

## Least Criminals for Some Given Conjectures in Group Theory

<sup>1</sup>Nor Haniza Sarmin & <sup>2</sup>Sheila a/p Ilangovan

<sup>1</sup>Department of Mathematics, Universiti Teknologi Malaysia, 81310 Skudai, Johor, Malaysia.

<sup>2</sup>Kolej Legenda, Bandar Universiti Teknologi Legenda, Mantin 71700, Negeri Sembilan, Malaysia

**Abstract** In the theory of finite groups, many major theorems have been proved by the minimum counterexample technique or sometimes called least criminals technique. It uses the fact that if there is a counterexample to a given theorem, then there is a counterexample of smallest possible order. In this paper, we will find the least criminals for five given conjectures in group theory.

**Keywords** Least Criminals, Group Theory, Conjectures

### 1 Introduction

The best way of proving that a theorem is false is to find an example where it is not true. Such example is called a counterexample or least criminals, as they are sometimes being used as ‘*Gegenbeispiel*’ by the German [1]. In the theory of finite groups, many major theorems have been proved false by the least criminal technique. In practice, the contradiction frequently arises from the existence of a counterexample of order smaller than that of the presumed least criminal. G. A. Miller [1] used this technique as early as 1916, though it may have been used by earlier writers. All conjectures given in this paper are not true in general and we will find a least criminal for each of the conjecture. We chose five of the 47 conjectures given in [2]. The five given conjectures are listed as follows:

Conjecture 1: If every group of order  $k$  is cyclic, then  $k$  must be a prime number.

Conjecture 2: If  $G$  is non-Abelian, then  $G$  is a 2-generator group.

Conjecture 3: If  $G$  is a simple group, then  $G$  is Abelian.

Conjecture 4: Every group  $G$  has a subgroup of prime index.

Conjecture 5: If  $G$  is a group with trivial center, then  $G = G'$ .

In order to find a least criminal for the given conjectures, we will work with groups given by generators and defining relations, which allows us to avoid the tedious use of multiplication tables. The process of finding a least criminal for each conjecture will be shown starting from the group of the smallest order.

## 2 Conjecture 1

In this section, we determine a least criminal for Conjecture 1, which is stated as follows:

**If every group of order  $k$  is cyclic, then  $k$  must be a prime number.**

We know that a group of prime order is necessarily cyclic, but we will show that a cyclic group need not necessarily be of prime order. In order to find a least criminal for this conjecture, we first eliminate all groups of prime order. Then, we determine whether all groups of a given order  $k$  is cyclic or not, where  $k$  is a non-prime number. If one of the groups of order  $k$  is not cyclic, then we will eliminate this group from being a least criminal.

We will see that the group of order 15 is our least criminal. First, we rule out groups of order 2, 3, 5, 7, 11, and 13 since all of these groups are groups of prime order less than 15, leaving groups of order 4, 6, 8, 9, 10, 12, 14 and 15. Next, we eliminate groups of order 4, 6, 8, 9, 10, 12 and 14 since there is a group that is not cyclic for each of these orders.

Now, there is only one group of order 15 that is  $C_{15}$ , the cyclic group of order 15. Thus  $C_{15}$  is the least criminal for Conjecture 1.

## 3 Conjecture 2

In this section, we will find a least criminal for Conjecture 2, which is stated below.

**If  $G$  is non-Abelian, then  $G$  is a 2-generator group.**

In order to find a counterexample for this conjecture, we first eliminate all Abelian groups. Next, we find the smallest non-Abelian group with more than two generators. We will show that the group of order 16 is our least criminal. Since groups of order 1, 2, 3, 4, 5, 7, 9 and 11 are Abelian groups of order up to 12, thus we rule them out from being a counterexample. Hence, only groups of order 6, 8, 10 and 12 would be left as non-Abelian groups of order up to 12.

We know that  $S_3 = \langle a, b | a^3 = b^2 = e, ba = a^{-1}b \rangle$  is the only non-Abelian group of order 6. Since  $S_3$  is a 2-generator group, it does not produce a counterexample. Next, we know that there are two non-Abelian groups of order 8, namely  $D_4$  and  $Q$ .

Now,  $D_4 = \langle a, b | a^4 = b^2 = e, ba = a^{-1}b \rangle$  is the dihedral group of order 8. Since  $D_4$  is a 2-generator group, it is ruled out as a counterexample.

Next,  $Q = \langle a, b | a^4 = e, b^2 = a^2, ba = a^{-1}b \rangle$  is the quaternion group of order 8. Since  $Q$  is a 2-generator group, it is also ruled out as a counterexample.

Now, we look at groups of order 10.  $D_5 = \langle a, b | a^5 = b^2 = e, ba = a^{-1}b \rangle$ , dihedral group of order 10, is the only non-Abelian group of order 10. Since  $D_5$  is a 2-generator group, it does not produce a counterexample.

Next, there are three non-Abelian groups of order 12, namely  $A_4$ ,  $D_6$  and  $Q_6$ . We know that all of them are 2-generator groups, so they do not produce a counterexample. Similarly,  $D_7$ , the dihedral group of order 14 is the only non-Abelian group of order 14. Since  $D_7$  is a 2-generator group, it does not produce a counterexample.

Finally, we look at groups of order 16. There are nine non-Abelian groups of order 16 ([3]). However, we will look at only one of them, namely  $D_4 \times C_2$ . We know that  $D_4 \times C_2$  is a 3-generator group. Thus,  $D_4 \times C_2$  is a least criminal for Conjecture 2.

## 4 Conjecture 3

Next, we will determine a least criminal for Conjecture 3, stated as follows:

**If  $G$  is a simple group, then  $G$  is Abelian.**

A simple group is a group whose only normal subgroups are the trivial subgroup of order one and the improper subgroup consisting of the entire original group. In other word, a simple group is a group with no proper nontrivial normal subgroups.

Since all subgroups of an Abelian group are normal and all cyclic groups are Abelian, the only simple cyclic groups are those that have no subgroups other than trivial subgroup and the improper subgroup consisting of the entire original group.

We have that  $A_5$ , the alternating group of order 60, is the smallest non-Abelian simple group [3]. Thus,  $A_5$  is a least criminal for Conjecture 3.

## 5 Conjecture 4

Our goal in this section is to determine a least criminal for Conjecture 4, which is stated below.

**Every group  $G$  has a subgroup of prime index.**

We first look at groups of prime order. Since the divisors of a prime number are the number itself and 1, thus its subgroups must be either the trivial subgroup or the group itself. Thus, for groups of prime order, there is a subgroup of prime index. So we rule out all groups of prime order from being a least criminal. Next, we look at cyclic groups. Every subgroup of a cyclic group is cyclic. Moreover, for a cyclic group, there is one subgroup for each divisor of the order of group. Thus we need to determine whether the group has at least one subgroup of prime index.

Finally, for a non-cyclic group, we need to find the order of each subgroup. Next, determine whether the group has at least one subgroup of prime index. If there is no subgroup of prime index for the appropriate group  $G$ , then we can conclude that  $G$  is a least criminal for Conjecture 4.

We will now show that  $C_4 \oplus C_4$ , the internal direct product of two cyclic groups of order 4 is a least criminal for Conjecture 4. First we eliminate all groups of prime order less than 16, namely groups of order 2, 3, 5, 7, 11 and 13. Hence, groups of order 4, 6, 8, 9, 10, 12, 14, 15 and 16 are the groups of order less than 17 that need to be examined. There are fourteen groups of order 16 ([4]), namely  $C_{16}$ ,  $C_8 \oplus C_2$ ,  $C_4 \oplus C_4$ ,  $C_4 \oplus C_2 \oplus C_2$ ,  $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ ,  $D_4 \times C_2$ ,  $Q_4 \times C_2$ , a subgroup of  $GL(2, \mathbb{Z}_5)$ , Sylow 2 subgroup of  $SL(2, \mathbb{Z}_4)$ ,  $C_4 \rtimes C_4$ , a subgroup of  $GL(2, \mathbb{Z}_5)$ , Sylow 2 subgroup of  $GL(2, \mathbb{Z}_3)$ ,  $D_8$  and  $Q_8$ . Firstly,  $C_{16}$  does not produce a counterexample since  $C_{16}$  has a subgroup of prime index, namely a subgroup of order 8.

Next,  $C_8 \oplus C_2$  also does not produce a counterexample since it has a subgroup of prime index, namely a cyclic subgroup of order 8. Finally,  $C_4 \oplus C_4$ , the internal direct product of two cyclic groups of order 4 has at least a cyclic subgroup for each of order 1, 2 and 4. However,  $C_4 \oplus C_4$  does not have a subgroup of order 8. So this group does not have at least a subgroup of prime index. Thus  $C_4 \oplus C_4$  is a least criminal for Conjecture 4.

## 6 Conjecture 5

In this last section, we will determine a least criminal for the last conjecture, which is stated as follows:

**If  $G$  is a group with trivial center, then  $G = G'$ .**

We know that  $p$ -groups have nontrivial centers. In order to find a minimum counterexample for this conjecture, we first eliminate all Abelian groups (including  $p$ -groups). Then we find  $G'$ , a commutator subgroup for a group  $G$ , with trivial center. If  $G = G'$ , then  $G$  does not produce a counterexample. However, if  $G \neq G'$ , then  $G$  is our least criminal for Conjecture 5.

Since we already eliminated all Abelian groups from being a counterexample, so  $S_3$ , the symmetric group of order 6, is the first non-Abelian group that needs to be examined. We have  $S_3 = \langle a, b \mid a^3 = b^2 = e, ba = a^{-1}b \rangle$ , and  $S'_3 = \{e, a^2, a\} \neq S_3$ . Thus  $S_3$  is our least criminal for Conjecture 5.

## 7 Conclusion

In this paper, minimum counterexamples or “least criminals” for five given conjectures were shown. Conjecture 1 states that if every group of order  $k$  is cyclic, then  $k$  must be a prime number. Since group of order 15 is cyclic but 15 is a non-prime number, thus group of order 15 is a least criminal for this conjecture. Furthermore Conjecture 2 states that if  $G$  is non-Abelian, then  $G$  is a 2-generator group.  $D_4 \times C_2$  is a least criminal for this conjecture since it is a non-Abelian 3-generator group. Conjecture 3 states that, if  $G$  is a simple group, then  $G$  is Abelian. Since  $A_5$  is our smallest non-Abelian simple group, thus  $A_5$  is a least criminal for Conjecture 3. Next is Conjecture 4, which states that every group  $G$  has a subgroup of prime index. But since  $C_4 \oplus C_4$  has no subgroup of prime index, thus  $C_4 \oplus C_4$  is a least criminal for this conjecture. Finally Conjecture 5 states that if  $G$  is a group with trivial center, then  $G = G'$  and since  $S_3 \neq S'_3$ , thus  $S_3$  is a least criminal for Conjecture 5.

## Acknowledgement

The authors would like to express their appreciation to Research Management Center (RMC), Universiti Teknologi Malaysia for the financial support of this research under Vote 71966.

## References

- [1] H.F. Blichfeldt, L.E. Dickson & G.A. Miller, *Theory and Applications of Finite Groups*, Dover Reprint, New York, 1961.
- [2] D. MacHale, *Minimum Counterexample in Group Theory*, Mathematics Magazine, 54(1981), 23–28.
- [3] E.M. Parker, *Laboratory Experiences in Group Theory*, The Mathematical Association of America, Inc., 1996.
- [4] J.A. Gallian, *Contemporary Abstract Algebra*, D.C. Heath and Co., 1994.