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Classes of Matroids

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Abstract This paper explores which classes of graphs and matroids are k-balanced. A connection between k-balanced graphs and k-balanced matroids was also obtained. In this paper, we continue our study of the class of k-balanced matroids in order to see what matroid operations preserve k-balance. Since strong maps of matroids are defined as analogues of continuous maps of topological spaces, it is natural to ask what other topological notions carry over to matroids. In characterizing strong maps from 2000 to 2003, Al-Hawary defined a closure matroid to be a matroid in which $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all subsets A and B of its ground set. We obtain a new classification of closure matroids. Moreover, necessary and sufficient conditions for the direct sum, parallel extension connection and series extension connection to preserve k-balance property are given.

Keywords Amalgam, Balance, K-density, Graph, Matroid, Closure matroid.

1 Background

The matroid terminology will follow [11]. Let $M = (E, \mathcal{F})$ denote the matroid on the ground set E with a collection of flats $\mathcal{F}$. The *direct sum* of $M_1 = (E_1, \mathcal{F}_1)$ and $M_2 = (E_2, \mathcal{F}_2)$ is the matroid $M_1 \oplus M_2 = (E, \mathcal{F})$ where $E = E_1 \mathbin{\dot\cup} E_2$, and

$$\mathcal{F} = \{F \subseteq E_1 \cup E_2 \,|\, F \cap E_i \in \mathcal{F}_i \text{ for } i = 1, 2\}. \tag{1.1}$$

In the first two sections of this paper, we only consider loopless matroids, that is, in which $\emptyset$ is a flat. If M_1 has a basepoint p_1 and M_2 has a basepoint p_2 are defined on disjoint ground sets such that neither p_i is an isthmus of M_i, then the *parallel connection* of M_1 and M_2 is the matroid $P(M_1, M_2)$ with ground set $(E_1 - p_1) \mathbin{\dot\cup} (E_2 - p_2) \mathbin{\dot\cup} p$ and flats

$$\{F : F \cap E_i \text{ is a flat in } M_i, i = 1, 2\}, \tag{1.2}$$

where we make the identification for its basepoint $p = p_1 = p_2$. The *series connection* of M_1 and M_2 is defined to be $S(M_1, M_2) := [P(M_1, M_2)]^*$. We remark that when p is an isthmus, $P(M_1, M_2) := (M_1 - p) \oplus M_2$ and $S(M_1, M_2) := M_1 \oplus (M_2 - p)$, see [10].

Let M be a matroid on $E = E_1 \cup E_2$ such that $M|_{E_1} = M_1$ and $M|_{E_2} = M_2$, then M is called an *amalgam* of M_1 and M_2. We remark that the amalgam of two matroids need not exist, see for example [10]. When E_1 and E_2 are disjoint, the amalgam is $M_1 \oplus M_2$, while it is $P(M_1, M_2)$ when $E_1 \cap E_2 = \{p\}$. Next, we recall the following result from [8].

Theorem 1 *The amalgam of the uniform matroids $U_{l,n}$ and $U_{r,m}$, exists.*

Let $M = (E, \mathcal{F})$ be a matroid of rank greater than k. The *k-density* of M is given by $d_k(M) := \frac{|M|}{r(M)-k}$, where $|M|$ is the size of the ground set E of M and $r(M)$ is the rank of the matroid M. A matroid M is *k-balanced* if $r(M) > k$ and

$$d_k(H) \leq d_k(M) \text{ for all non-empty submatroids } H \text{ of } M \text{ of rank greater than } k, \quad (1.3)$$

and *strictly k-balanced* if the inequality is strict for all such $H \neq M$. When $k = 0$, M is called *balanced*, see for example [4, 5, 7, 8].

The following quote from [5] provides a good motivation for k-balanced matroid:

> The concept of balanced graphs was introduced by Erdös and Rényi in the 1950's and the concept of k-balanced graphs was introduced by Veerapandiyan, Ramachandran and Arumugam [9]. Since this time the theory of k-balanced graphs has undergone enormous growth because it is often easier to find a proof for the k-balanced graph case and then extend it to the general graph case. Since matroids are generalizations of graphs, it is natural to see which results for graphs may be extended to matroids.

Just a few matroid operations preserve k-balance, for example in [9] it was shown that the dual of a 0-balanced matroid without isthmuses is 0-balanced and the union of two 0-balanced matroids defined on the same ground set is 0-balanced. In the class of k-balanced matroids, the preceding two operations are not preserved as shown in the following two examples.

Example 1 *The matroid M in Figure 1 is 1-balanced but M^* is not as M^* has 1-density $5/2$ while the 1-density of $M|_{\{1,2,3\}}$ is 3.*

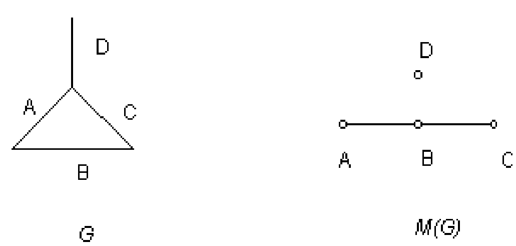

Figure 1: A 1-balanced matroid whose dual is not 1-balanced

Example 2 *Let $M_1 = M_2 = U_{2,3}$. Clearly M_1 and M_2 are 1-balanced while the union $M_1 \vee M_2$ is not.*

In section 2, we show that the amalgam A of the uniform matroids M_1 and M_2 is k-balanced if and only if the k-density of M_i, $i = 1, 2$ is at most the k-density of A. We then obtain conditions for the parallel connection (and consequently the series connection) of uniform matroids to be k-balanced. For our purposes, we recall the following result from [5].

Lemma 1 *A matroid M with $r(M) > k$ is k-balanced if the k-density of every flat in M of rank at least $k+1$ is less than the k-density of M.*

In section 3, we classify all closure matroids and show that this property is stronger than modularity.

2 K-balance Preserving Operations

We begin this section by giving a characterization of k-balanced matroids, which consequently gives conditions for the direct sum to preserve k-balance.

Lemma 2 *Let $M = (E, \mathcal{F})$ be a connected matroid with $r(M) > k$. Then M is k-balanced if and only if each component H of M with $r(H) > k$ is k-balanced and $d_k(H) = d_k(M)$.*

Proof. Let M be a connected k-balanced matroid and let H be a component of M with $r(H) > k$. Then $H = \{e\} \cup \{f \in E : M$ has a circuit containing both e and $f\}$ for some $e \in E$. As M is connected, $H = M$. Thus, $d_k(H) = d_k(M)$ and as M is k-balanced, H is k-balanced.

Conversely, let H be a component of M. As M is connected, $H = M$ and as H is k-balanced, M is k-balanced. ■

The following result which follows immediately from Lemma 2 was also proved in [5].

Corollary 1 *Let M_1 and M_2 be connected k-balanced matroids such that $r(N \cap E(M_i)) > k$ for all submatroids N of $M_1 \oplus M_2$. Then $M_1 \oplus M_2$ is k-balanced if and only if $d_k(M_1) = d_k(M_2) = d_k(M_1 \oplus M_2)$.*

To characterize the flats of the amalgam of two uniform matroids $M_1 = (E_1, \mathcal{F}_1) \cong U_{l,n}$ and $M_2 = (E_2, \mathcal{F}_2) \cong U_{r,m}$ where $E_1 \cap E_2 = T \cong U_{\alpha,\alpha}$, we follow the procedure in [8]. Let $E = E_1 \cup E_2$ and

$$\mathcal{L}(M_1, M_2) = \{Y \subseteq E : E|Y \cap E_i \in \mathcal{F}_i, i = 1, 2\}.$$

We denote by $\mathcal{A}_S(M_1, M_2)$ the amalgam of M_1 and M_2 where $E_1 \cap E_2 = S$. Using the following indices: $\alpha = 0, 1, ..., r + l - 2$; $\beta = 1, 2, ..., \alpha$; $l_\alpha = \alpha, \alpha + 1, ..., (l - 1)$ $l_\beta = \beta, \beta + 1, ..., (r - 1)$; and $r_\beta = \beta, \beta + 1, ..., (r - 1)$, we recall the following result from [8] to prove our main result in this section, that the amalgam A of two uniform matroids each of rank greater than k is k-balanced if and only if each has k-density at most the k-density of A.

Lemma 3 *The flats of $\mathcal{L}(M_1, M_2)$ are of the following types: $U_{\alpha,\alpha}$; $\mathcal{A}_T(M_1, M_2)$; $\mathcal{A}_S(U_{l_\beta,l_\beta}, U_{r_\beta,r_\beta})$; $\mathcal{A}_S(U_{l_\alpha,l_\alpha}, M_2)$; and $\mathcal{A}_S(M_1, U_{r_\alpha,r_\alpha})$ for some $S \subseteq T$ with $S \cong U_{\beta,\beta}$.*

Next, we prove that the amalgam A of the uniform matroids M_1 and M_2 is k-balanced if and only if the k-density of M_i, $i = 1, 2$ is at most the k-density of A, this is a generalization of the main result in [8].

Theorem 2 *Let k be a positive integer such that $k < l$ and $k < r$. The amalgam A of the uniform matroids $M_1 \cong U_{l,n}$ and $M_2 \cong U_{r,m}$ with $E_1 \cap E_2 = T \cong U_{\alpha,\alpha}$ for some $\alpha > k$ is k-balanced if and only if $d_k(M_i) \leq d_k(A)$, $i = 1, 2$. The amalgam is strictly k-balanced if and only if $d_k(M_i) \leq d_k(A)$, $i = 1, 2$ and $1 < d_k(A)$.*

Proof. Suppose $d_k(M_i) \leq d_k(A)$, $i = 1, 2$. Then since M_1 and M_2 are submatroids of A, it is trivial that A is k-balanced. Conversely, If F is a nonempty proper flat of the amalgam A such that $r(F) > k$, then F must be congruent to one of the matroids given in Lemma 3. Thus, F must have k-density $\dfrac{\alpha}{\alpha - k}, \dfrac{m + l_\alpha - \alpha}{r + l_\alpha - k}$ or $\dfrac{n + r_\alpha - \alpha}{l + r_\alpha - \alpha - k}$ for some l_α, r_α as defined above. Note that the maximum of $\dfrac{m + l_\alpha - \alpha}{r + l_\alpha - k}$ for $l_\alpha \in [\alpha, l-1]$ is $\dfrac{m}{r-k}$ and the maximum of $\dfrac{n + r_\alpha - \alpha}{l + r_\alpha - \alpha - k}$ for $r_\alpha \in [\alpha, r-1]$ is $\dfrac{n}{l-k}$. As $r \leq m$ and $l \leq n$, then $r - k \leq m$ and $l - k \leq n$. thus, $1 \leq \dfrac{m}{r-k}$ and $1 \leq \dfrac{n}{l-k}$ and hence $d_k(F) \leq \max\{\dfrac{m}{r-k}, \dfrac{n}{l-k}\} \leq d_k(A)$ and A is k-balanced by Lemma 1. The proof for the condition of strictly k-balanced is similar. ■

The following two results, in which we get conditions for the parallel connection (and consequently the series connection) of uniform matroids to be k-balanced, follow immediately from Theorem 1 combined with Theorem 2 when taking $\alpha = 1$.

Corollary 2 *The parallel connection $P(M_1, M_2)$ of $M_1 \cong U_{l,n}$ and $M_2 \cong U_{r,m}$ with $k < l < n$ and $k < r < m$ is k-balanced if and only if $d_k(M_i) \leq d_k(P(M_1, M_2))$ for $i = 1, 2$, that is if and only if*

$$n(r - k - 1) \leq l(m - k - 1) \text{ and}$$
$$m(l - k - 1) \leq r(n - k - 1).$$

Since the series connection operation is the dual of the parallel connection operation, we obtain the following result.

Corollary 3 *The series connection $S(M_1, M_2)$ of $M_1 \cong U_{l,n}$ and $M_2 \cong U_{r,m}$ with $k < l < n$ and $k < r < m$ is k-balanced if and only if*

$$n(r - k) \geq l(m - k - 1) \text{ and}$$
$$m(l - k) \geq r(n - k - 1).$$

3 Closure Matroids

We begin this section with the following definition.

Definition 1 *A matroid M is called* a closure matroid *if $\overline{A} \cup \overline{B} = \overline{A \cup B}$ for all subsets A and B of $E(M)$.*

In [6], it was shown that a closure matroid can also be defined in terms of flats and this definition was used to characterize the class of all closure matroids as given next.

Lemma 4 [6] *A matroid M is a closure matroid if and only if unions of flats of M are again flats of M.*

Theorem 3 [6] *Let M_1 and M_2 be matroids on disjoint ground sets. Then $M_1 \oplus M_2$ is a closure matroid if and only if M_1 and M_2 are closure matroids.*

Corollary 4 [6] *Free matroids are closure matroids.*

Lemma 5 [6] *A matroid M is a closure matroid if and only if $\widetilde{M}$ is a closure matroid.*

Next, a comparison of modular and closure matroids is given. Recall that a matroid M is modular if and only if for every flat X in M and for every other flat Y, $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$, see [10].

Theorem 4 [6] *A closure matroid is a modular matroid.*

Next, we recall Birkhoff's Theorem that classifies all modular matroids. A similar classification for closure matroids is given next.

Theorem 5 [10] *A matroid M is a modular if and only if for every connected component N of M, $\widetilde{N}$ is either a free matroid or a finite projective space.*

Theorem 6 [6] *A matroid M is a closure matroid if and only if $\widetilde{M}$ is free.*

From Theorems 3 and 6, we immediately deduce the following classification of closure matroids.

Corollary 5 [6] *A matroid M is a closure matroid if and only if M is the direct sum of a parallel extension of a free matroid and $U_{0,m}$ for some positive integer m.*

Finally, we give one more classification of the class of all closure matroids.

Theorem 7 *A matroid M is a loopless closure matroid if and only if all circuits have size exactly 2.*

Proof. Since M is loopless all circuits have size at least 2. Say C is a circuit of size $k > 2$. Choose $x \in C$ and let A and B be a partition of $C - x$ where both A and B are non-empty.. Now $x \notin \overline{A} \cup \overline{B}$, but $x \in \overline{A \cup B}$. Thus M is not a closure matroid.

The converse is easy to show. ■

Thus, closure matroids can all be obtained from free matroids (where all subsets are independent) by replacing elements by parallel classes.

4 Conclusion

It was shown that the amalgam of uniform matroids is k-balanced if and only if each matroid is k-balanced, which is an extension of the result obtained in [8]. In addition, k-balanced matroids were characterized and a new classification of them was obtained. Thus necessary and sufficient conditions for the direct sum, parallel extension connection and series extension connection to preserve k-balance property are given.

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