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A Note on the Differential Equations with Distributional Coefficients

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Abstract A differential equation is a relationship between a function and its derivatives which are naturally appeared in many disciplines. They might have coefficients as constant and polynomials. There are several methods to solve a differential equation. But there is no general method to solve all the differential equations. Different problem might require different techniques. In this work after reviewing the present solution techniques we introduce some of the differential equations with distributional coefficients and make some comparisons between the solutions.

Keywords Distributions, Integral Transforms, Differential Equations with Distributional Coefficients.

1 Ordinary Differential Equations

A differential equation is a relationship between a function of time and its derivatives and naturally appear in many disciplines. The homogenous differential equations they might appear in the most general case as follows

$$a_s(x)\frac{d^s y}{dx^s} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and if the general solution exists then it looks like

$$Y(c_1, c_2, c_3, c_4, \ldots, c_n) \quad = \quad c_1 y_1(x) + c_2 y_2(x) + \ldots + c_s y_s(x)$$

where $y_i(x)$ are independent set of particular solutions. The constants c_i can be adjusted so that the solution satisfies specified initial values

$$y_0(x_0) \quad = \quad b_0, \quad y_1(x_0) = b_1 \quad y_2(x_0) = b_2 \quad y_3(x_0) = b_3 \cdots y_{s-1}(x_0) = b_{s-1}$$

at any point x_0 where the equation coefficient $a_i(x)$ are continuous and $a_s(x)$ is non zero. The general form of the linear ordinary differential equation is given by the equation

$$a_s(x)\frac{d^s y}{dx^s} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

in short form we write

$$\mathcal{P}(\mathcal{D})\, y = f(x)$$

To solve a differential equation there are several methods and each method requires different techniques and there are no general method that will solve all the differential equations. We list the common methods:

(a) Exact Solutions Method: This method is a more general method that it can also work for some nonlinear differential equations. Each of these options recognizes some functions that the other may not.

(b) Integral Transform Methods, Laplace transforms, Fourier transform, Mellin transform, sin and cos transforms that solve either homogeneous or non homogeneous systems in which the coefficients are all constants. Initial conditions appear explicitly in the solutions.

(c) Numerical Solutions, Some Appropriate systems can be solved numerically. These numeric solutions are functions that can be evaluated at points or plotted.

(d) Series Solutions, For many applications, a few terms of a Taylor series solution are sufficient. We can also control the number of terms that appear in the solution by changing series order.

Example 1 . Consider the following differential equations

$$\frac{dy}{dx} = x \sin \frac{1}{x}$$

then the exact solution is given by :

$$y\,(x) = \frac{1}{2}\left(\sin \frac{1}{x}\right) x^2 + \frac{1}{2}\left(\cos \frac{1}{x}\right) x + \frac{1}{2}\,\mathrm{Si}\left(\frac{1}{x}\right) + C_1$$

Example 2 . Similar to the previous example, consider to find the general solution of differential equation

$$x^2 \frac{dy}{dx} + xy = \sin x$$

Then exact solution is given by

$$y\,(x) = \frac{1}{x}\left(\mathrm{Si}\,(x) + C_1\right)$$

Example 3 . Consider the initial-value problem

$$\frac{d^2y}{dx^2} + y = \sum_{k=0}^{\infty} \delta(x - k\pi), \qquad y(0) = y'(0) = 0$$

then we give the solution as

$$\begin{aligned} y(x) \quad &= \quad \sum_{k=0}^{\infty} (-1)^k (H(x - k\pi)) \sin(x) \\ &+ \quad C_1 \sin(x) + C_2 \cos(x) \end{aligned}$$

However there are no Serial solution for these differential equations.

In the literature, we can find several examples in different books. In fact when we try to solve the differential equation $\mathcal{P}(\mathcal{D})\, y = f(x)$, we might have either of the following cases, see the details [4].

(i) The solution y is a smooth function such that the operation can be performed in the classical sense and the resulting equation is an identity. Then y is a classical solution.

(ii) The solution y is not smooth enough, so that the operation can not be performed but satisfies as a distributions.

(iii) The solution y is a singular distribution then the solution is a distributional solution.

There is no general method that can solve all the differential equations. Each might require different methods. In this work we study the Transforms Analysis Method.

2 Integral Transforms Methods

Integral transforms are extensively used and very useful tools in solving several kind of boundary problems and integral equations. First of all we have the following definition.

Definition 1 . The transform

$$g(\alpha) = \int_a^b f(x)\, K(\alpha, x)\, dx$$

is called the Integral transform and $K(\alpha, x)$ is called the Kernel of the transform.

Now by changing the kernel we will have several types of the Integral transforms such as; If $K(\alpha, x) = e^{-\alpha x}$ it will result Laplace transform,

$$F(\alpha) = \int_0^\infty f(x)\, e^{-\alpha x}\, dx$$

if we consider the kernel $K(\alpha, x) = x\, J_\nu(\alpha x)$ then we obtain the Hankel transform

$$F_\nu(\alpha) = \int_0^\infty f(x)\, x\, J_\nu(\alpha x)\, dx$$

and if $K(x, \alpha) = \dfrac{1}{x - \alpha}$ then we obtain Hilbert transform

$$F(\alpha) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x)}{x - \alpha}\, dx$$

provided that integrals exists. In the next definition we define Laplace transform by using the Improper Integrals.

Definition 2 . Let $f(x)$ be defined on $0 \leq x < \infty$. The Laplace transform of $f(x)$ which is denoted by $L(s)$ or $\mathcal{L}\{f(x)\}$ is given by the following formula

$$L(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx \tag{1}$$

where

$$\int_0^\infty e^{-sx} f(x)\,dx = \lim_{n\to\infty} \int_0^n e^{-sx} f(x)\,dx$$

provided that the limit exists.

Definition 3 . A function f is said to be of exponential order m, written $O(e^{mt})$, if there exist constant T and M such that

$$\Big| f(t) \Big| < Me^{mt}$$

for all $t > T$.

We note that when f is piecewise continuous on every finite interval $0 \leq t \leq T$ and f is of exponential order m then its Laplace transform exists for $s > m$. We also note that the functions such as $f(x) = e^{px}$, $g(x) = \cos x$ and $h(x) = x^n$ where n is a nonnegative integer, are having exponential order.
Then we have the following implication

bounded function $\Longrightarrow$ is of exponential order $\Longrightarrow$ Laplace transform exists

converse does not necessarily true as the function $f(x) = \dfrac{1}{\sqrt{x}}$ fails to be bounded at the neighborhood of $x = 0$ but its Laplace transform

$$\mathcal{L}\left(\frac{1}{\sqrt{x}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}}$$

Now if L is the Laplace transform of f, we call f the inverse Laplace transform of L and is written

$$\mathcal{L}^{-1}[L] = f(x)$$

that is the inverse Laplace transform recovers the continuous function f. Thus if we know the Laplace transforms of several functions, we can combine them in order to obtain a new functions since Laplace transformation is linear operation. Many theorem exists in the Computation of Laplace transform of more complicated functions we skip the details and we summarized some of the properties in the next theorem.

Theorem 1 . Let f be $(n-1)$-times continuously differentiable and piecewise smooth for $t \geq 0$ and each of these functions having exponential order, further f and all its derivatives

Laplace transformable function defined on $\mathbb{R}_+$ for all $n \in \mathbb{N}$. Then if differentiation under the integral sign is allowed,

$$\begin{aligned} L\left(f^{(n)}\right)(s) &= \int_0^\infty f^{(n)}(x)\, e^{-sx}\, dx \\ &= s^n\, L(s) - s^{n-1} f(0) \ldots s\, f^{(n-2)}(0) - f^{(n-1)}(0) \\ L\left(x^n\, f^{(n)}\right)(s) &= \int_0^\infty x^n\, f^{(n)}(x)\, e^{-sx}\, dx = (-1)^n L(f^{(n)}) \\ L\left(x^n\, f\right)(s) &= \int_0^\infty x^n\, f(x)\, e^{-sx}\, dx = (-1)^n\, L^{(n)}(s) \end{aligned}$$

We note that the Laplace transforms solve either homogeneous differential equations or non homogeneous linear ODE systems in which the coefficients are all constants. Initial conditions appear explicitly in the solution, this is the advantage of the Laplace transform. It can also be applied in the certain types of differential equation with polynomial coefficients.

Example 4 . The differential equation

$$x^2 \frac{dy}{dx} + xy = \sin x$$

By using the Laplace Method we obtain

$$y\,(x) = \frac{1}{x}\left(\mathrm{Si}\,(x) + C_1\right)$$

this same as exact solution.

In the next two examples we use exact method for nonlinear equations. In both cases the Laplace method produces no result for these equations, as Laplace transforms are appropriate for linear equations only. Series method would also fail in the second example, since $\ln x$ does not have a series expansion about $x = 0$ in powers of x.

Example 5 . Consider the initial-value problem

$$\begin{aligned} y' &= y^2 + 4 \\ y(0) &= -2 \end{aligned}$$

And the exact solution can be given by

$$y\,(x) = 2\tan\left(2x - \frac{1}{4}\pi\right)$$

Example 6 . The equation

$$\begin{aligned} (x+1)y' + y &= \ln x \\ y(0) &= 10 \end{aligned}$$

has exact solution as

$$y\,(x) = \frac{1}{x+1}\left(x \ln x - x + 21\right)$$

The problems with constant coefficients are easier to solve but if we have a differential equations with arbitrary coefficients then we can still apply the transform method as follows:

Example 7 . Consider the equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$$

and the solution will be

$$y\left(x\right)=C_1\text{BesselJ}_0\left(x\right)+C_2\text{BesselY}_0\left(x\right)$$

where BesselJ_0 and BesselY_0 are the Bessel Functions.

Definition 4 . The Bessel function of order 0 is defined by

$$J_0\left(x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{2^{2n}\left(n!\right)^2}$$

The series converges for all values of x. That is, the domain of the Bessel function is $\mathbb{R}$.

The Bessel functions are rather complicated oscillatory functions with many interesting properties. The functions BesselY and BesselJ are solutions of the first and second kind, respectively, to the modified Bessel equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-\left(z^2+v^2\right)w=0$$

and the functions $J_v(z)$ and $Y_v(z)$ are solutions of the first and second kind, respectively, to the Bessel equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-v^2\right)w=0$$

As we can see to solve a differential equation with polynomial coefficient becomes more and more difficult in Laplace method then we introduce another Integral transform:
The Mellin transform is defined by

$$M[f:s] \quad = \quad \int_0^{\infty} f(x)\,x^{s-1}\,dx=\int_0^{\infty} f(x)\,x^{s}\,\frac{dx}{x},\quad s=a+it,\quad \text{see [7].}$$

We note that if the integral is bounded then the transform exists. But converse is not necessarily to be true. The Inverse Mellin transform is defined by the contour integral

$$M^{-1}[M(f:s)](x) \quad = \quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} M(f:s)x^{-s}\,ds=f(x)$$

where $a>0$.

Example 8 . If $f(x) = e^{-x}$ then the Mellin transform is the well known Gamma Function and one can easily make a statement that the Mellin transform of $f(x) = e^{-\lambda x}$ times any polynomials in the same variable x of the same degree in s, multiplied by the Gamma Function $\Gamma(s)$. In fact, if we let $P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$ be some polynomials then we obtain

$$\int_0^\infty P_n(x)\, e^{-\lambda x}\, x^{s-1}\, dx \;=\; \sum_{k=0}^{n} \int_0^\infty e^{-\lambda x}\, x^{k+s-1}\, dx = \frac{\Gamma(s)}{\lambda^{k+s-2}} \sum_{k=0}^{n} a_{n,k} (s)_k$$

Similar to the Laplace transform we have the following theorem.

Theorem 2 . Let f be Mellin transformable function defined on $\mathbb{R}_+$. Then if differentiation under the integral sign is allowed we have

$$\begin{aligned}
M\left(f^{(n)}\right)(s) &= \int_0^\infty f^{(n)}(x)\, x^{s-1}\, dx = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} M[f : s-n] \\
M\left(x^n\, f^{(n)}\right)(s) &= \int_0^\infty x^n\, f^{(n)}(x)\, x^{s-1}\, dx = (-s)^n\, M[f : s] \\
\left(\frac{d}{ds}\right)^r M(f)(s) &= \int_0^\infty f(x)\, \log^r x\ x^{s-1}\, dx = M[\log^r\ f : s-1] \\
M\left(\int_0^x f(t)\, dt\right)(s) &= \int_0^\infty \left(\int_0^x f(t)\, dt\right) x^{s-1}\, dx = -\frac{1}{s} M[f : s+1]
\end{aligned}$$

The Mellin transform might also be used in solving the ordinary differential equations with polynomial coefficients which are usually difficult to be solved by using the Laplace transform, see [6].

Example 9 . Consider the differential equation

$$Y_\nu''(x) - xY_\nu'(x) + \nu Y(x) = 0$$

this is difficult to solve by Laplace transform, however one can solve by applying Mellin transform and the solutions is given as Hermite functions

$$Y_\nu(x) = \frac{K}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(-\frac{s}{2} - \frac{\nu}{2} - 1\right)! \left(\frac{s}{2} - \frac{1}{2}\right)! \left(\frac{s}{2} - 1\right)! \left(\frac{x^2}{2}\right)^{\frac{s}{2}}\, ds$$

where

$$K = \frac{2^{\frac{\nu}{2}}}{\left(-\frac{\nu}{2} - \frac{1}{2}\right)! \left(-\frac{\nu}{2} - 1\right)}$$

see [1].

The proof of the next theorem is straight forward and easy on using the Theorem 2.

Theorem 3 . Let f be a Laplace transformable function then f Mellin transformable.

We note that in general converse is not correct, consider $f(x) = \dfrac{\ln x}{x^2}$ then f is Mellin transformable but Laplace transform of f does not exists. We also note that there are some differential equations are not possible to solve them by using the above mentioned Integral Transforms we can apply different Integral Transforms.

3 Differential Equations with Distributional Coefficients

Distribution theory in its full scope is quite complex, subtle and difficult branch of mathematical analysis and requires a sophisticated mathematical background. For the detail we refer to Hoskins [3] or Schwartz [8] and we restrict ourself to the parts that are effective tools in applied mathematics.
Now we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and we also let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then reconsider the following differential equation

$$a_s(x)\frac{d^s y}{dx^s} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

where $a_i(x) \in \mathcal{D}'$ and $f \in \mathcal{D}'$.
The same procedure applies in the case of more general differential equations. For example, suppose that we want to find the distribution g satisfying

$$P(D)g = f, \tag{2}$$

where $P(D)$ is the generalized differential operator given by

$$P(D) = a_0(x)\frac{d^s}{dx^s} + a_1(x)\frac{d^{s-1}}{dx^{s-1}} + \ldots + a_s(x)$$

We interpret (2) as a differential equation involving distributional derivative of g, and seek g such that

$$\langle P(D)g, \phi\rangle = \langle f, \phi\rangle \quad \text{for} \quad \text{all} \quad \phi \in \mathrm{D}$$

which is equivalent to

$$\langle g, P(D)^*\phi\rangle = \langle f, \phi\rangle \quad \text{for} \quad \text{all} \quad \phi \in \mathrm{D}$$

the operator $P(D)^*$ is the distributional derivative and is given by

$$P(D)^* = \frac{d^s}{dx^s}(a_0\,\phi) + \frac{d^{s-1}}{dx^{s-1}}(a_1\,\phi) + \ldots + a_s\,\phi$$

Note if f is a regular distribution generated by a locally integrable function but not continuous or if it is a singular distribution then equation (2) has no meaning in the classical sense. The solution in this case is called a weak solution or distributional solution.
Now if we try to find a solution for

$$x\,f' = \delta(x) \Rightarrow \quad f' = x^{-1}\,\delta(x)$$

or more general form of

$$x^s\,f^{(n)} = \delta^{(r)}(x) \Rightarrow f^{(n)} = x^{-s}\;\delta^{(r)}(x), \tag{3}$$

for $n, r, s = 0, 1, 2, 3, 4, \ldots,$.
Consider to find the general solution of the differential equation

$$x^n\,f' = 0, \quad \text{for} \;\; n \in \mathbb{N} \tag{4}$$

then the solutions is given by

$$f(x) = c_1 + c_2 H(x) + c_3\delta(x) + c_4\delta'(x) + \ldots + c_{n+1}\delta^{(n-2)}(x)$$

Even in the $n = 1$ case we have $f(x) = c_1 + c_2 H(x)$ which is the distributional solution since $H(x)$ is not differentiable at $x = 0$ in the classical sense.
We note that with application of the distributional method we have advantage to cover the discontinuous function. In particular case we have the following definition.

Definition 5 . Let $P(D)$ be a differential operator with constant coefficients. A distributions g such that

$$P(D)\, g = \delta$$

is called a fundamental (or elementary) solution of the operator $P(D)$.

Example 10 . Consider the differential equation

$$x\,\frac{dy}{dx} + y = \delta(x)$$

Then the fundamental solution can be given by

$$y\,(x) = \frac{H(x)}{x} + \frac{C}{x}$$

where C arbitrary constant and H Heaviside function.

The following theorem was proved in [2].

Theorem 4 . Let $f \in D'$ and g be a fundamental solution of $P(D)$. Suppose that $f * g$ is defined. Then $u = f * g$ is a solution of

$$P(D)\, g = f$$

If u_0 is any solution of $P(D)\, g = f$, then u_0 is in the form of $u_0 = u + H = f * g + H$ where H is the function that satisfies $P(D)H = 0$.

Thus we let f be the given distribution and g is a fundamental solution $P(D)\, g = \delta(x)$, then we are able to solve the equation

$$P(D)\, g = f$$

provided that $f * g$ is defined.
The equation $y'' = x^{-1}\,\delta$ has no classical solution on $(-1, 1)$. However on using the distributional approach then we have

$$\begin{aligned} y(x) = (f * g)\,(x) &= \left(x^{-1}\,\delta(x)\right) * (\text{Heaviside}(x)x + C_1 x + C_2) \\ y(x) &= \int \left(\int \left(\frac{\delta(x)}{x} \right) dx + x\, C_1\, dx \right) + C_2 \end{aligned}$$

as a distributional solution since

$$y(x) = \text{Heaviside}(x)x + C_1 x + C_2$$

is a solution for elementary equation

$$y''(x) = \delta(x)$$

4 Conclusion

So we can conclude that there is no classical solution for the differential equations with distributional coefficients. However we can solve certain differential equations with distributional coefficients by using the distributional approach. In fact the solution is a singular distribution as long as distributional products exists.

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