0, f(x) \in \mathcal{C}_{-1}^n$.

Definition 3 *The Laplace transform for fractional derivatives defined by the Caputo derivative can be written as:*

$$\mathscr{L}[D_*^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), m-1 < \alpha \leq m, \tag{6}$$

where $\mathscr{L}[*]$ is Laplace transform

Definition 4 *If $\mathscr{L}\{G_1(t)\} = g_1(v)$ and $\mathscr{L}\{G_2(t)\} = g_2(v)$ then:*

$$\begin{aligned} \mathscr{L}\{G_1(t) * G_2(t)\} &= \mathscr{L}\{G_1(t)\} * \mathscr{L}\{G_2(t)\}, \\ &= g_1(v) * g_2(v), \end{aligned} \tag{7}$$

where $G_1(t) * G_2(t)$ is called the convolution of $G_1(t)$ and $G_2(t)$ and it is defined as [18]:

$$G_1(t) * G_2(t) = \int_0^t G_1(t-x) G_2(t) dx = \int_0^t G_1(t) G_2(t-x) dx. \tag{8}$$

3 Laplace Transform Variational Iteration Method (LT-VIM)

The basic principle of VIM to solve Equation (1) is to construct the following correction function:

$$\begin{aligned} u_{n+1} &= u_n + \int_0^t \lambda(\xi)(D_*^\alpha u(\xi) - f(\xi) - \gamma\tilde{u}(\xi) - \int_0^\xi [g(x)\tilde{u}(x) + h(x)F(\tilde{u}(x))]dx)d\xi, \\ & n = 1, 2, 3..., \end{aligned} \tag{9}$$

where λ is a general Lagrange multiplier. Applying the Laplace transform to both sides of Equation (9), we get:

$$\begin{aligned} \mathscr{L}[u_{n+1}] &= \mathscr{L}[u_n] + \mathscr{L}[\int_0^t \lambda(t-\xi)(D_*^\alpha u(\xi) - f(\xi) - \gamma\tilde{u}(\xi) \\ &- \int_0^\xi [g(x)\tilde{u}(x) + h(x)F(\tilde{u}(x))]dx)d\xi], \end{aligned} \tag{10}$$

The convolution method was applied to Equation (10) instead of, we obtain:

$$\begin{aligned} \mathscr{L}[u_{n+1}] &= \mathscr{L}[u_n] + \mathscr{L}[\lambda(t) * (D_*^\alpha u(t) - f(t) - \gamma\tilde{u}(t) - \int_0^t [g(x)\tilde{u}(x) + h(x)F(\tilde{u}(x))]dx)], \\ &= \mathscr{L}[u_n] + \mathscr{L}[\lambda(t)][\mathscr{L}(D_*^\alpha u(t) - f(t) - \gamma\tilde{u}(t) \\ &- \int_0^t [g(x)\tilde{u}(x) + h(x)F(\tilde{u}(x))]dx]. \end{aligned} \tag{11}$$

Take the variation with respect $u_n(t)$ to get the optimal value of $\lambda(\xi)$ as:

$$\begin{aligned} \frac{\delta}{\delta u_n}\mathscr{L}[u_{n+1}] &= \frac{\delta}{\delta u_n}\mathscr{L}[u_n] + \frac{\delta}{\delta u_n}\mathscr{L}[\lambda(t)][(s^\alpha \mathscr{L}[u_n] - \sum_{k=0}^{3} s^{\alpha-k-1}u^{(k)}(0) + \mathscr{L}(-f(t) - \gamma\tilde{u}(t) \\ &- \int_0^t [g(x)\tilde{u}(x) + h(x)F(\tilde{u}(x))]dx)), \end{aligned} \tag{12}$$

where $\mathscr{L}[D_*^\alpha u(t)] = s^\alpha \mathscr{L}[u_n] - \sum_{k=0}^{3} s^{\alpha-k-1}u^{(k)}(0), 3 < \alpha \le 4$.
Hence upon applying the variation, this simplifies to

$$\mathscr{L}[\delta u_{n+1}] = \mathscr{L}[\delta u_n] + s^\alpha \mathscr{L}[\lambda(t)][\mathscr{L}(\delta u_n)]. \tag{13}$$

The right hand side of Equation (13) should be set to zero because the extremum condition of u_{n+1} requires that $\delta u_{n+1} = 0$. Then we obtain:

$$\mathscr{L}[\lambda(t)] = \frac{-1}{s^\alpha}. \tag{14}$$

Substituting (14) into function (11), we obtain:

$$\mathscr{L}[u_{n+1}] = \mathscr{L}[u_n] + \frac{-1}{s^\alpha}[\mathscr{L}(D_*^\alpha u(t) - f(t) - \gamma\tilde{u}(t) - \int_0^t [g(x)\tilde{u}(x) + h(x)F(\tilde{u}(x))]dx)]. \tag{15}$$

Taking the inverse Laplace of Equation (15) we get:

$$u_{n+1} = u_n + \mathscr{L}^{-1}[\frac{-1}{s^\alpha}[\mathscr{L}(D_*^\alpha u(t) - f(t) - \gamma u(t) - \int_0^t [g(x)u(x) + h(x)F(u(x))]dx)]], \quad (16)$$

$$= \gamma_0 + \gamma_1 t + \frac{\gamma_2 t^2}{2} + \frac{\gamma_3 t^3}{6} + \mathscr{L}^{-1}[\frac{-1}{s^\alpha}\mathscr{L}(-f(t) - \gamma u_n(t) - \int_0^t [g(x)u_n(x) + h(x)F(u_n(x))]dx). \quad (17)$$

The initial approximation can be chosen which satisfies the initial conditions (2) as:

$$u_0 = \gamma_0 + \gamma_1 t + \frac{\gamma_2 t^2}{2} + \frac{\gamma_3 t^3}{6}, \quad (18)$$

where $\gamma_1 = u'(0)$ and $\gamma_3 = u'''(0)$.

The first order approximation can be presented as:

$$u_1 = \gamma_0 + \gamma_1 t + \frac{\gamma_2 t^2}{2} + \frac{\gamma_3 t^3}{6} + \mathscr{L}^{-1}[\frac{-1}{s^\alpha}\mathscr{L}(-f(t) - \gamma u_0(t) - \int_0^t [g(x)u_0(x) + h(x)F(u_0(x))]dx). \quad (19)$$

4 Laplace Transform with Modified Variational Iteration Method (LT-MVIM)

The initial approximation plays an important role in approximate analytical methods. The researchers are still actively trying to find an appropriate initial approximation in increasing the accuracy of approximate solution. The choice of initial approximation in LT-VIM depends on the initial conditions, however, there is a drawback when the initial condition is equal to zero. In the current study, a rule is proposed to adjust the selection of the initial approximation that satisfies the boundary conditions to obtain high accuracy approximate solutions. This method starts with selecting the initial approximation in a power series form as:

$$z(t) = \sum_{i=0}^{n} A_i t^i, \quad (20)$$

where A_i are unknown parameters to be determined by system of algebraic equations. In this study, only linear trial function is used , (i.e. for $n = 1$). As a result Equation (20) becomes:

$$z(t) = A_0 + A_1 t. \quad (21)$$

To illustrate the main idea of LT-MVIM, the researchers depend on the same LT-VIM algorithm, but unlike of LT-VIM, the LT-MVIM method exploits the freedom of LT-VIM, by replacing initial approximation for an arbitrary linear trial function $z(t)$ by substituting the Equation (21) into Equation (16) to obtain the first-order approximation as:

$$u_1 = \gamma_0 + \gamma_1 t + \frac{\gamma_2 t^2}{2} + \frac{\gamma_3 t^3}{6} + \mathscr{L}^{-1}[\frac{-1}{s^\alpha}\mathscr{L}(-f(t) - \gamma(A_0 + A_1 t) - \int_0^t [g(x)(A_0 + A_1 x) + h(x)F(A_0 + A_1 x)]dx). \quad (22)$$

The approximate solution (22) can be expressed as:

$$u_1 = u(t, \gamma_1, \gamma_3, A_0, A_1). \tag{23}$$

To determine the values of γ_1, γ_3, A_0 and A_1, an algebraic system will be formed: to find the first and the second equation, Equation (22) is required to satisfy the boundary conditions (3). In finding the total parameter values, we require the addition of two algebraic equations generated by using boundary condition. We should combine the following two equations $u(t_1, \gamma_1, \gamma_3, A_0, A_1) = u(t_2, \gamma_1, \gamma_3, A_0, A_1) = 0$, where the residual is defined, by substituting Equation (22) into Equation (1), to obtain

$$\begin{aligned} D_*^\alpha u(t, \gamma_1, \gamma_3, A_0, A_1) &= f(t) + \gamma u(t, \gamma_1, \gamma_3, A_0, A_1) + \int_0^t [g(x)u(t, \gamma_1, \gamma_3, A_0, A_1) \\ &+ h(x)F(u(t, \gamma_1, \gamma_3, A_0, A_1))]dx, \end{aligned} \tag{24}$$

where $0 \leq t_1, t_2 \leq b$.

5 Numerical Examples

In this section, a linear and nonlinear examples of fourth order FIDEs will be solved by the standard LT-VIM and the MLT-VIM. The results will be compared with the exact solution to show the efficiency of the proposed method.

Example 1 Let the fourth order FIDEs as following [20]:

$$D_*^\alpha u(t) = t(1 + e^t) + 3e^t + u(t) - \int_0^t u(x)dx, \tag{25}$$

with the boundary conditions:

$$u(0) = 1, \quad u(1) = 1 + e, \quad u''(0) = 2, \quad u''(1) = 3e \tag{26}$$

When $\alpha = 4$, the exact solution is $u(t) = 1 + te^t$.

LT-VIM

According to LT-VIM, the iteration formula (16) for Equation (25) can be expressed as:

$$u_{n+1} = u_n + \mathscr{L}^{-1}[\frac{-1}{s^\alpha}[\mathscr{L}(D_*^\alpha u(t) = t(1 + e^t) + 3e^t + u(t) - \int_0^t u(x)dx)]]. \tag{27}$$

Let $e^t \sim 1 + t + \frac{t^2}{2}$ to avoid complexity in integration, and suppose that the initial approximate solution is in the form $u_0 = u(0) + u'(0)t + u''(0)t^2 + u'''(0)t^3$, let $u'(0) = \beta_0$ and $u'''(0) = \beta_1$, also applying $u(0) = 1$ and $u''(0) = 2$, this in turn gives the approximate solution for $\alpha = 4$:

$$\begin{aligned} u(t) &= 1 + \beta_0 t + t^2 + 1/6\,\beta_1 t^3 - \frac{1}{10080} t^4 (\beta_1 t^4 - 2\,\beta_1 t^3 + 6\,t^3 + 28\,\beta_0 t^2 - 98\,t^2 \\ &- 84\,\beta_0 t - 336\,t - 1680). \end{aligned} \tag{28}$$

Using $u(1) = 1 + e$ and $u''(1) = 3e$ into Equation (28) to obtain the values of β_0 and β_1 as:

$$\beta_0 = 0.9815547028, \beta_1 = 3.131018652. \tag{29}$$

Substituting Equation (29) into Equation (28), the approximate solution using one iteration is:

$$\begin{aligned} u(t) &= 1 + 0.9815547028\,t + t^2 + 0.5218364420\,t^3 - \frac{1}{10080}t^4(3.131018652\,t^4 \\ &- 0.262037304\,t^3 - 70.51646832\,t^2 - 418.4505950\,t - 1680). \end{aligned} \tag{30}$$

LT-MVIM

Applying the LT-MVIM to obtain the an approximate solution for Equation (25), let $z[t] = A_0 + A_1 t$ as the initial approximation, where we define $\beta_0 = u'(0)$ and $\beta_1 = u'''(0)$, also applying $u(0) = 1$ and $u''(0) = 2$, this in turn gives the approximate solution for $\alpha = 4$:

$$\begin{aligned} u(t) &= 1 + \beta_0 t + t^2 + 1/6\,\beta_1 t^3 + t^\alpha(9\,\frac{t^3}{\Gamma(4+\alpha)} + 8\,\frac{t^4}{\Gamma(5+\alpha)} + \frac{A_0\,(2+\alpha)\,(1+\alpha)}{\Gamma(3+\alpha)} \\ &+ \frac{(-tA_1 - A_0\,(2+\alpha) + A_1\,(2+\alpha) + 5\,t + 5\,\alpha + 10)\,t}{\Gamma(3+\alpha)} + 3\,\frac{(1+\alpha)\,(2+\alpha)}{\Gamma(3+\alpha)}). \end{aligned} \tag{31}$$

To determine the values of β_0, β_1, A_0, and A_1, let $t = 0.45$ and $t = 0.48$ for residual error cancellation, and when we satisfy the boundary conditions $u(1) = 1+e$ and $u''(1) = 3e$. Table 1 illustrates the values of β_0, β_1, A_0, and A_1, for different values of α, which are randomly selected between 3 and 4.

Table 1: Values of β_0, β_1, A_0, and A_1 for different values of α of Example 1

	$\alpha = 3.2$	$\alpha = 3.4$	$\alpha = 3.6$	$\alpha = 3.8$	$\alpha = 4$
β_0	1.112728247	1.109723458	1.083962214	1.044864418	.9995736204
β_1	-.5582745325	.4974874727	1.456401505	2.306335439	3.043255978
A_0	.643230186	.642391858	.634645563	.622846100	.609201707
A_1	2.260193805	2.259840370	2.257147249	2.252345493	2.245955936

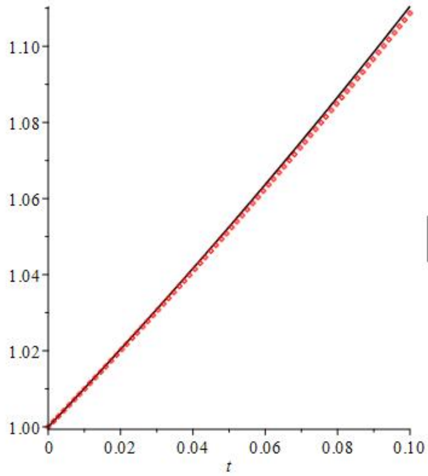
Substitute the values in the Table 1 into Equation (31), for $\alpha = 4$ we get:

$$\begin{aligned} u(t) &= 1 + 0.9995736204\,t + t^2 + 0.5072093297\,t^3 + t^4(\frac{t^3}{560} + \frac{t^4}{5040} + 0.1503834045 \\ &+ \frac{(2.754044064\,t + 39.82052538)\,t}{720}). \end{aligned} \tag{32}$$

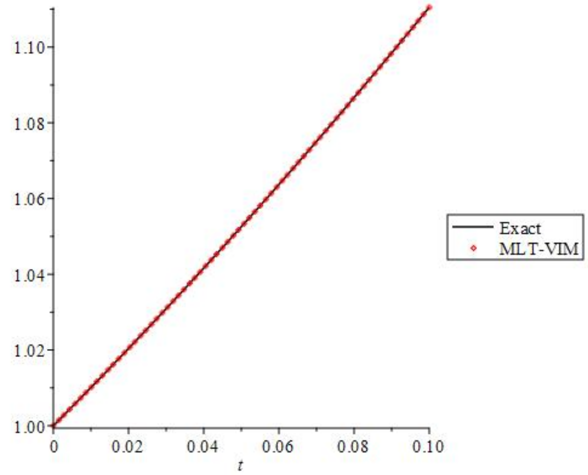
Figures 1(a) and (b) show the approximate solutions obtained by LT-MVIM and LT-VIM respectively, are compared with the exact solutions. As we see the LT-MVIM gives an approximate solution agreement with the exact solution better than the approximate solution obtained by LT-VIM. It is clear that the advantage of the LT-MVIM over the LT-VIM is through the reduction amount of computational work to obtain the first order approximate solution which

Table 2: Compare absolute error between LT-VIM and LT-MVIM on [0,1] of Example 1 .

	t=0	t=0.2	t=0.4	t=0.6	t=0.8	t=1
LT-VIM	0	1.822E-3	4.945E-3	6.531E-3	5.747E-3	0
LT-MVIM	0	3.692E-5	3.521E-5	2.911E-5	5.833E-5	0



(a) Comparison between the exact solution and LT-VIM.



(b) Comparison between the exact solution and LT-MVIM.

Figure 1: Comparison between LT-MVIM and LT-VIM when $\alpha = 4$ of Example 1.

greatly accelerates the convergence of the solution. The difference between the accuracy of the solutions from both methods LT-VIM and LT-MVIM can be better observed through Table 2 which represents the absolute error for the LT-MVIM and the standard LT-VIM when $\alpha = 4$. Fig. 2(a) shows the approximate solutions obtained by LT-MVIM with different values of α. It can be noted that all approximate solutions are compatible with each other, which is a good indicator to portray the quality of the approximate solutions obtained by LT-MVIM. The accuracy can be better observed through Fig. 2(b) which shows the absolute error for LT-MVIM when α is not an integer with the exact solution when α is an integer. We can note that all the curves in the figure have the same behavior and satisfy the boundary.

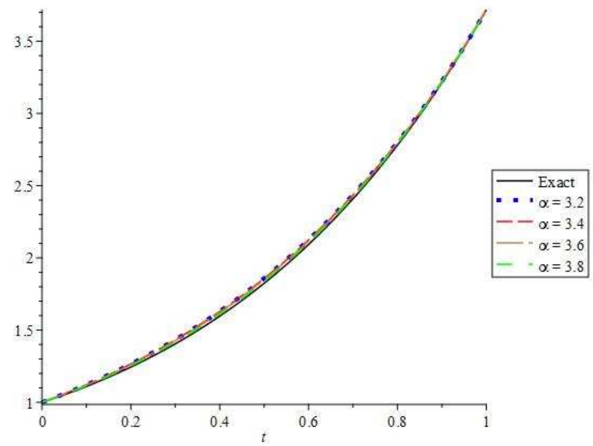
Example 2 Let fourth-order FIDEs as following [20]:

$$D_*^\alpha u(t) = 1 + \int_0^t e^{-x} u^2(x) dx, \tag{33}$$

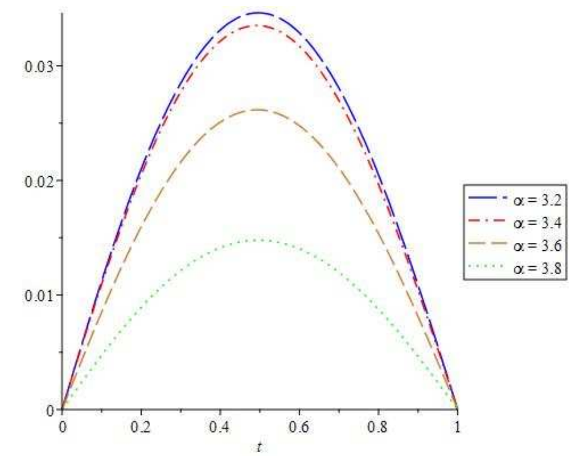
with the boundary conditions:

$$u(0) = 1, \quad u''(0) = 1, \quad u(1) = e, \quad u''(1) = e. \tag{34}$$

When $\alpha = 4$, the exact solution is $u(t) = e^t$.



(a) Approximate solution by LT-MVIM for different values of α.



(b) Absolute error of LT-MVIM for different values of α.

Figure 2: Approximate solution and absolute error of LT-MVIM for different values of α of Example 1.

The LT-VIM

Applying the LT-VIM method to obtain an approximate solution for Equation (25), we get:

$$\begin{aligned} u(t) &= 1 + 1.000536540\,t + 1/2\,t^2 + 0.1660788254\,t^3 + \frac{1}{311351040}(t^4 27.80283365\,t^9 \\ &+ 181.6779066\,t^8 + 857.4019321\,t^7 + 2574.927197\,t^6 + 5151.672784\,t^5 + 15389.51494\,t^4 \\ &+ 61776.0356\,t^3 + 432896.0341\,t^2 + 2594592\,t + 12972960). \end{aligned} \tag{35}$$

The LT-MVIM

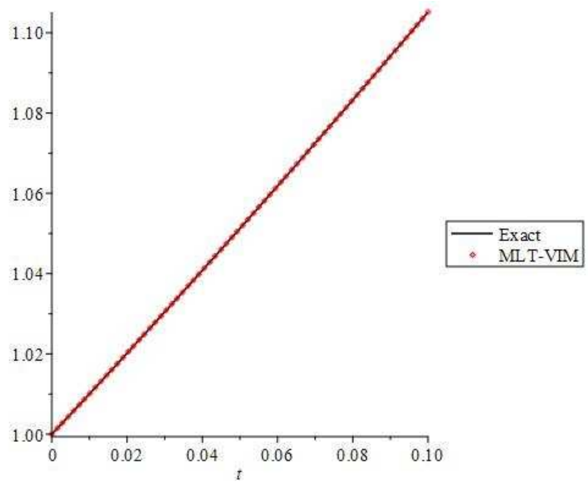
Applying the LT-MVIM method to obtain an approximate solution for Equation (25), we get:

$$\begin{aligned} u(t) &= 1 + 0.9999846397\,t + 1/2\,t^2 + 0.1666558342\,t^3 + t^4(1/24 \\ &+ 0.00005416673790\,t^5 + 0.007982871264\,t + 0.002149086182\,t^2 \\ &- 0.0001540921317\,t^3 - 0.00005734506744\,t^4). \end{aligned} \tag{36}$$

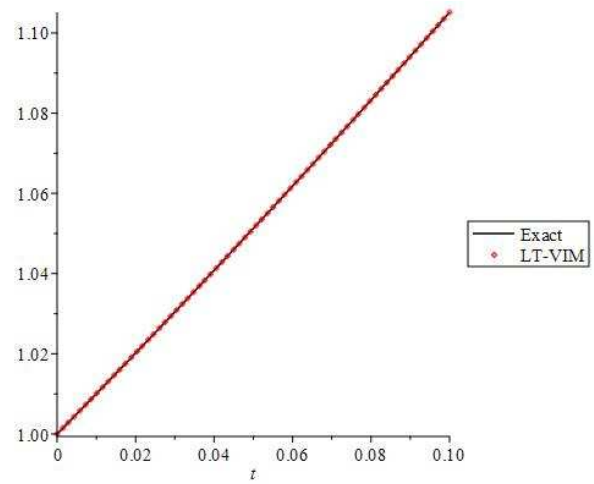
Table 3: Compare absolute error between LT-VIM and LT-MVIM on [0,1] of Example 1 .

	t=0	t=0.2	t=0.4	t=0.6	t=0.8	t=1
LT-VIM	0	1.026E-4	1.949E-4	1.763E-4	7.491E-5	0
LT-MVIM	0	3.226E-6	1.108E-5	1.563E-5	8.815E-6	0

Figures 3(a) and 3(b) show a good compatibility between the approximate solutions obtained by LT-VIM and LT-MVIM with the exact solution, in which it is difficult to notice which is more accurate. In order to better observe the difference between the accuracy of the solutions from both methods LT-VIM and LT-MVIM, it is recommended to look at Table 3, which represents the absolute error for the LT-MVIM and the standard LT-VIM when $\alpha = 4$. It is noted that the LT-MVIM gives absolute error less than LT-VIM, which indicates the success of the trail function in improving the approximate solutions obtained by LT-VIM. The exact solution of Equation (33) is not known when α is not an integer, but it can be noted from Fig. 4(a) the approximate solutions obtained by LT-MVIM are applicable to each other with different values of α. The accuracy can be better observed through Fig. 4(b) which shows the absolute error for LT-MVIM when $\alpha = 3.2, 3.4, 3.6$ and 3.8. It can be observed that all the curves in the figure have the same behavior and satisfy the boundary conditions, which are good indicators of the quality of the approximate solutions obtained by LT-MVIM.



(a) Comparison between the exact solution and LT-MVIM.

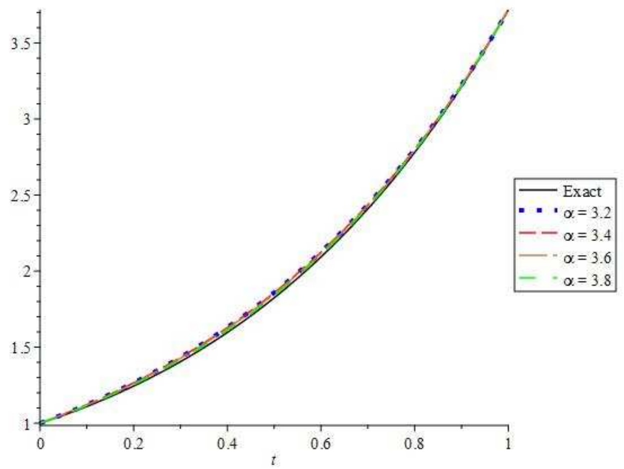


(b) Comparison between the exact solution and LT-VIM

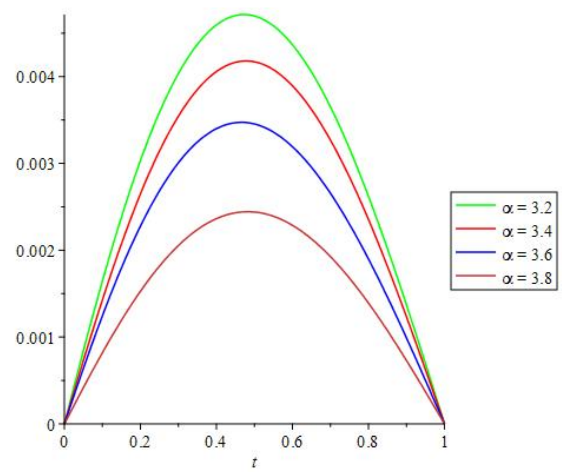
Figure 3: Comparison between LT-MVIM and LT-VIM when $\alpha = 4$ of Example 2.

6 Conclusion

The main objective of the present study is to obtain the analytical approximate solutions for linear and nonlinear fourth-order FIDEs with boundary value problems by applying LT-MVIM which is a modification of the LT-VIM. The analytical results obtained by the LT-MVIM showed the accelerating convergence to the exact solution by replacing the initial approximation with a linear trial function, taking into account that the linear trial function contributed to accelerating the convergence of the approximate solution to the exact solution by reducing the computational work to obtain the approximate solution using one iteration. The proposed method proved its efficiency in all the examples in comparison with the LT-VIM. In future work, it is possible to introduce the concept of the trial function to other existing numerical and analytical methods, so that it might become a rule or standard that can be dependent on to make the methods more convenient.



(a) Approximate solution by LT-MVIM for different values of α.



(b) Absolute error of LT-MVIM for different values of α.

Figure 4: Approximate solution and absolute error of LT-MVIM for different values of α of Example 2.

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References

[1] Johansyah, M. D., Supriatna, A. K., Rusyaman, E., Saputra, J. *Application of fractional differential equation in economic growth model: A systematic review approach. AIMS Math*. 2021. 6, 1026610280.

[2] Zhang, Y., Zhou, D., Wei, W., Frame, J. M., Sun, H., Sun, A. Y., Chen, X. *Hierarchical fractional advection-dispersion equation (FADE) to quantify anomalous transport in river corridor over a broad spectrum of scales: Theory and applications. Mathematics*. 2021. 9(7), 790.

[3] Ate, I., Zegeling, P. A. *A homotopy perturbation method for fractional-order advection-diffusion-reaction boundary-value problems. Applied Mathematical Modelling*. 2017. 47, 425-441.

[4] Ziane, D., Cherif, M. H. *Variational iteration transform method for fractional differential equations. Journal of Interdisciplinary Mathematics*. 2018. 21(1), 185-199.

[5] Haq, F., Shah, K., ur Rahman, G., Shahzad, M. *Numerical solution of fractional order smoking model via Laplace Adomian decomposition method. Alexandria Engineering Journal*. 2018. 57(2), 1061-1069.

[6] Gurbuz, M., Tasdan, Y., Set, E. *Ostrowski type inequalities via the Katugampola fractional integrals. AIMS Mathematics*. 2020. 5(1), 42-54.

[7] Kumar, S., Pandey, R. K., Srivastava, H. M., Singh, G. N. *A convergent collocation approach for generalized fractional integro-differential equations using Jacobi poly-fractonomials. Mathematics*. 2021. 9(9), 979.

[8] Panda, A., Santra, S., Mohapatra, J. *Adomian decomposition and homotopy perturbation method for the solution of time fractional partial integro-differential equations. Journal of Applied Mathematics and Computing*. 2022. 68(3), 2065-2082.

[9] Entesar, A., Mahmood, H. S., Qasim, O. S. *Solving nonlinear differential equations using a hybrid method between the variational iteration method and grey wolf optimization algorithm. Journal of Interdisciplinary Mathematics*. 2022. 25(2), 573-585.

[10] Eshkuvatov, Z., Laadjal, Z., Ismail, S. *Numerical treatment of nonlinear mixed Volterra-Fredholm integro-differential equations of fractional order. In AIP Conference Proceedings*. 2021. Vol. 2365, No. 1, p. 020006.

[11] He, J. H. *Approximate analytical solution for seepage flow with fractional derivatives in porous media. Computer Methods in Applied Mechanics and Engineering*. 1998. 167(1-2), 57-68.

[12] He, J. H. *Variational iteration methoda kind of non-linear analytical technique: some examples. International journal of non-linear mechanics*. 1999. 34(4), 699-708.

[13] He, J. H., Wu, X. H. *Variational iteration method: new development and applications. Computers and Mathematics with Applications*. 2007. 54(7-8), 881-894.

[14] Wu, G. C., Baleanu, D. *Variational iteration method for the Burgers flow with fractional derivativesnew Lagrange multipliers. Applied Mathematical Modelling*. 2013. 37(9), 6183-6190.

[15] Shang, X., Han, D. *Application of the variational iteration method for solving nth-order integro-differential equations. Journal of Computational and Applied Mathematics*. 2010. 234(5), 1442-1447.

[16] Yuliyanto, B. D., Mungkasi, S. *Variational iteration method for solving the population dynamics model of two species. In Journal of Physics: Conference Series*. 2017. Vol. 795, No. 1 p. 012044 IOP Publishing.

[17] Abassy, T. A., El-Tawil, M. A., El Zoheiry, H. *Toward a modified variational iteration method. Journal of computational and applied Mathematics*. 2007. 207(1), 137-147.

[18] Chauhan, R., Kumar, N., Aggarwal, S. *Dualities between Laplace-Carson transform and some useful integral transforms. International journal of innovative technology and exploring engineering*. 2019. 8(12), 1654-1659.

[19] Zhang, X., Tang, B., He, Y. *Homotopy analysis method for higher-order fractional integro-differential equations. Computers and Mathematics with Applications*. 2011. 62(8), 3194-3203.

[20] Nawaz, Y. *Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. Computers and Mathematics with Applications*. 2011. 61(8), 2330-2341.