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B-spline Collocation Methods for Solving Linear Two-Point Boundary Value Problems

Seherish Naz Khalid Ali Khan and Md Yushalify Misro*
School of Mathematical Sciences, Universiti Sains Malaysia
11800 Pulau Pinang, Malaysia
*Corresponding author: yushalify@usm.my

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Abstract Generating numerical solutions for boundary value problems (BVPs) using numerical methods can be a difficult task since the solutions may involve complex mathematical formulations. Hence, various types of B-splines collocation methods were developed to produce better numerical approximations with much simpler approaches. The present study introduced two new types of extended Cubic Hybrid B-spline Collocation Method (CHBSM) namely the Extended Cubic Hybrid B-spline Collocation Method (ECHBSM) and Extended Trigonometric Cubic Hybrid B-spline Collocation Method (ETCHBSM) for solving second order linear two-point BVPs. These methods were tested on three examples of linear two-point BVPs of order two. For comparison purposes, three established collocation methods which are the Cubic B-spline Collocation Method (CBSM), Cubic Trigonometric B-spline Collocation Method (CTBSM) and Cubic Hybrid B-spline Collocation Method (CHBSM) were also applied on these examples. The numerical results were tabulated and analysed to make comparisons with the analytical solutions and the conventional methods from past literatures. For CHBSM, ECHBSM and ETCHBSM, optimization was applied to the free parameter, γ using a simple proposed approach. The result demonstrated that CHBSM yields the best approximation with the analytical solutions.

Keywords
Two-point boundary value problem; Numerical approximation; Collocation method; Cubic B-spline; Optimization.

Mathematics Subject Classification 34K10; 34K28; 65D05, 65D07

1 Introduction

BVPs in ordinary differential equations (ODEs) frequently arise in modeling numerous practical applications within the fields of applied mathematics, physics, chemistry, and engineering. Examples of such applications include heat transmissions [1], traffic flow [2-6] and transports in fluid mechanics [7-11]. The reason behind this prevalence is simply that these problems can be modeled as BVPs and solved in this form. In this study, the second order linear BVPs of the following form were considered:

$$\begin{gathered} y''(x) + q(x)y'(x) + r(x)\,y(x) = f(x), \quad x \in [a,b], \\ y(a) = \alpha_1, \quad y(b) = \alpha_2. \end{gathered} \tag{1}$$

Given that $y(x)$ is a function in terms of x, where $y'(x)$ and $y''(x)$ are the first and second-order derivative functions with respect to x, and $f(x)$ is a continuous function in the domain $[a, b]$. The coefficients of the functions $y(x)$and $y'(x)$ are denoted by $r(x)$ and $q(x)$, respectively, while a, b, α, and α are known real numbers. It is noteworthy that the solution is specified at the boundary points a and b. This type of boundary condition is referred to as the Dirichlet condition.

In the past century, numerous types of numerical methods have been established to solve differential equations (DEs), including ODEs and partial differential equations (PDEs). However, the approximation of the solutions is not continuous and can only be computed at defined grid points in the domain. It was in 1968 when a study was conducted on Cubic Splines Interpolation, giving rise to the concept of splines [12]. In contrast to numerical methods, spline approximations can provide continuous approximations in the domain of the problem. The following year revealed that the spline approach is more precise than the conventional finite difference method (FDM) using the same knots [13]. Subsequently, the authors in [14] proposed the use of one-dimensional Cubic B-spline Interpolation (CBI) for two-point BVPs. The study laid the foundation for ideas extensively investigated in CBSM. A few years later, CTBSM and ECBSM were discovered [15]. The author concluded that ECBSM was superior to CBI and other numerical approaches, while CTBSM provided better approximations for problems involving trigonometric functions.

Following the work in [15], B-splines have been utilized to generate the numerical solutions of linear and non-linear PDEs. This is due to the favourable characteristics of B-spline functions, such as their composition of piecewise polynomials and local support within specific intervals, making them effective in producing numerical solutions for DEs. The initial implementation was demonstrated by the authors in [16], where CBSM were employed to solve one-dimensional heat and advection-diffusion equations. More recent applications can be observed in [2, 4, 8, 17, 18]. Similar to CBSM, promising results were obtained when CTBSM and Extended Cubic B-spline Methods (ECBSM) were examined in the context of non-linear PDEs, including fractional PDEs [3, 6, 7, 11, 20-25]. In 2016, a new type of B-spline, the CHBSM, was proposed for solving non-linear Klein-Gordon and Korteweg-de Vries equations [26]. This method hybridizes the basis functions of cubic B-spline and cubic trigonometric B-spline, resulting in precise accuracy. The same idea presented in [26] was echoed in [27], where CHBSM surpassed the accuracy of other B-spline methods. [10] further verified the superiority of CHBSM when the author tested it on non-linear PDEs.

To the best of the authors' knowledge, no methods involving the extension of CHBSM through the hybridization of basis functions had been undertaken prior to this date. Therefore, this study aims to propose new methods, namely ECHBSM and ETCHBSM, for solving problems in the form of Equation (**??**). The new methods are established by hybridizing the extended cubic B-spline basis with the cubic B-spline basis and the trigonometric cubic B-spline basis, respectively. Theoretically, the outcomes were anticipated to be comparable to those of CHBSM. Additionally, this study presents a simple and direct approach for optimizing a free parameter, γ, in CHBSM, ECHBSM, and ETCHBSM to enhance the accuracy of the approximation.

2 Methodology

The basis functions for the Cubic Hybrid B-spline, Extended Cubic Hybrid B-spline, and Extended Trigonometric Cubic Hybrid B-spline were developed. Subsequently, this concept was applied to establish collocation methods, namely CHBSM, ECHBSM, and ETCHBSM, for solving BVPs. A straightforward approach was employed to optimize the parameter γ in all three methods.

2.1 Basis Functions

In this section, four types of basis functions namely the Cubic Hybrid B-spline, Extended Cubic Hybrid B-spline and Extended Trigonometric Cubic Hybrid B-spline were presented.

2.1.1 Cubic Hybrid B-spline

The Hybrid B-spline is defined as the linear combination of the Cubic B-spline function, $B_j^k(x)$ and Trigonometric B-spline function, $T_j^k(x)$. The j-th Hybrid B-spline basis function of k-th order is defined in [26] as:

$$H_j^k(x) = \gamma B_j^k(x) + (1-\gamma) T_j^k(x), \ \gamma \in \mathbb{R}. \tag{2}$$

It is important to point out that the value of the parameter, γ has significant importance in the Hybrid B-spline basis function. When $\gamma = 1$, the basis function is reduced to the B-spline basis function and when $\gamma = 0$, the basis function becomes the Trigonometric B-spline basis function. The relationship between the order, k, and degree, d, of a basis function is defined by $k = d + 1$ [28]. For instance, if a B-spline function of degree, $d = 4$, is used, the value of k will be 5, as the order is simply the degree plus 1. In our study, B-spline basis functions of degree, $d = 3$, were used; hence, the order $k = d + 1 = 4$ was substituted. However, it is possible to use other values of k, as a B-spline curve can also be defined by the control points, n. Based on [28], a B-spline curve can be constructed up to order $n + 1$ by adjusting the locality of the B-spline. Subsequently, the Cubic Hybrid B-spline basis can be obtained by substituting $k = 4$ into Equation (2) yielding

$$H_j^4(x) = \gamma B_j^4(x) + (1-\gamma) T_j^4(x), \quad \gamma \in \mathbb{R}. \tag{3}$$

....[29]. The definitions of $B_j^4(x)$ and $T_j^4(x)$ are as follows:

$$B_j^4(x) = \frac{1}{6h^3}\begin{cases} (x-x_i)^3, & x \in \left[x_j, x_{j+1}\right], \\ h^3 + 3h^2\left(x - x_{j+1}\right) + 3h\left(x-x_{j+1}\right)^2 \\ -3\left(x-x_{j+1}\right)^3, & x \in \left[x_{j+1}, x_{j+2}\right], \\ h^3 + 3h^2(x_{i+3}-x) + 3h(x_{i+3}-x)^2 \\ -3(x_{i+3}-x)^3, & x \in \left[x_{j+2}, x_{j+3}\right], \\ (x_{i+4}-x)^3, & x \in \left[x_{j+3}, x_{j+4}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

$$T_j^4(x) = \frac{1}{\varphi}\begin{cases} \delta^3\left(x_j\right), & x \in \left[x_j, x_{j+1}\right], \\ \delta\left(x_j\right)\left[\delta\left(x_j\right)\theta\left(x_{j+2}\right) + \theta\left(x_{j+3}\right)\delta\left(x_{j+1}\right)\right] \\ +\theta\left(x_{j+4}\right)\delta^2\left(x_{j+1}\right), & x \in \left[x_{j+1}, x_{j+2}\right], \\ \delta\left(x_j\right)\theta^2\left(x_{j+3}\right) + \theta\left(x_{j+4}\right)[\delta\left(x_{j+1}\right)\theta\left(x_{j+3}\right) \\ +\theta\left(x_{j+4}\right)\delta\left(x_{j+2}\right)], & x \in \left[x_{j+2}, x_{j+3}\right], \\ \theta^3(x_{j+4}), & x \in \left[x_{j+3}, x_{j+4}\right], \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where $\delta\left(x_j\right) = \sin\left(\frac{x-x_j}{2}\right), \theta\left(x_j\right) = \sin\left(\frac{x_j - x}{2}\right)$ and $\varphi = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)$ [25].

2.1.2 Extended Cubic Hybrid B-spline

The Extended Cubic Hybrid B-spline is simply an extension of CHBSM, where it consists of the linear combination of the Extended Cubic B-spline function, $E_j^k(x)$ [15] with the Cubic B-spline function, $B_j^k(x)$. The j-th Extended Hybrid B-spline basis function, $EH_j^k(x)$ of order k is defined as:

$$EH_j^k(x) = \gamma E_j^k(x) + (1-\gamma) B_j^k(x), \quad \gamma \in \mathbb{R}. \tag{6}$$

By substituting $k = 4$ into Equation (6), the Cubic Hybrid B-spline basis can be obtained as:

$$EH_j^4(x) = \gamma E_j^4(x) + (1-\gamma) B_j^4(x), \quad \gamma \in \mathbb{R}, \tag{7}$$

where

$$E_j^4(x) = \begin{cases} 4h(1-\lambda)\left(x-x_j\right)^3 + 3\lambda\left(x-x_j\right)^4, & x \in \left[x_j, x_{j+1}\right], \\ h^4(4-\lambda) + 12h^3\left(x-x_{j+1}\right) + 6h^2(2+\lambda)\left(x-x_{j+1}\right)^2 \\ -12h\left(x-x_{j+1}\right)^3 - 3\lambda\left(x-x_{j+1}\right)^4, & x \in \left[x_{j+1}, x_{j+2}\right], \\ h^4(4-\lambda) + 12h^3\left(x_{j+3}-x\right) + 6h^2(2+\lambda)\left(x_{j+3}-x\right)^2 \\ -12h\left(x_{j+3}-x\right)^3 - 3\lambda\left(x_{j+3}-x\right)^4, & x \in \left[x_{j+2}, x_{j+3}\right], \\ 4h(1-\lambda)\left(x_{j+4}-x\right)^3 + 3\lambda\left(x_{j+4}-x\right)^4, & x \in \left[x_{j+3}, x_{j+4}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

and $B_j^4(x)$ as shown previously in Equation (4).

2.1.3 Extended Trigonometric Cubic Hybrid B-spline

The Extended Trigonometric Cubic Hybrid B-spline is also an extension of CHBSM such that it consists of the linear combination of the Extended Cubic B-spline function, $E_j^k(x)$ [15] with the Trigonometric B-spline function, $T_j^k(x)$. The j-th Extended Trigonometric Hybrid B-spline basis function, $ET_j^k(x)$ of order k is defined as:

$$ET_j^k(x) = \gamma E_j^k(x) + (1-\gamma)\, T_j^k(x), \quad \gamma \in \mathbb{R}. \tag{9}$$

Subsequently, the 4-th order basis function can be written as:

$$ET_j^4(x) = \gamma E_j^4(x) + (1-\gamma)\, T_j^4(x), \quad \gamma \in \mathbb{R}, \tag{10}$$

where $E_j^4(x)$ is defined in Equation (8) and $T_j^4(x)$ as shown in Equation (5).

2.2 Collocation Method

This section explains only one hybrid collocation method which is CHBSM as the same procedures were implemented on ECHBSM and ETCBSM. The only difference between all three collocation methods are the basis functions as discussed earlier. An approximate solution of Equation (1) using CHBSM can be written as:

$$S(x) = \sum_{j=-3}^{n-1} C_j H_j^4(x) = f(x), \quad x \in [x_0, x_n], \tag{11}$$

where $C_j(x)$ are the unknown real coefficients and $H_j^4(x)$ is the Cubic Hybrid B-spline basis functions presented in Equation (3). The term $f(x)$ is the continuous function in the domain $[x_0, x_n]$ located at the right-hand side of the BVP. Analogous to CBSM and CTBSM, there are three nonzero terms at each knot, x_j namely $H_{j-3}^4(x)$, $H_{j-2}^4(x)$, and $H_{j-1}^4(x)$ [26]. The values are shown in Table 1.

Table 1: Values of $H_j^4(x), H_j'^4(x)$ and $H_j''^4(x)$

	x_j	x_{j+1}	x_{j+2}	x_{j+3}	x_{j+4}
$H_j^4(x)$	0	h_1	h_2	h_1	0
$H'^4_j(x)$	0	h_3	0	h_4	0
$H''^4_j(x)$	0	h_5	h_6	h_5	0

The value of h_j for $j = 1, 2, \ldots, 6$ are as follows:

$$h_1 = \frac{\gamma}{6} + \frac{(1-\gamma)\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, \qquad h_2 = \frac{4\gamma}{6} + \frac{2(1-\gamma)\sin\left(\frac{h}{2}\right)}{\sin\left(\frac{3h}{2}\right)}$$

$$h_3 = \frac{\gamma}{2h} + \frac{3(1-\gamma)}{4\sin\left(\frac{3h}{2}\right)}, \qquad h_4 = \frac{-\gamma}{2h} - \frac{3(1-\gamma)}{4\sin\left(\frac{3h}{2}\right)},$$

$$h_5 = \frac{\gamma}{h^2} + \frac{3(1-\gamma)\left[\sin\left(\frac{h}{2}\right) - 2\sin^3\left(\frac{h}{2}\right) + \sin\left(\frac{3h}{2}\right)\right]}{8\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)}, \qquad h_6 = \frac{-2\gamma}{h^2} - \frac{3(1-\gamma)\left[\sin(2h) + 2\sin^2\left(\frac{h}{2}\right) + \sin(h)\right]}{4\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)}.$$

Thus, the approximated solution in Equation (11) and its derivatives with respect to x up to order two can be obtained as follows:

$$\begin{gathered} S(x_j) = C_{j-3}H_{j-3}^4(x_j) + C_{j-2}H_{j-2}^4(x_j) + C_{j-1}H_{j-1}^4(x_j) = f(x_j), \\ \text{or } S(x_j) = C_{j-3}(A_1) + C_{j-2}(A_2) + C_{j-1}(A_1) = f(x_j), \end{gathered} \tag{12}$$

$$S'\left(x_j\right) = C_{j-3}H'^{4}_{j-3}\left(x_j\right)+C_{j-2}H'^{4}_{j-2}\left(x_j\right)+C_{j-1}H'^{4}_{j-1}\left(x_j\right) = f'\left(x_j\right),$$
$$\text{or } S'\left(x_j\right) = C_{j-3}\left(A_3\right)+C_{j-2}\left(0\right)+C_{j-1}\left(A_3\right)=f'\left(x_j\right), \tag{13}$$

$$S''\left(x_j\right) = C_{j-3}H''^{4}_{j-3}\left(x_j\right)+C_{j-2}H''^{4}_{j-2}\left(x_j\right)+C_{j-1}H''^{4}_{j-1}\left(x_j\right) = f''\left(x_j\right)$$
$$\text{or } S''\left(x_j\right)=C_{j-3}\left(A_4\right)+C_{j-2}\left(A_5\right)+C_{j-1}\left(A_4\right)=f''\left(x_j\right), \tag{14}$$

where

$$A_i = \gamma\sigma_i+\left(1-\gamma\right)\eta_i, \quad \text{for } i = 1, 2, \ldots, 5 \tag{15}$$

and

$$\sigma_1=\frac{1}{6}, \quad \sigma_2=\frac{4}{6}, \quad \sigma_3=\frac{-1}{2h}, \quad \sigma_4=\frac{1}{h^2}, \quad \sigma_5=\frac{-2}{h^2},$$
$$\kappa_1 = \sin\left(\frac{h}{2}\right), \quad \kappa_2 = \sin\left(h\right), \quad \kappa_4 = \sin\left(2h\right). \quad \kappa_4 = \sin\left(2h\right).$$
$$\eta_1=\frac{{\kappa_1}^2}{\kappa_2\kappa_3}, \quad \eta_2=\frac{2\kappa_2}{\kappa_3}, \quad \eta_3=\frac{-3}{4\kappa_3}, \quad \eta_4=\frac{3\left(\kappa_1-2{\kappa_1}^3+\kappa_3\right)}{8\kappa_1\kappa_2\kappa_3}, \quad \eta_5 = \frac{-3\left(\kappa_4+2{\kappa_1}^2\kappa_2\right)}{4\kappa_1\kappa_2\kappa_3}.$$

Since $S\left(x\right)$ is assumed to be the approximated solution of the analytical solution $y\left(x\right)$, Equation (1) can be written as:

$$S''\left(x_j\right)+q\left(x_j\right)S'\left(x_j\right)+r\left(x_j\right)S\left(x_j\right)=f\left(x_j\right), \quad x\in\left[x_0, x_n\right],$$
$$S\left(x_0\right) = \alpha_1, \quad S\left(x_n\right) = \alpha_2, \tag{16}$$

for $j = 0, 1, \ldots, n-1$. Equation (12) – Equation (14) is then substituted in Equation (16) and upon simplifications, Equation (17) is obtained as:

$$C_{j-3}\left[\gamma\sigma_4+\left(1-\gamma\right)\eta_4-q\left(x_j\right)\left(\gamma\sigma_3+\left(1-\gamma\right)\eta_3\right)+r\left(x_j\right)\left(\gamma\sigma_1+\left(1-\gamma\right)\eta_1\right)\right]$$
$$+C_{j-2}\left[\gamma\sigma_5+\left(1-\gamma\right)\eta_5+r\left(x_j\right)\left(\gamma\sigma_2+\left(1-\gamma\right)\eta_2\right)\right]$$
$$+C_{j-1}\left[\gamma\sigma_4+\left(1-\gamma\right)\eta_4+q\left(x_j\right)\left(\gamma\sigma_3+\left(1-\gamma\right)\eta_3\right)+r\left(x_j\right)\left(\gamma\sigma_1+\left(1-\gamma\right)\eta_1\right)\right]=f\left(x_j\right), \tag{17}$$

with boundary conditions

$$S_H\left(x_0\right) = C_{-3}\left[\gamma\sigma_1+\left(1-\gamma\right)\eta_1\right]+C_{-2}\left[\gamma\sigma_2+\left(1-\gamma\right)\eta_2\right]+C_{-1}\left[\gamma\sigma_1+\left(1-\gamma\right)\eta_1\right] = \alpha_1, \tag{18}$$
$$S_H\left(x_n\right) = C_{n-3}\left[\gamma\sigma_1+\left(1-\gamma\right)\eta_1\right]+C_{n-2}\left[\gamma\sigma_2+\left(1-\gamma\right)\eta_2\right]+C_{n-1}\left[\gamma\sigma_1+\left(1-\gamma\right)\eta_1\right] = \alpha_2. \tag{19}$$

Equation (17)–Equation (19) results in a tridiagonal matrix system of size $(n+3)\times(n+3)$. This system can be denoted as BC_j= F. The matrix C is a column matrix of the coefficients $C = \left(C_0, C_1, \ldots, C_{n-1}, C_n\right)$, the right hand side matrix is the function $F = \left(\alpha_1, f\left(x_0\right), f\left(x_1\right), \ldots, f\left(x_n\right), \alpha_2\right)^T$ whereas the coefficients of matrix B is presented as follows:

$$B = \begin{bmatrix} Q_1 & Q_2 & Q_1 & 0 & \ldots & \ldots & 0 \\ \lambda_0\left(x_0\right) & \mu_0\left(x_0\right) & \rho_0\left(x_0\right) & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_1\left(x_1\right) & \mu_1\left(x_1\right) & \rho_1\left(x_1\right) & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ldots & 0 & \lambda_n\left(x_n\right) & \mu_n\left(x_n\right) & \rho_n\left(x_n\right) \\ 0 & \ldots & \ldots & 0 & Q_1 & Q_2 & Q_1 \end{bmatrix}.,$$

with

$$Q_1 = \gamma\sigma_1+\left(1-\gamma\right)\eta_1, \tag{20}$$
$$Q_2 = \gamma\sigma_2+\left(1-\gamma\right)\eta_2, \tag{21}$$
$$\lambda_j\left(x_j\right) = \left[\gamma\sigma_4+\left(1-\gamma\right)\eta_4\right]-q\left(x_j\right)\left[\gamma\sigma_3+\left(1-\gamma\right)\eta_3\right]+r\left(x_j\right)\left(Q_1\right), \tag{22}$$
$$\mu_j\left(x_j\right) = \left[\gamma\sigma_5+\left(1-\gamma\right)\eta_5\right]+r\left(x_j\right)\left(Q_2\right), \tag{23}$$
$$\rho_j\left(x_j\right) = \left[\gamma\sigma_4+\left(1-\gamma\right)\eta_4\right]+q\left(x_j\right)\left[\gamma\sigma_3+\left(1-\gamma\right)\eta_3\right]+r\left(x_j\right)\left(Q_1\right), \tag{24}$$

for $j = 0, 1, \ldots, n-1$. The values of the unknown C_j for $j = 0, 1, \ldots, n-1$ can be obtained by solving the matrix system using

$$C_j = B^{-1}F, \tag{25}$$

which are then substituted back in Equation (11) to get the approximated solution of the second order linear two-point BVP in Equation (1).

2.3 Optimization of γ

The accuracy level of the approximated solutions of second order linear two-point BVPs using CHBSM, ECHBSM and ETCHBSM is highly dependent on the value of the free parameter γ. This section explains the method of determining γ without involving the use of an analytical solution for CHBSM, which was adopted from [15]. The approach applies for ECHBSM and ETCHBSM as well but is omitted here to avoid redundancies.

In contrast to the approach in [15], a much simpler method, which does not include Newton's method for minimization, was applied. Firstly, the approximated solution of CHBSM given in Equation (11) is written in the form of γ such that

$$S(x,\gamma) = \sum_{j=-3}^{n-1} C_j H_j^4(x,\gamma), \quad x \in [x_0, x_n], \tag{26}$$

for $j = 1, 2, \ldots, n-1$ where C_i's are the unknown real coefficients and $H_j^4(x)$ is the Cubic Hybrid B-spline basis function. Now, the general form of the second order linear two-point BVP as in Equation (1) can be written as:

$$y''(x) + q(x)y'(x) + r(x)y(x) - f(x) = 0. \tag{27}$$

Then, substituting the approximated solution, $S(x,\gamma)$ and its respective derivatives into Equation (27) yields

$$S''(x,\gamma) + q(x)S'(x,\gamma) + r(x)S(x,\gamma) - f(x) \approx 0. \tag{28}$$

Equation (28) is a form of error formula, denoted by $E(x,\gamma)$ such that

$$E(x,\gamma) = S''(x,\gamma) + q(x)S'(x,\gamma) + r(x)S(x,\gamma) - f(x). \tag{29}$$

Further expanding the previous equation for each piecewise polynomial, $S_j(x,\gamma)$ yields

$$S''E_j(x,\gamma) = S_j''(x,\gamma) + q(x)S_j'(x,\gamma) + r(x)S_j(x,\gamma) - f(x), \tag{30}$$

where $j = 1, 2, \ldots, n-1$. Next, for a sequence $\left\{x_j^*\right\}_{j=1}^{n-1}$ with $x_j^* \in [x_0, x_n]$, a collocation point representative from each sub-interval is chosen to be evaluated such that

$$x_j^* = \frac{x_j + x_{j+1}}{2},$$

for $i = 1, 2, \ldots, n-1$. The error at collocation points can be written as:

$$E\left(x_j^*, \gamma\right), \qquad x \in \left[x_j, x_{j+1}\right]. \tag{31}$$

The previous expression is further combined using the Euclidean norm (L_2) formula defined as:

$$L_2 = \sqrt{\sum_{j=1}^{n-1} \left[S\left(x_j\right) - y\left(x_j\right)\right]^2}, \tag{32}$$

which yields

$$e_1(\gamma) = \sqrt{\sum_{j=1}^{n-1} \left[E\left(x_j^*\right), \gamma\right]^2}. \tag{33}$$

The former equation measures the accuracy of the approximated solution without relying on the exact solution. Now, applying minimization to Equation (32) results in

$$e_2(\gamma) = \sum_{j=1}^{n-1} \left[E\left(x_j^*\right), \gamma\right]^2. \tag{34}$$

It is evident that Equation (34) is essentially the square of Equation (33), making it easier to compute than the former equation. Another alternative is to combine the expressions using the one-norm formula, yielding a more simplified equation

$$e_3(\gamma) = \sum_{j=1}^{n-1} \left|E\left(x_j^*\right), \gamma\right|. \tag{35}$$

Theoretically, finding the optimized value of γ implies that $e_1(\gamma)$, $e_2(\gamma)$ and $e_3(\gamma)$ in Equation (33), Equation (34) and Equation (35) need to be minimized as well. This can be achieved by setting the error at collocation points to be zero, as follows.

$$E\left(x_j^*, \gamma\right) = 0,$$

$$e_1(\gamma) = \sqrt{\sum_{j=1}^{n-1} \left[E\left(x_j^*\right), \gamma\right]^2} = 0,$$

$$e_2(\gamma) = \sum_{j=1}^{n-1} \left[E\left(x_j^*\right), \gamma\right]^2 = 0,$$

$$e_3(\gamma) = \sum_{j=1}^{n-1} \left|E\left(x_j^*\right), \gamma\right| = 0.$$

Note that the error equations, $e_1(\gamma)$, $e_2(\gamma)$ and $e_3(\gamma)$ are in terms of two unknowns: the collocation points representative, x_j^*, and the parameter, γ. The collocation points representative was defined earlier as the midpoint value of the collocation points in each sub-interval. Then, by substituting the values of x_j^*, the error equations will be left in terms of γ. Now, the value of γ in the error equations $e_1(\gamma)$, $e_2(\gamma)$ and $e_3(\gamma)$ can be found by using Mathematica 12.3 with a simple built-in function "*Solve*". The value found is considered the optimized value of γ since the error equations have been minimized.

In this study, Equation (34), representing the Euclidean norm (L_2), was considered to determine the value of γ. This choice was made because the L_2-norm is capable of accurately measuring the dispersion of a feature. Another reason is that the square root, as seen in Equation (33) and Equation (34), implies a continuously differentiable function, in contrast to the absolute function in Equation (35), which introduces a discontinuity. Generally, being continuously differentiable is a crucial criterion for optimization problems. It is worth mentioning that ECHBSM and ETCHBSM undergo this optimization process twice since the basis of the Extended Cubic B-spline itself has a free parameter, λ. Therefore, the first optimization is carried out on the Extended Cubic B-spline basis to obtain an optimized value of λ before proceeding to the second optimization for the bases of Extended Cubic Hybrid B-spline and Extended Trigonometric Cubic Hybrid B-spline to obtain an optimized value of γ.

3 Results and Discussion

The findings of this study were presented and discussed in this section. The discussion focuses on the numerical results of all five types of B-spline collocation methods compared to the analytical solutions when implemented on three chosen problems of second order linear two-point BVPs.

Example 1: $y''(x) - y(x) = 2e^{x-1}$, $0 \le x < 1$, with $y(0) = 0$ and $y(1) = 1$. [29]

The analytical solution is $y(x) = x\,e^{x-1}$. Table 2 presents the absolute errors and norms upon applying CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM to Example 1.

From Table 2, the corresponding maximum absolute error (L_∞) for CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM are found to be 2.66489×10^{-4}, 6.62037×10^{-4}, 1.45553×10^{-5}, 8.11253×10^{-1} and 8.07113×10^{-1}, respectively. The results of CBSM agree with those of [29]. On the other hand, CHBSM produces better approximations of the solutions than CTBSM and the other two new hybrid methods. It also has smaller error compared to CBSM.

Table 2: The absolute errors and norms upon applying CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM to Example 1

x	**CBSM [29]**	**CBSM**	**CTBSM**	**CHBSM** ($\gamma = -0.70786$)	**ECHBSM** ($\lambda = -0.0356$, $\gamma = -0.0486$)	**ETCHBSM** ($\lambda = -0.0356$, $\gamma = -0.07426$)
0.1	8.20×10^{-5}	8.20393×10^{-5}	1.98907×10^{-4}	6.86513×10^{-7}	3.96483×10^{-2}	3.83012×10^{-2}
0.2	1.15×10^{-4}	1.15096×10^{-4}	3.67729×10^{-4}	2.48477×10^{-6}	8.86978×10^{-2}	8.71384×10^{-2}
0.3	2.06×10^{-4}	2.05606×10^{-4}	5.03258×10^{-4}	5.09038×10^{-6}	1.47626×10^{-1}	1.45824×10^{-1}
0.4	2.44×10^{-4}	2.44454×10^{-4}	6.01275×10^{-4}	8.12575×10^{-6}	2.17968×10^{-1}	2.15891×10^{-1}
0.5	2.66×10^{-4}	2.65555×10^{-4}	6.56420×10^{-4}	1.11229×10^{-5}	3.01474×10^{-1}	2.99082×10^{-1}
0.6	2.66×10^{-4}	2.66489×10^{-4}	6.62037×10^{-4}	1.35035×10^{-5}	4.00133×10^{-1}	3.97383×10^{-1}
0.7	2.44×10^{-4}	2.44305×10^{-4}	6.09998×10^{-4}	1.45553×10^{-5}	5.16209×10^{-1}	5.13054×10^{-1}
0.8	1.95×10^{-4}	1.95449×10^{-4}	4.90498×10^{-4}	1.34050×10^{-5}	6.52276×10^{-1}	6.48659×10^{-1}
0.9	1.16×10^{-4}	1.15684×10^{-4}	2.91808×10^{-4}	8.98678×10^{-6}	8.11253×10^{-1}	8.07113×10^{-1}
L_∞	2.66×10^{-4}	2.66489×10^{-4}	6.62037×10^{-4}	1.45553×10^{-5}	8.11253×10^{-1}	8.07113×10^{-1}
L_2	6.20×10^{-4}	6.19963×10^{-4}	1.53512×10^{-3}	2.96136×10^{-5}	1.29607×10^{0}	1.28330×10^{0}

Figure 1 depicts the error for the approximated analytical solution using CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM with $n = 10$. From the figure, it is evident that the error of approximation is the smallest for CHBSM, followed by CBSM, ECHBSM, CTBSM and ETCHBSM. Since the error is the smallest for CHBSM, it implies that the approximated solution is the closest to the analytical solution compared to the other four methods. Figure 2 validates that the approximated solution is in good agreement with the analytical solution, as both solutions are seen to overlap.

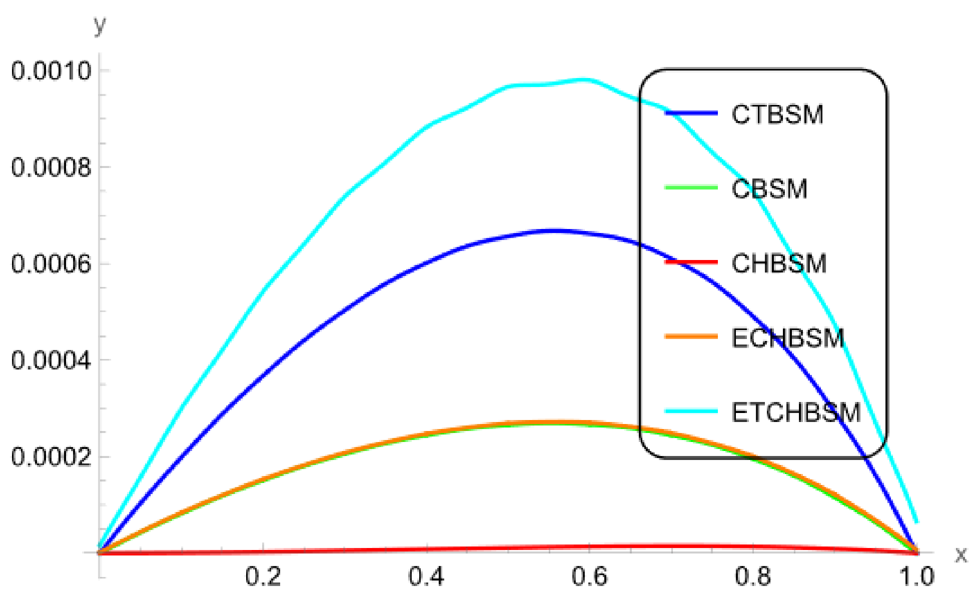

Figure 1: Error plots of the approximated analytical solution for Example 1 with $n = 10$ and $h=\frac{1}{10}$.

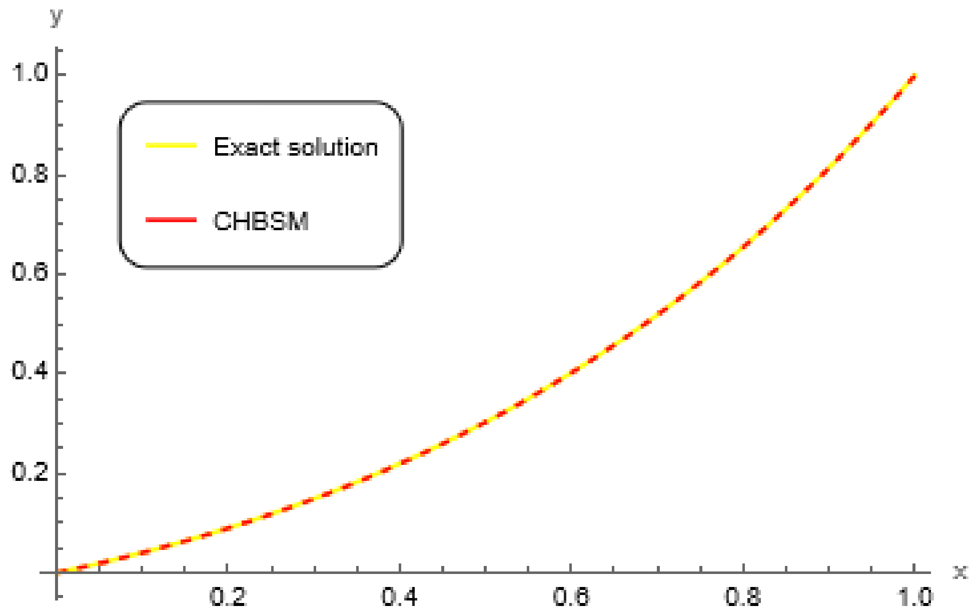

Figure 2: Plot of exact solution with approximated solution for Example 1 by CHBSM.

Example 2: $y''(x) + \frac{4x}{1+x^2}y'(x) + \frac{2}{1+x^2}y(x) = 0, \quad 0 \le x < 2$, with $y(0) = 1$ and $y(2) = \frac{1}{5}$. [29]

The analytical solution is $y(x) = \frac{1}{1+x^2}$. Table 3 presents the absolute errors and norms upon applying CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM to Example 2.

Table 3: Absolute errors and norms at collocation points using CBSM, CTBM, CHBSM, ECHBSM and ETCHBSM for Example 2 with n= 10 and h=$\frac{1}{5}$

x	CBSM [29]	CBSM	CTBSM	CHBSM ($\gamma = 1.98222$)	ECHBSM ($\lambda = -1.89774$, $\gamma = 0.02036$)	ETCHBSM ($\lambda = -1.89774$, $\gamma = 0.01364$)
0.2	1.52×10^{-3}	1.51882×10^{-3}	1.22065×10^{-3}	9.27793×10^{-4}	6.3540×10^{-1}	4.23380×10^{-1}
0.4	4.05×10^{-3}	4.04647×10^{-3}	3.13880×10^{-3}	2.24726×10^{-3}	3.31356×10^{-1}	4.84335×10^{-2}
0.6	5.34×10^{-3}	5.34080×10^{-3}	3.86910×10^{-3}	2.42357×10^{-3}	2.74447×10^{-1}	9.78919×10^{-3}
0.8	5.00×10^{-3}	5.00427×10^{-3}	3.24409×10^{-3}	1.51522×10^{-3}	3.51896×10^{-1}	7.47952×10^{-1}
1.0	3.84×10^{-3}	3.83821×10^{-3}	2.09235×10^{-3}	3.77544×10^{-4}	5.13389×10^{-1}	8.68052×10^{-1}
1.2	2.58×10^{-3}	2.57757×10^{-3}	1.06527×10^{-3}	4.20136×10^{-4}	7.36301×10^{-1}	6.33041×10^{-1}
1.4	1.55×10^{-3}	1.54831×10^{-3}	3.88934×10^{-4}	7.49821×10^{-4}	4.89627×10^{-1}	4.99794×10^{-1}
1.8	3.15×10^{-4}	3.14843×10^{-4}	5.31774×10^{-5}	4.14653×10^{-4}	3.12447×10^{0}	8.31428×10^{-1}
L_∞	5.30×10^{-3}	5.34080×10^{-3}	3.86910×10^{-3}	2.42357×10^{-3}	3.12447×10^{0}	1.35569×10^{0}
L_2	9.84×10^{-3}	9.83740×10^{-3}	6.51960×10^{-3}	3.95312×10^{-3}	3.62754×10^{0}	2.16202×10^{0}

Based on the results in Table 3, the corresponding maximum absolute error (L_∞) for CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM are found to be 5.34080×10^{-3}, 3.86910×10^{-3}, 2.42357×10^{-3}, 3.12447×10^{0} and 1.35569×10^{0}, respectively. The results of CBSM align with those in [29]. It can be observed that the approximations by CTBSM have slightly smaller errors compared to CBSM. However, CHBSM gives the smallest error compared to both CBSM and CTBSM, although the differences are very small. Table 3 also demonstrates that ETCHBSM provides better results compared to ECHBSM. Even so, CHBSM surpasses all the other methods.

For better visualisation, the error for the approximated analytical solution using CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM for Example 2 is presented in Figure 3. The plot of the approximated solution using CHBSM alongside the analytical solution is also presented in Figure 4.

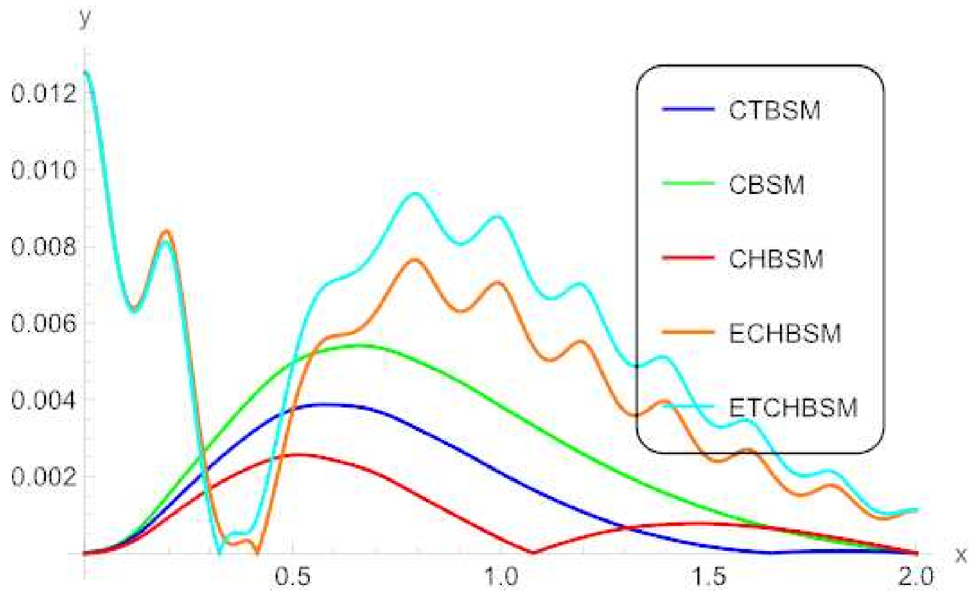

Figure 3: Error plots of the approximated analytical solution for Example 2 with $n = 10$ and $h = \frac{1}{5}$

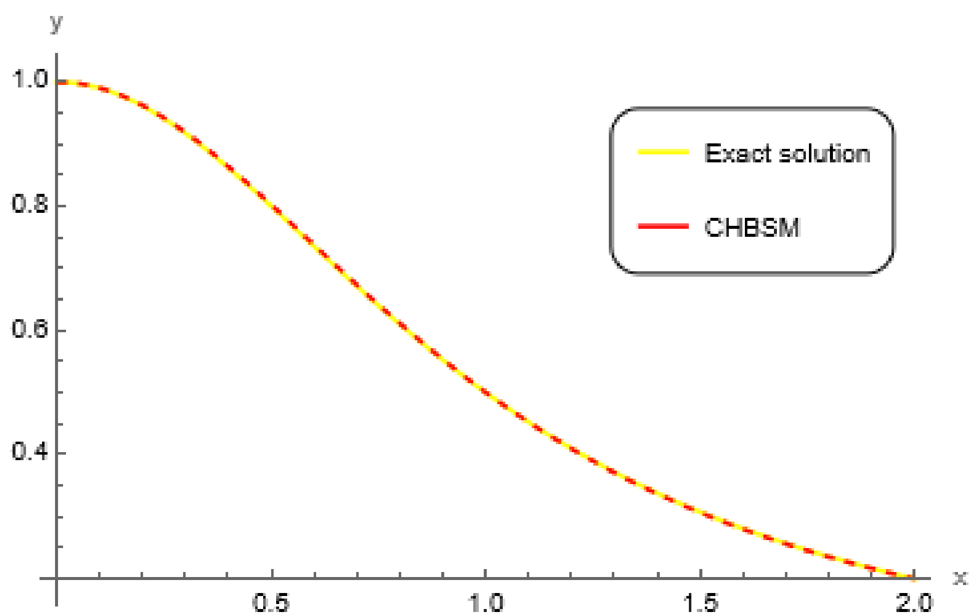

Figure 4: Plot of exact solution with approximated solution for Example 2 by CHBSM

Figure 3 clearly illustrates the irregular behaviour of the errors for ECHBSM and ETCHBSM. One possible reason for this behaviour is the unreliability of the optimization method proposed for ECHBSM and ETCHBSM. Meanwhile, the overlap of the approximated solution with the analytical solution in Figure 4 suggests that there are insignificant differences between both plots. However, from both figures, it is safe to conclude that CHBSM produces the best approximation for this problem.

Example 3: $y''(x) + 2\,y'(x) + 5\,y(x) = 6\cos(2x) - 7\sin(2x)$, $0 \leq x < \frac{\pi}{4}$, with $y(0) = 4$ and $y\left(\frac{\pi}{4}\right) = 1$. [30]

The analytical solution is $y(x) = 2\,(1 + e^{-x})\cos(2x) + \sin(2x)$.

Table 4: Absolute errors and norms at collocation points using CBSM, CTBM, CHBSM, ECHBSM and ETCHBSM for Example 3 with n= 20 and $h=\frac{\pi}{80}$

x	**CBSM [30]**	**CBSM**	**CTBSM**	**CHBSM** (γ=**0.14871**)	**ECHBSM** ($\lambda = 0.81650$, $\gamma = -0.000192$)	**ETCHBSM** ($\lambda = 0.81650$, $\gamma = -0.000787$)
$\frac{\pi}{80}$	0	1.4211×10^{-14}	4.1619×10^{-5}	1.13760×10^{-5}	2.0769×10^{-5}	4.2125×10^{-5}
$\frac{\pi}{40}$	2.0634×10^{-5}	2.0634×10^{-5}	7.6890×10^{-5}	1.96254×10^{-5}	3.6615×10^{-5}	7.7331×10^{-5}
$\frac{3\pi}{80}$	3.6486×10^{-5}	3.6486×10^{-5}	1.0598×10^{-4}	2.52119×10^{-5}	4.8252×10^{-5}	1.0636×10^{-4}
$\frac{\pi}{20}$	4.8129×10^{-5}	4.8129×10^{-5}	1.2911×10^{-4}	2.85565×10^{-5}	5.6216×10^{-5}	1.2945×10^{-4}
$\frac{\pi}{16}$	5.6010×10^{-5}	5.6010×10^{-5}	1.4659×10^{-4}	3.00401×10^{-5}	6.1005×10^{-5}	1.4688×10^{-4}
$\frac{3\pi}{40}$	6.0895×10^{-5}	6.0895×10^{-5}	1.5874×10^{-4}	3.00065×10^{-5}	6.3082×10^{-5}	1.5899×10^{-4}
$\frac{7\pi}{80}$	6.2978×10^{-5}	6.2978×10^{-5}	1.6594×10^{-4}	2.87656×10^{-5}	6.2876×10^{-5}	1.6616×10^{-4}
$\frac{\pi}{10}$	6.2779×10^{-5}	6.2779×10^{-5}	1.6860×10^{-4}	2.65967×10^{-5}	6.08786×10^{-5}	1.6879×10^{-4}
$\frac{9\pi}{80}$	6.0696×10^{-5}	6.0696×10^{-5}	1.6715×10^{-4}	2.37513×10^{-5}	5.7182×10^{-5}	1.6731×10^{-4}
$\frac{\pi}{8}$	5.7099×10^{-5}	5.7099×10^{-5}	1.6201×10^{-4}	2.04561×10^{-5}	5.2406×10^{-5}	1.6215×10^{-4}
$\frac{11\pi}{80}$	4.6707×10^{-5}	4.6707×10^{-5}	1.5362×10^{-4}	1.69162×10^{-5}	4.6776×10^{-5}	1.5375×10^{-4}
$\frac{3\pi}{20}$	4.0526×10^{-5}	4.0526×10^{-5}	1.4244×10^{-4}	1.33171×10^{-5}	4.0587×10^{-5}	1.4255×10^{-4}
$\frac{13\pi}{80}$	3.4059×10^{-5}	3.4059×10^{-5}	1.2888×10^{-4}	9.82776×10^{-6}	3.4113×10^{-5}	1.2898×10^{-4}
$\frac{7\pi}{40}$	2.7560×10^{-5}	2.7560×10^{-5}	1.1337×10^{-4}	6.60233×10^{-6}	2.7607×10^{-5}	1.1345×10^{-4}
$\frac{3\pi}{16}$	2.1267×10^{-5}	2.1267×10^{-5}	9.6306×10^{-5}	3.78222×10^{-6}	2.1306×10^{-5}	9.6376×10^{-5}
$\frac{\pi}{5}$	1.5397×10^{-5}	1.5397×10^{-5}	7.8066×10^{-5}	1.4977×10^{-6}	1.5429×10^{-5}	7.8125×10^{-5}
$\frac{17\pi}{80}$	1.0154×10^{-5}	1.0154×10^{-5}	5.9004×10^{-5}	1.3064×10^{-7}	1.0179×10^{-5}	5.9052×10^{-5}
$\frac{9\pi}{40}$	5.7265×10^{-6}	5.7265×10^{-6}	3.3944×10^{-5}	9.9109×10^{-7}	5.7443×10^{-6}	3.9483×10^{-5}
$\frac{19\pi}{80}$	2.2885×10^{-6}	2.2885×10^{-6}	1.9689×10^{-5}	9.7972×10^{-7}	2.2992×10^{-6}	1.9715×10^{-5}
L_∞	6.2978×10^{-5}	6.2978×10^{-5}	1.6860×10^{-4}	3.0040×10^{-5}	6.3082×10^{-5}	1.6879×10^{-4}
L_2	1.8733×10^{-4}	1.8733×10^{-4}	5.3528×10^{-4}	7.6599×10^{-5}	1.8770×10^{-4}	5.3605×10^{-4}

According to Table 4, the corresponding maximum absolute error (L_∞) for CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM are 6.2978×10^{-5}, 1.6860×10^{-4}, 3.0040×10^{-5}, 6.3082×10^{-5} and 1.6879×10^{-4}, respectively. The results of CBSM align with those in [30], and it was observed that CBSM produces slightly smaller error compared to CTBSM. Similar to Example 1 and Example 2, CHBSM has the smallest errors, yielding the most accurate approximation of the solution compared to the other four methods. This can be clearly seen in Figure 5. The plot of the analytical solution with the approximated solution using CHBSM for this example is also shown in Figure 6.

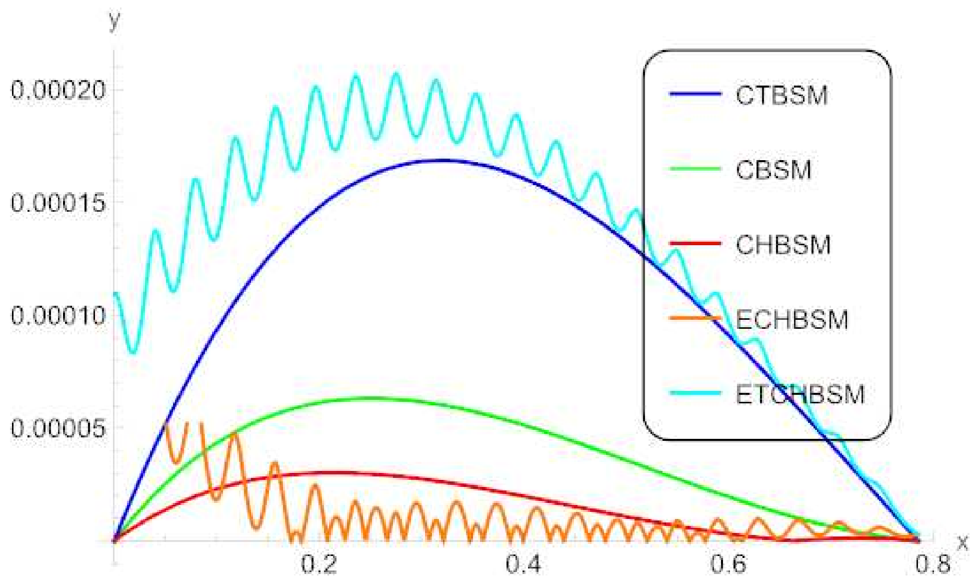

Figure 5: Error plots of the approximated analytical solution for Example 3 with $n = 20$ and $h = \frac{\pi}{80}$

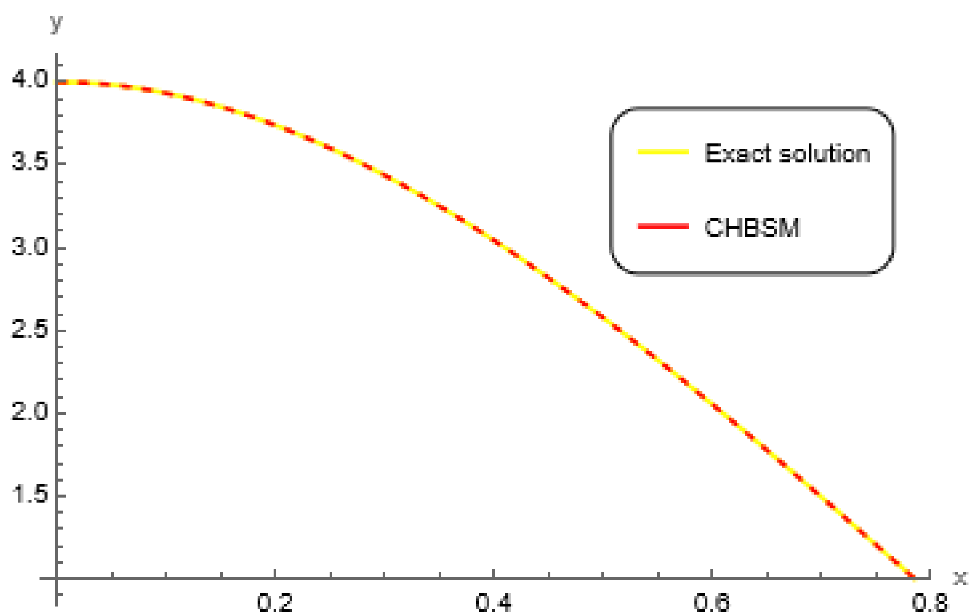

Figure 6: Error plots of the approximated analytical solution for Example 3 with $n = 20$ and $h = \frac{\pi}{80}$

From Table 2 to Table 4 and Figure 1 to Figure 6, it is evident that CHBSM has indeed provided more accurate approximations of the solutions compared to CBSM, CTBSM, ECHBSM and ETCHBSM. The presence of the free parameter γ contributes to this improved accuracy by offering greater flexibility in the approximations. Additionally, the optimization technique successfully enhances the numerical solutions for CHBSM. In summary, for all the examples, there exist values of γ that lead to better approximations of the solutions. For the purpose of comparison, the maximum absolute error (L_∞) and Euclidean norm (L_2) for Examples 1, 2, and 3 using CBSM, CTBSM, CHBSM, ECHBSM, and ETCHBSM, along with other methods from the literature, are summarized in Table 5.

Table 5: Absolute errors and norms for Examples 1, 2 and 3 using CBSM, CTBM, CHBSM, ECHBSM, ETCHBSM and other methods from literatures

Example	**Method**	**Maximum Absolute Error (L_∞)**	**Euclidean Norm (L_2)**
1	LSM [29]	3.66×10^{-7}	8.50×10^{-7}
	FDM [29]	2.66×10^{-4}	6.18×10^{-4}
	CBSM [29]	2.66×10^{-4}	6.20×10^{-4}
	CBSM	2.66489×10^{-4}	6.19963×10^{-4}
	CTBSM	6.62037×10^{-4}	1.53512×10^{-3}
	CHBSM ($\gamma = -0.70786$)	1.45553×10^{-5}	2.96136×10^{-5}
	ECHBSM ($\lambda = -0.0356$, $\gamma = -0.0486$)	8.11253×10^{-1}	1.29607×10^{0}
	ETCHBSM ($\lambda = -0.0356$, $\gamma = -0.07426$)	8.07113×10^{-1}	1.28330×10^{0}
2	LSM [29]	2.98×10^{-5}	3.97×10^{-5}
	FDM [29]	2.00×10^{-3}	4.18×10^{-3}
	CBSM [29]	5.30×10^{-3}	9.84×10^{-3}
	CBSM	5.34080×10^{-3}	9.83740×10^{-3}
	CTBSM	3.86910×10^{-3}	6.51960×10^{-3}
	CHBSM ($\gamma = 1.98222$)	2.42357×10^{-3}	3.95312×10^{-3}
	ECHBSM ($\lambda = -1.89774$, $\gamma = 0.02036$)	3.12447×10^{0}	3.62754×10^{0}
	ETCHBSM ($\lambda = 0.01364$, $\gamma = 0.02036$)	1.35569×10^{0}	2.16202×10^{0}
3	CBSM [30]	6.29781×10^{-5}	1.87331×10^{-4}
	CBSM	6.29781×10^{-5}	1.87331×10^{-4}
	CTBSM	1.68603×10^{-4}	4.07593×10^{-4}
	CHBSM ($\gamma = 0.14871$)	3.00401×10^{-5}	7.65993×10^{-5}
	ECHBSM ($\lambda = 0.81650$, $\gamma = -0.000192$)	6.3082×10^{-5}	1.8770×10^{-4}
	ETCHBSM ($\lambda = 0.81650$, $\gamma = -0.000787$)	1.6879×10^{-4}	5.3605×10^{-4}

From Table 5, the errors produced by LSM appear to be better than FDM and all the B-spline collocation methods presented. One reason is that LSM involves the use of the fourth order Runge-Kutta method, which yields a truncation error of $O\left(h^4\right)$. On the contrary, all the B-spline collocation methods and FDM, derived from finite difference approximation, have a truncation error of $O\left(h^2\right)$.

Even though the approximations by LSM are better than CBSM, CTBSM, CHBSM, ECHBSM and ETCHBSM, the approximation by CBSM is comparable to LSM when the subinterval, h is increased [29]. Therefore, the author concludes that the accuracy of CHBSM can be improved by having smaller step size, h. Additionally, a better optimization approach for the parameter γ can enhance the accuracy of the method, as well as for ECHBSM and ETCHBSM. Furthermore, B-spline collocation methods can provide continuous approximation in the domain of the problem, whereas numerical methods only give approximations at defined grid points. Therefore, CHBSM is the preferred method in approximating second order linear two-point BVPs.

4 Conclusion

In conclusion, the numerical results reveal that CHBSM is the preferred method for approximating second order linear BVPs compared to other spline methods. The primary reason for this is because CHBSM has a free parameter, γ, which can be optimized using the proposed method to provide more accurate approximations. As for future work, two major suggestions can be made. First, the optimization of the free parameter, γ can be further improved and explored to give more promising results for hybridization methods. This is supported by the finding that Extended Cubic B-spline Collocation Method (ECBSM) achieved more accurate solutions for second-order linear BVPs than CBSM and CTBSM [15]. Additionally, CTBSM was proven to offer better approximations compared to CBSM for certain equations [15]. Therefore, theoretically, the approximation using ETCHBSM would provide the best accuracy. However, the results obtained in this study showed otherwise, presumably due to the efficacy of the proposed optimization method. Lastly, due to the efficiency of CHBSM, this method can be applied to solve more complex multidimensional BVPs in future works. For instance, CHBSM can be applied to non-linear PDEs, such as those in fluid dynamics and heat transfer problems.

Acknowledgments

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