MATEMATIKA, MJIM, Volume 39, Number 2, 159–171

Elzaki Transform Homotopy Perturbation Method for Solving Two-dimensional Time-fractional Rosenau-Hyman Equation

Inderdeep Singh* **and Umesh Kumari**
Department of Physical Sciences
Sant Baba Bhag Singh University, Jalandhar, Punjab-144030, India
*Corresponding author: inderdeeps.ma.12@gmail.com

Article history
Received: 10 December 2022
Received in revised form: 28 May 2023
Accepted: 3 July 2023
Published on line: 1 August 2023

Abstract Two-dimensional fractional Rosenau-Hyman equation has significant applications in applied sciences and engineering. In this research paper, we investigated solutions of two-dimensional time- fractional Rosenau- Hyman equation with the implementation of a powerful technique based on an integral transform known as Elzaki transform and Homotopy perturbation method (HPM). Convergence analysis has been discussed in this research. To illustrate the simplicity and accuracy of the presented analytical technique, few numerical experiments have been performed.

Keywords Elzaki transform; Homotopy perturbation method (HPM); Two-dimensional Rosenau-Hyman equations; Numerical examples.

Mathematics Subject Classification 35A08, 44A05

1 Introduction

The fractional partial differential equations attracted the interest of scientific researchers, due to its wide and significant applications in several areas of sciences and engineering. Many mathematical models have been found in literature and several analytical, semi- analytical and computational techniques have been established and implemented, for solving these nonlinear models. In many cases, we observed that the numerical solutions of non-linear fractional partial differential equations have not been easily achieved. Analytical techniques are proven to be a milestone for solving these problems. Some of these techniques includeHomotopy perturbation method (HPM), Homotopy analysis method (HAM), tanh method, Adomian decomposition method (ADM), tau method, Variational iteration method (VIM) and many more. In this research we have discussed an efficient classical technique based on the Homotopy perturbation technique and Elzaki transform, for solving two-dimensional Rosenau-Hyman equations. The general form of two- dimensional Rosenau-Hyman equation is:

$$\frac{\partial^\alpha \omega}{\partial t^\alpha} + a\left(\frac{\partial}{\partial x}(\omega^n) + \frac{\partial}{\partial y}(\omega^n)\right) + \left(\frac{\partial^3}{\partial x^3}(\omega^n) + \frac{\partial^3}{\partial y^3}(\omega^n)\right) = 0,$$

in some continuous domain with initial conditions $\omega(x, y, 0) = f(x, y)$. Here, '$\alpha$' is any real number. Elzaki introduced a novel transform called Elzaki transform, and applied it for solving differential equations(see for example[1-2]).In [3], authors investigated the correlation between the Laplace and Elzaki transforms, when solving first and second- order ODEs. A comparative analyses and brief discussion of the Sumudu transform and the Laplace transform has been presented, and gained the insights into the distinct characteristics of these two transforms in [4-5].

In [6], Elzaki and Sumudu transforms have been utilized for solving few differential equations. Homotopy decomposition method has been used for solving partial differential equations with time- and space- fractional derivatives in [7].Combination of Laplace transform and homotopy perturbation method is utilized for solving fractional non-linear reaction diffusion system of Lotka-Volterra type differential equation in [8]. In [9], Homotopy perturbation method has been used for solving fractional Fornberg-Whitham equation. An analytical approach has been developed for solving fractional partial differential equations arising in fluid mechanics in [10]. Sumudu transform based homotopy perturbation method has been established for solving nonlinear fractional differential equations in [11]. Fractional Rosenau Hyman equation has been solved with the aid of variational iteration method and homotopy perturbation method in [12]. A homotopy perturbation technique has been used for solving nonlinear partial differential equations in finite domain as discussed in [13]. Application of homotopy perturbation method has been used for solving nonlinear wave equations in [14]. Introduction about homotopy perturbation method has been discussed in [15]. Application of nonlinear fractional differential equations and their approximations has been discussed in [16]. Coupling scheme of homotopy and perturbation techniques has been developed for solving nonlinear problems in [17]. Rosenau and Hyman study and introduced Korteweg- de Varies like equations to understand the role of nonlinear dispersion in pattern formation in [18]. In [19], authors developed a hybrid method based on Sumudu transform and homotopy perturbation method (HPM) for solvingnonlinear differential equations and derived the convergence and error analysis of the solution. Series solutions of fractional partial differential equations have been established in [20] with the aid of homotopy perturbation method. In [21], homotopy perturbation method has been developed to find the series solutions of fourth order parabolic equations with variable coefficients. Hybrid method based on the combination of homotopy perturbation method and Sumudu transform for solving Burger equation has been presented in [22]. Elzaki transform based homotopy perturbation method has been established in [23] for solving linear and nonlinear differential equations. Modified homotopy perturbation method has been introduced for solving interal equations in [24]. New homotopy perturbation method has been used for solving various linear and nonlinear differential equations in [25-28].

This research paper is arranged as follows. Introduction to Elzaki integral transform and their properties has been discussed in Section 2. Section 3 contains the complete discussion of Homotopy perturbation method (HPM). Elzaki integral transformation based Homotopy perturbation method (ETHPM) has been discussed in Section 4. Some theorems have been presented for convergence analysis in Section 5. In Section 6, some numerical problems have been performed to illustrate the simplicity and accuracy of the proposed technique and investigate the solution of two- dimensional fractional Rosenau-Hyman equation. Conclusion is discussed in Section 7.

2 Some Basic Definitions

Fractional calculus extends the concept of differentiation and integration to non-integral or fractional order. In this paper we utilize fundamental definitions and properties from the theory of fractional calculus.

Definition 1 *A real function* $h(t) \in C_\mu, \quad t > 0, \quad \mu \in \mathcal{R}$ *if* $\exists q \in \mathcal{R}; (q > \mu)$, *s.t* $h(t) = t^q m(t)$, *where* $m(t) \in C[0,)$ *and* $h(t) \in C_\mu^n$*if* $h^{(n)} \in C_\mu, \ n \in N$.

Definition 2 *The Caputo fractional derivative of* $h(\tau)$*(as discussed in[20]):*

$$\frac{\partial^\alpha}{\partial \tau^\alpha} h(\tau) = J^{(n-\alpha)} \frac{\partial^n}{\partial \tau^n} h(\tau) = \frac{1}{I(n-\alpha)} \int_0^\tau (\tau -)^{n-\alpha-1} h^n(I)\, dI,$$

where $h \in C_{-1}^n, \ n-1 < \alpha, \ n \in \mathbb{N}, \ \tau > 0$. *Here,* $\frac{\partial^\alpha}{\partial \tau^\alpha}$ *is Caputo derivative operator and is a Gamma function.*

Definition 3 *The Elzaki transform of the function* $g_1(t)$ *is defined as(see [1]):*

$$E_L\{g_1(t)\} = v \int^a z_0 g_1(t) . e^{-\frac{t}{v}} dt, \qquad t > 0, \ \ k_1 \le v \le k_2$$

where k_1 *and* k_2 *may be finite or infinite.*

Definition 4 *The Mittag-Leffler function in term of two parameters a and b is defined as (see [20]):*

$$E_{a,\,b}(\tau) = \sum^a z_{n=0} \frac{\tau^n}{I(an+b)}, \quad a, b > 0$$

Definition and results related to Elzaki transform of fractional order:

- The Elzaki transform of the Caputo fractional derivative is defined as:

$$E_L\left\{\frac{\partial^\alpha}{\partial \tau^\alpha} h(\tau)\right\} = \frac{E_L\{h(\tau)\}}{v^\alpha} - \sum_{k=0}^{n-1} v^{k-\alpha+2} h^k(0), \quad n-1 < k \tag{1}$$

- The Elzaki transformation of some partial derivative are given below:
- $E_L\left[\frac{\partial}{\partial t} f(x,t)\right] = \frac{E_L[f(x,t)]}{v} - v.f(x,0)$,
- $E_L\left[\frac{\partial^2}{\partial t^2} f(x,t)\right] = \frac{1}{v^2} E_L[f(x,t)] - f(x,0) - v.\frac{\partial f}{\partial t}(x,0)$,
- $E_L\left[\frac{\partial}{\partial x} f(x,t)\right] = \frac{d}{dx} E_L[f(x,t)]$,
- $E_L\left[\frac{\partial^2}{\partial x^2} f(x,t)\right] = \frac{d^2}{dx^2} E_L[f(x,t)]$.
- Elzaki transform of some functions are listed here:

$$E_L(1) = v^2, \quad E_L(t) = v^3, \quad E_L(t^n) = n! v^{n+2}, \quad E_L(e^{at}) = \frac{v^2}{1-av}, \quad E_L(\sin at) = \frac{av^3}{1+a^2v^2}$$

3 Homotopy Perturbation Method (HPM)

To discuss the main concept of the method, let us take a non-linear differential equation

$$\psi(u) = g(r), \qquad r \in \Omega \tag{2}$$

Consider the boundary condition as

$$\gamma\left(u, \frac{\partial u}{\partial x}\right) = 0, \qquad r \in \Gamma, \tag{3}$$

where the general differential operator is represented as ψ, the boundary operator is represented as γ, $g(r)$ is a known analytical function and the boundary of the domain Ω is represented as Γ. Divide the operator ψ into two parts $\mathcal{A}$and N, where $\mathcal{A}$ is a linear operator and N is a non-linear operator. The equation (2) can be written as:

$$\mathcal{A}(u) + N(u) - g(r) = 0$$

According to Homotopy technique, we need to establish a Homotopy $w(r, p) : \Omega \times [0,\ 1] \to \mathcal{R}$ which satisfies

$$H(w, p) = (1 - p)[\mathcal{A}(w) - \mathcal{A}(u_0)] + p[\psi(w) - g(r)] = 0, \quad p \in [0, 1], \ r \in \Omega$$

Or

$$\mathcal{A}(w) - \mathcal{A}(u_0) - p\mathcal{A}(w) + p\mathcal{A}(u_0) + p[\mathcal{A}(w) + N(w) - g(r)] = 0$$

$$H(w, p) = \mathcal{A}(w) - \mathcal{A}(u_0) + p\mathcal{A}(u_0) + p[N(w) - g(r)] = 0 \tag{4}$$

Here $p \in [0,\ 1]$ is an embedding parameter. The initial approximation of Equation (2) is considered as u_0, which satisfies the boundary conditions. From Equation (4), we have

$$H(w, 0) = \mathcal{A}(w) - \mathcal{A}(u_0) = 0$$

and

$$H(w,\ 1) = \psi(w) - g(r) = 0$$

As p changes from zero to one, similarly $w(r,\ p)$ will change from $u_0(r)$ to $u(r)$, and this process is called deformation. The quantities $\mathcal{A}(w) - \mathcal{A}(u_0)$ and $\psi(w) - g(r)$ are known as Homotopy. Suppose solution of Equation (2) can be presented as a power series in terms of p :

$$w = w_0 + pw_1 + p^2 w_2 + \ldots$$

Letting $p = 1$, then the approximate solution of Equation (2) is:

$$u = \lim_{p \to 1} w = w_0 + w_1 + w_2 + \ldots$$

4 Elzaki Transform Homotopy Perturbation Method (ETHPM)

Consider the general form of two – dimensional Rosenau-Hyman equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}\omega + a\left\{(\omega^n)_x + (\omega^n)_y\right\} + \left\{(\omega^n)_{xxx} + (\omega^n)_{yyy}\right\} = 0. \tag{5}$$

Taking Elzaki transform of caputo fractional derivatives on both the sides of Equation (1), we obtain

$$E_L\left\{\frac{\partial^\alpha}{\partial t^\alpha}\omega + a\left\{(\omega^n)_x + (\omega^n)_y\right\} + \left\{(\omega^n)_{xxx} + (\omega^n)_{yyy}\right\}\right\} = 0. \tag{6}$$

Using Equation (1), we obtain

$$E_L\{\omega\} = \sum_{k=0}^{n-1} v^{k+2}\omega^{(k)}(x,y,0) - v^\alpha E_L\left\{a\left\{(\omega^n)_x + (\omega^n)_y\right\} + \left\{(\omega^n)_{xxx} + (\omega^n)_{yyy}\right\}\right\}$$

Applying inverse Elzaki transformation on both the sides of the above Equation, we obtain

$$\omega = \sum_{k=0}^{n-1} \frac{t^k}{k!}\omega^{(k)}(x,y,0) - E_L{}^{-1}\left\{v^\alpha E_L\left\{a\left\{(\omega^n)_x + (\omega^n)_y\right\} + \left\{(\omega^n)_{xxx} + (\omega^n)_{yyy}\right\}\right\}\right\} \tag{7}$$

By applying Homotopy Perturbation Method, we obtain

$$\omega(x,y,t) = \sum_{n=0}^{\infty} p^n \omega_n(x,y,t). \tag{8}$$

The decomposition of non-linear term can be as follows:

$$N[\omega(x,y,t)] = \sum_{n=0}^{\infty} p^n H_n(\omega), \tag{9}$$

where $H_n(\omega)$) is He's polynomial and is given as:

$$H_n(\omega_0,\omega_1,\omega_2,\ldots,\omega_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\left(\sum_{i=0}^{\infty} p^i\omega_i\right)\right]_{p=0},\ n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{10}$$

From Equation (7), we obtain

$$\sum_{n=0}^{\infty} p^n\omega_n = \omega(x,y,0) + p\left(\sum_{k=1}^{n-1}\frac{t^k}{k!}\omega^{(k)}(x,y,0) - E_L{}^{-1}\left\{v^\alpha E_L\left\{\sum_{n=0}^{\infty} p^n H_n(\omega)\right\}\right\}\right) \tag{11}$$

Comparing the like powers of p, we obtain

$$p^0 : \omega_0 = \omega(x,y,0),$$

$$p^1 : \omega_1 = \sum_{k=1}^{n-1}\frac{t^k}{k!}\omega^{(k)}(x,y,0) - E_L{}^{-1}\{v^\alpha E_L(H_0(\omega))\},$$

$$p^2 \;:\; \omega_2 = -{E_L}^{-1}\left\{v^{\alpha}E_L\left(H_1\left(\omega\right)\right)\right\},$$

$$p^3 \;:\; \omega_3 = -{E_L}^{-1}\left\{v^{\alpha}E_L\left(H_2\left(\omega\right)\right)\right\},$$

$$\vdots$$

Therefore, the series solution is given by

$$\omega\left(x,y,t\right) = \lim_{p\to 1}\omega_n(x,y,t)$$

$$\omega\left(x,y,t\right) = \omega_0 + \omega_1 + \omega_2 + \omega_3 + \ldots \tag{12}$$

5 Convergence Analysis

In this Section, we discussed the theorems to demonstrate the convergence of the proposed method for solving the nonlinear partial differential equations [20].

Theorem 1 *Let $w(x,y,t)$ and $w_n(x,y,t)$ be defined in the Banach space, then the condition that the series solution given by*

$$w\left(x,y,t\right) = \sum_{n=0}^{a} p^n w_n\left(x,y,t\right), \tag{13}$$

converges to the solution of the Equation (5), if there exist $\chi \in (0,\ 1)$ such that

$$\|w_{n+1}\|\chi\,\|w_n\|$$

Proof: For the convergence of sequence $\{s_n\}$ of the partial sums of the series (12) and we prove that $\{s_n\}$ is a Cauchy sequence in $(C\left[0,1\right], \|.\|)$.
As

$$\|s_{n+1} - s_n\| = \|w_{n+1}\| \le \chi\,\|w_n\| \le \chi^2\,\|w_{n-1}\| \le \cdots \le \chi^{n+1}\,\|w_0\|$$

Therefore,

$$\|s_n - s_m\| = \left\|\sum_{i=m+1}^{n} w_i\right\| \le \sum_{i=m+1}^{n}\|w_i\| \le\ \chi^{m+1}\left(\sum_{i=0}^{n-m}\chi^0\right)\|w_0\| = \chi^{m+1}\left(\frac{1-\chi^{n-m}}{1-\chi}\right)\|w_0\|,$$

$$n,m\ \varepsilon\ N$$

Since $0 < \chi < 1$, then

$$\|s_n - s_m\| \le\ \frac{\chi^{m+1}}{1-\chi}\,\|w_0\|$$

Since w_0 is bounded. Therefore,

$$\|s_{n+1} - s_n\| \to 0 \ \text{ as m, n}\to\infty.$$

So $\{s_n\}$ is a Cauchy sequence in $[0,1]$. Hence $\sum_{n=0}^{\infty} w_n(x,t)$ is convergent. □
Note**:** The maximum absolute truncation error of the series solution, given in Equation (13) is :

$$\left|w\left(x,y,t\right) - \sum_{k=0}^{n} w_k(x,y,t)\right|\frac{\chi^{n+1}}{1-\chi}\,\|w_0\|.$$

6 Numerical Experiments and Discussion

Here in this section, we use Elzaki transform Homotopy perturbation method in two- dimensional fractional Rosenau-Hyman equation to understand the procedure of proposed scheme:

Example 1: Consider the nonlinear two- dimensional fractional Rosenau-Hyman equation

$$\frac{\partial^\alpha}{\partial t^\alpha}\omega + \left(\omega^2\right)_x + \left(\omega^2\right)_y + \left(\omega^2\right)_{xxx} + \left(\omega^2\right)_{yyy} = 0 \tag{14}$$

with initial condition $\omega(x, y, 0) = x + y$. The exact solution (for $\alpha = 1$) is:

$$\omega(x, y, t) = \frac{x + y}{1 + 4t}.$$

By applying the Elzaki transformation based Homotopy perturbation method, we obtain

$$\sum_{n=0}^{\infty} p^n \omega_n = (x + y) + p\left(\sum_{k=1}^{n-1} \frac{t^k}{k!}\omega^{(k)}(x, y, 0) - {E_L}^{-1}\left\{v^\alpha E_L\left\{\sum_{n=0}^{\infty} p^n H_n(\omega)\right\}\right\}\right) \tag{15}$$

where $H_n(\omega)$ represents the He's polynomials. The first few components of He's polynomials are:

$$\begin{cases} H_0(\omega) = 4(x + y), \\ H_1(\omega) = -32(x + y)t, \\ H_2(\omega) = 192(x + y)t^2, \\ \qquad\vdots \end{cases} \tag{16}$$

Comparing the same powers of p, we obtain

$$\begin{aligned} p^0 &: \omega_0 = \omega(x, y, 0) = (x + y), \\ p^1 &: \omega_1 = -E^{-1}\{v^\alpha E_L(H_0(\omega))\} = -4(x + y)\frac{t^\alpha}{\Gamma(\alpha + 1)}, \\ p^2 &: \omega_2 = -{E_L}^{-1}\{v^\alpha E_L(H_1(\omega))\} = 32(x + y)\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}, \\ p^3 &: \omega_3 = -{E_L}^{-1}\{v^\alpha E_L(H_2(\omega))\} = -384(x + y)\frac{t^{\alpha+2}}{\Gamma(\alpha + 3)}, \\ &\vdots \end{aligned}$$

Therefore, the series solution is given by

$$\begin{aligned} \omega(x, y, t) &= \lim_{p\to 1} \omega_n(x, y, t) \\ \omega(x, y, t) &= \omega_0 + \omega_1 + \omega_2 + \omega_3 + \dots \end{aligned}$$

This implies

$$\omega(x, y, t) = (x + y) - 4(x + y)\frac{t^\alpha}{\Gamma(\alpha + 1)} + 32(x + y)\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} - 384(x + y)\frac{t^{\alpha+2}}{\Gamma(\alpha + 3)} + \dots \tag{17}$$

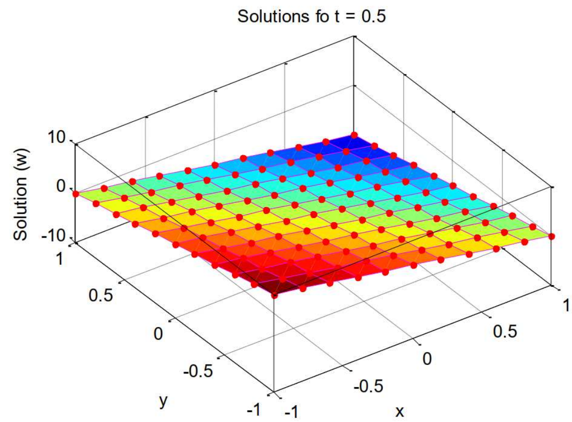

Figure 1: For $\alpha = 1$ and $t = 0.5$

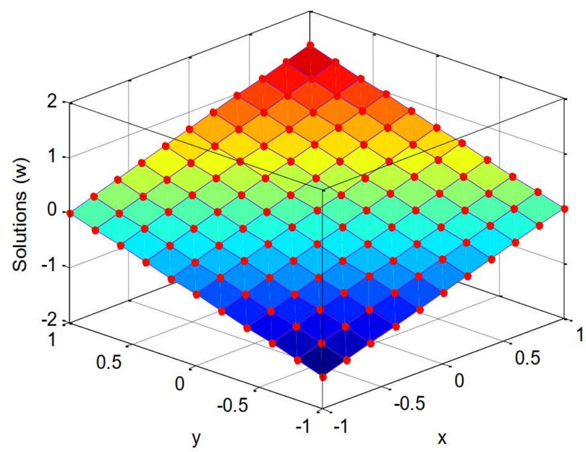

Figure 2: For $\alpha = 1$ and $t = 0.1$

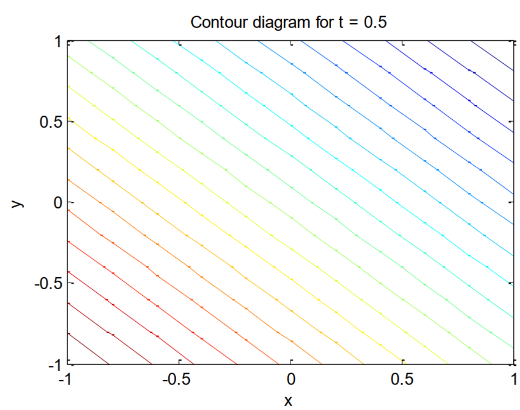

Figure 3: For $\alpha = 1$ and $t = 0.5$

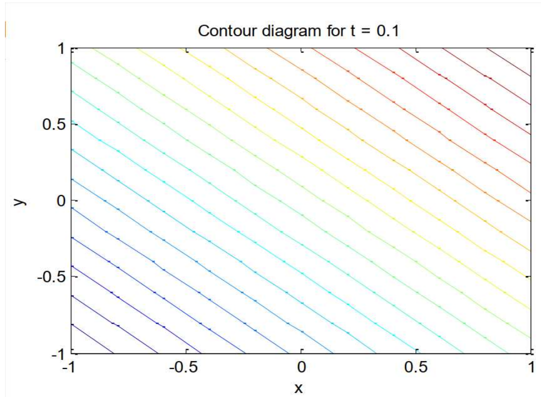

Figure 4: For $\alpha = 1$ and $t = 0.1$

Figure 1 and Figure 2 show the physical behavior of solutions for $t = 0.5$ and $t = 0.1$at $\alpha = 1$, when four terms of the series are considered. Figure 3 and Figure 4 show the contour diagram or $t = 0.5$ and $t = 0.1$ at $\alpha = 1$, when four terms of the series are utilized.

Example 2: Consider the nonlinear two- dimensional fractional Rosenau-Hyman equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\omega = \omega\omega_x + \omega\omega_y + \omega\omega_{xxx} + \omega\omega_{yyy} + 3\omega_x\omega_{xx} + 3\omega_y\omega_{yy} \tag{18}$$

with initial condition $\omega(x, y, 0) = -\frac{8}{3}cos^2\left(\frac{x+y}{4}\right)$. The exact solution (for $\alpha = 1$) is:

$$\omega(x, y, t) = -\frac{8}{3}\cos^2\left(\frac{x + y - 2t}{4}\right).$$

By applying Elzaki transformation based Homotopy perturbation method, we obtain

$$\sum_{n=0}^{\infty} p^n\omega_n = -\frac{8}{3}cos^2\left(\frac{x+y}{4}\right) + p\left(\sum_{k=1}^{n-1}\frac{t^k}{k!}\omega^{(k)}(x, y, 0) - E_L^{-1}\left\{v^{\alpha}E_L\left\{\sum_{n=0}^{\infty}p^nH_n(\omega)\right\}\right\}\right)$$

where $H_n(\omega)$ represents the He's polynomials. The first few components of He's polynomials are given by:

$$\begin{cases} H_0(\omega) = -\frac{4}{3}\sin\left(\frac{x+y}{2}\right), \\ H_1(\omega) = \frac{4t}{3}\cos\left(\frac{x+y}{2}\right), \\ H_2(\omega) = \frac{2t^2}{3}\sin\left(\frac{x+y}{2}\right), \\ \quad\vdots \end{cases} \tag{19}$$

Comparing the like powers of p, we obtain

$$
\begin{aligned}
p^0 &: \omega_0 = \omega(x,y,0) = -\frac{8}{3}cos^2\left(\frac{x+y}{4}\right),\\
p^1 &: \omega_1 = E_L{}^{-1}\{v^\alpha E_L(H_0(\omega))\} = -\frac{4}{3}\sin\left(\frac{x+y}{2}\right).\ \frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2 &: \omega_2 = E_L{}^{-1}\{v^\alpha E_L(H_1(\omega))\} = \frac{4}{3}\cos\left(\frac{x+y}{2}\right).\ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\\
p^3 &: \omega_3 = E_L{}^{-1}\{v^\alpha E_L(H_2(\omega))\} = \frac{4}{3}\sin\left(\frac{x+y}{2}\right).\ \frac{t^{\alpha+2}}{\Gamma(\alpha+3)},\\
&\ \ \vdots
\end{aligned}
$$

Therefore, the series solution is given by

$$
\begin{aligned}
\omega(x,y,t) &= \lim_{p\to 1}\omega_n(x,y,t)\\
\omega(x,y,t) &= \omega_0 + \omega_1 + \omega_2 + \omega_3 + \dots
\end{aligned}
$$

This implies

$$
\omega(x,y,t) = -\frac{8}{3}cos^2\left(\frac{x+y}{4}\right) - \frac{4}{3}\sin\left(\frac{x+y}{2}\right).\ \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{4}{3}\cos\left(\frac{x+y}{2}\right).\ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \dots \tag{20}
$$

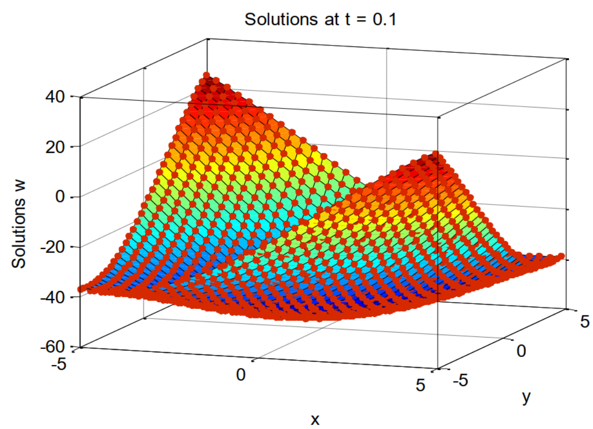

Figure 5: For $\alpha = 1$ and $t = 0.1$

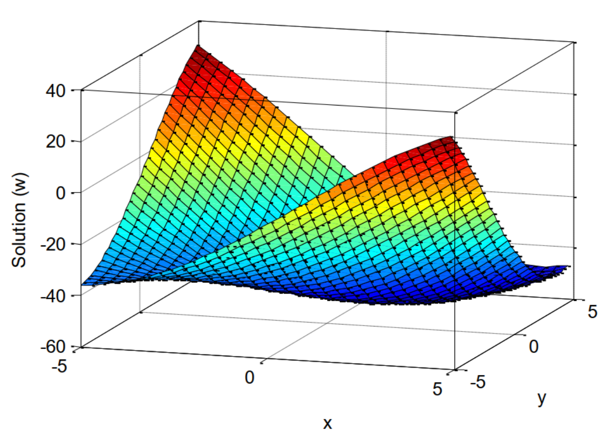

Figure 6: For $\alpha = 1$ and $t = 1$

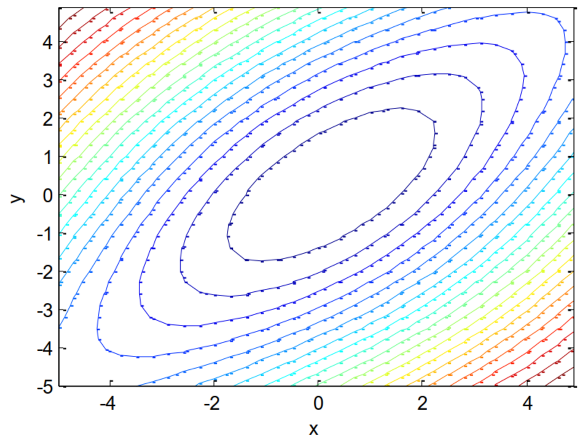

Figure 7: For $\alpha = 1$ and $t = 0.1$

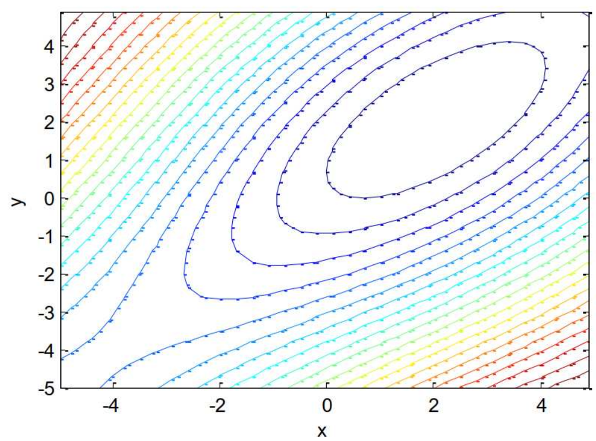

Figure 8: For $\alpha = 1$ and $t = 1$

Figure 5 and Figure 6 show the physical behavior of solutions for $t = 0.1$ and $t = 1$at $\alpha = 1$, when four terms of the series are considered. Figure 7 and Figure 8 show the contour diagram for $t = 0.1$ and $t = 1$at $\alpha = 1$, when four terms of the series are utilized.

7 Conclusions

From the results of numerical problems, we concluded that the hybrid Homotopy perturbation method based on Elzaki transforms is a powerful analytical techniqueas this method is applied on linear, non-linear, fractional order differential equations and integral equations arising in real life problems. For future scope, this technique will be effective and applicable to find the solution of three-dimensional nonlinear partial differential equations and also the system of partial differential equations.

References

[1] Elzaki, T.M. The new integral transform "Elzaki Transform" Global Journal of Pure and Applied Mathematics. 2011, 1: 57-64.

[2] Elzaki, T.M. and Elzaki, S. M. Application of new transform "Elzaki Transform" to partial differential equations, Global Journal of Pure and Applied Mathematics. 2011, 1: 65-70.

[3] Elzaki, T.M. and Elzaki, S. M. On the Connections between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics. 2011, 6(1): 1-11.

[4] Eltayeb, H. and Kilicman, A. A note on the Sumudu transforms anddifferential equations, Applied Mathematical Sciences. 2010, 4(22): 1089-1098

[5] Kilicman A. and ELtayeb, H. A note on Integral transform and partial differential equation, Applied Mathematical Sciences. 2010, 4(3): 109-118.

[6] Elzaki, T.M., Elzaki, S.M., and Hilal, M.A. Elzaki and Sumudu transforms for solving some differential equations, Global Journal of Pure and Applied Mathematics. 2012, 8(2): 167-173.

[7] Abdon, A. and Brahim, B.S. Solving partial differential equation with space- and time-fractional derivatives via homotopy decomposition method, Math. Probl. Eng. 2013, 2013: 318590–318599.

[8] Tiwana, M.H., Maqbool, K. and Mann, A.B.Homotopy perturbation Laplace transform solution of fractional non-linear reaction diffusion system of Lotka-Volterra type differential equation, Engineering Science and Technology, An International Journal. 2017, 20: 672-678.

[9] Gupta, P.K. and Singh, M. Homotopy perturbation method for fractional Fornberg-Whitham equation, Comput. Math. Appl. 2011, 61(2): 250-254.

[10] Yıldırım, A. Analytical approach to fractional partial differential equations in fluid mechanics by means of the homotopy perturbation method, Int. J. Numer. Meth. Heat Fluid Flow. 2010, 20(2): 186-200.

[11] Yousif, E.A. and Hamed, S.H.M. Solution of nonlinear fractional differential equations using the homotopy perturbation Sumudu transform method, Appl. Math. Sci. 2014, 8(44): 2195-2210.

[12] Molliq, R.Y. and Noorani, M.S.M.Solving the fractional Rosenau-Hyman equation via variational iteration method and Homotopy perturbation method, International Journal of Differential Equations. 2012, 2012: 1-15.

[13] El-Sayed, M.A., Elsaid, A., El-Kalla, I.L. and Hammad, D. A homotopy perturbation technique for solving partial differential equations of fractional order in finite domains, Applied Mathematics and Computation. 2012, 218(17): 8329–8340.

[14] He, J.H.Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons and Fractals. 2005, 26(3): 695–700.

[15] He, J.H. Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering. 1999, 178 (3-4) :257–262.

[16] He, J.H.Some applications of nonlinear fractional differential equations and their approximations, Bulletin of Science, Technology & Society. 1999, 15(2): 86–90.

[17] He, J.H. A coupling method of a homotopy technique and a perturbation technique for non-linear problems, International Journal of Non-Linear Mechanics. 2000, 35(1): 37–43.

[18] Rosenau, P. and Hyman, J.M. Compactons: solitons with finite wavelength, Physical Review Letters. 1993, 70(5): 564–567.

[19] Singh, P. and Sharma, D.Convergence and error analysis of series solution of nonlinear partial differential equation, Nonlin. Eng. 2018, 7(4): 303-308.

[20] Singh,P. and Sharma, D.On the problem of convergence of series solution of non-linear fractional partial differential equation, In: AIP Conf. Proc. 2017, 1860: 020027.

[21] Elbadri M. and Elzaki T.M. New modification of homotopy perturbation method and the fourth –order parabolic equation with variable coefficients, Pure and Applied Mathematics Journal. 2015, 4(6): 242-247.

[22] Hamza A.E. and Elzaki T.M. Application of homotopy perturbation and sumudu transform method for solving Burgers equations, American Journal of Theoretical and Applied Statistics. 2015, 4(6): 480-483.

[23] Eljaily M.H. and Elzaki T.M. Solution of linear and non linear Schrodinger equations by combine Elzaki transform and homotopy perturbation method. American Journal of Theoretical and Applied Statistics. 2015, 4(6): 534-538.

[24] Elzaki T.M. and Alamri B.A.S. Note on new homotopy perturbation method for solving non linear integral equations. Journal of Mathematical and Computational Science. 2016, 1: 149-155.

[25] Gad-Allah M.R. and Elzaki T.M. A new homotopy perturbation method for solving system of nonlinear equation of Emden-Fowler type. Journal of Progressive Research in Mathematics. 2017, 11(2).

[26] Gad-Allah M.R. and Elzaki T.M. Application of new homotopy perturbation method (NHPM) for solving non linear partial differential equations. Journal of Computational and Theoretical Nanoscience. 2017, 14: 1-9.

[27] Gad-Allah M.R. and Elzaki T.M. An application of improvement of new homotopy perturbation method for solving third order nonlinear singular partial differential equations. Universal Journal of Mathematics. 2017, 2(1).

[28] Gad-Allah M.R. and Elzaki T.M. Application of new homotopy perturbation method for solving partial differential equations. Journal of Computational and Theoretical Nanoscience. 2018, 15(2): 500-108.