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Analytical Solution of Generalized Fractional Integro-Differential Equations via Shifted Gegenbauer Polynomials

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Abstract In this paper, we proposed an analytical solution for generalized fractional order integro-differential equations with non-local boundary conditions via shifted Gegenbauer polynomials as an approximating polynomial using the Galerkin method and collocation techniques involving operational matrix that make use of the Liouville-Caputo operator of differentiation in combination with Gegenbauer polynomials. Shifted Gegenbauer polynomial properties were exploited to transform fractional order integro-differential equation and its non-local boundary conditions into an algebraic system of equations. Shifted Gegenbauer polynomial $C_m^{(\alpha)}(x)$ was used in order to generate and generalize the results of some other orthogonal polynomials by varying the value of parameter α. The accuracy and effectiveness of the proposed method are tested on some selected examples from the literature. We observed that, when the exact solution is in polynomial form, the approximate solution gives a closed form solution, and non-polynomial exact solution, also give better results compared to the existing results in the literature.

Keywords Caputo fractional derivative; Gegenbauer polynomial; Fractional integro-differential equation; collocation method; Galerkin method

Mathematics Subject Classification 34A08, 34K28, 41A10, 45J05, 76M22.

1 Introduction

Fractional calculus represent strong tools in applied Mathematics to study a myriad of problems emanate from different fields of studies, such as science, engineering, Physics which gives rise to a progressive results in Mathematical Physics [1], hydrology, biophysics, finance, statistical mechanics, control theory, cosmology, bio-engineering. [2] presents synopsis of fractional calculus tools for characterising respiratory mechanics.

With the attraction of fractional calculus, researchers are giving attention to the studies of fractional differential equations and fractional order integro-differential equations (FOIEs). In the recent years, different approaches have been developed to solve FOIEs some of these methods are, [3] employed improved tau method to investigate the accuracy of multi-dimensional fractional Rayleigh-Stokes problem, [4] employed Taylor expansion method to find an approximate solution to FOIEs, [5] proposed Euler wavelet operational matrix method to solve nonlinear Volterra integro-differential equations, [6, 7] investigate the accuracy of fractional linear Volterra and Fredholm integro-differential equations using Laguerre polynomials as an approximating polynomials, [8] adopted Adomian decomposition method (ADM) to find an approximate solution to FIOEs. [9] employed Vieta-lucas polynomial as basis functions in obtaining the approximate solution of generalized fractional-order integro-differential equation, [10,11] investigate the convergence of the Jacobi spectral collocation method for the solution of FOIEs, collocation method with convergence for the generalized fractional integro-differential equation was studied in [12], [13] employed legendre wavelets method to solve FOIEs with weakly singular kernel, spline collocation method was used to solve fractional weakly singular integro-differential equations in [14]. Two-dimensional non-linear Volterra-Fredholm integro-differential equations was investigated using variational Adomian decomposition method in [15], [16] proposed modified Laplace decomposition method for solving fractional Volterra-Fredholm integro-differential equations, [17] developed adaptive huber method for weakly singular fractional intrgro-differential equations, analysis of the error involve in 1D Fredholm integro-differential equations was studied in [18] using Volterra-transformation method [19] proposed a method based on Laplace transform for finding an approximate solution to Fredholm-type integro-differential equation with Atangana-Baleanu fractional derivative in Caputo sense, [20] proposed that the unknown function $S^{\alpha} f(x)$ be written as a linear combination of new hybrid fractional function consisting of block-pulse functions and Fibonacci, then used collocation in the Newton-Cotes nodes to transform the integro-differential equation to algebraic equations which was solved using Newtons iterative method, [21] adopted pade approximation technique to solve fractional integro-differential equation with non-local boundary conditions. The main focus in this paper is to find the solution of generalized FOIEs via shifted Gegenbauer polynomials as an approximating polynomial using Galerkin method and collocation method, since Gegenbauer approach generalize the results of some commonly used orthogonal polynomials such as Legendre polynomial $P_n(x)$, shifted Chebyshev polynomials of certain kinds, shifted Jacobi polynomials $P_n^{\alpha,\beta}(x)$ with $\alpha = \beta$.

This paper is organized as follows. In Section 2, we give preliminaries of the proposed method, which include statement of the problem in 2.1, review of Gegenbauer polynomials are presented in 2.2 and Caputo fractional differentiation operator are presented in 2.3. We formulate the scheme for the proposed method in Section 3 for both Gegenbauer-Galerkin and Gegenbauer-collocaltion methods. In Section 4, numerical examples are presented with computational results and graphical representation to show the effectiveness and the accuracy of the proposed methods. Discussion of results and concluding remarks are given in the last two sections of the paper.

2 Preliminaries

2.1 Statement of the problem

In this paper, the problem under consideration is FOIEs of the form:

$$D^{\delta}y(x) = G(x)y(x) + f(x) + \nu_1 \int_0^x k_1(x,t)y(t)dt + \nu_2 \int_0^1 k_2(x,t)y(t)dt, \tag{1}$$

such that $k-1 < \delta \leq k$, $a \leq x \leq b$, $k \in \mathbb{N}$, together with non-local boundary conditions given by:

$$\sum_{j=1}^{k} \left(\alpha_{ij} y^{(j-1)}(a) + \gamma_{ij} y^{(j-1)}(b)\right) + \rho_i \int_a^b S_i(x)y(x)dx = k_i, \; i = 1, 2, \ldots, k, \tag{2}$$

where $S_i(x)$ is a continuous function, $G(x), f(x), k_1(x,t), k_2(x,t)$ are holomorphic functions, $\nu_1, \nu_2, \alpha_{ij}, \gamma_{ij}$, ρ_i and k_i are constants, D^{δ} is the fractional derivative operator of order δ and $y(x)$ is the unknown function. If $\nu_1 = 0$ or $\nu_2 = 0$, Eq. (1) is becomes fractional Fredholm or Volterra integro-differential equation, respectively. Here, we try to solve FOIEs by transforming the equation into system of algebraic equations using Galerkin method and collocation methods via Gegenbauer polynomials as an approximation.

2.2 Shifted Gegenbauer polynomials

Gegenbauer polynomials $C_m^{(\alpha)}(u), u \in [-1,\ 1]$ with respect to the weight function $\omega(u) = (1-u^2)^{(\alpha-\frac{1}{2})}$ is defined as:

$$C_m^{(\alpha)}(u) = \sum_{n=0}^{m} \frac{(-1)^n \Gamma(2\alpha + 2m - n)\Gamma(\alpha + \frac{1}{2})}{(m-n)!\Gamma(2\alpha)\Gamma(n+1)\Gamma(m-n+\alpha+\frac{1}{2})} u^{m-n}. \tag{3}$$

and the recurrence relation is:

$$C_{m+1}^{(\alpha)}(u) = \frac{1}{m+1}\left[2(m+\alpha)uC_m^{(\alpha)}(u) - (m+2\alpha-1)C_{m-2}^{(\alpha)}(u)\right], m \geq 1, \tag{4}$$

where $C_0^{(\alpha)}(u) = 1$, $C_1^{(\alpha)}(u) = 2\alpha u$.

The corresponding shifted Gegenbauer polynomial $C_m^{(\alpha)*}(u), u \in [a,b]$ is derived using the transformation $x = \frac{2u-(a+b)}{b-a}$ as:

$$C_{m+1}^{(\alpha)*}(u) = \frac{1}{m+1}\left[2(m+\alpha)\left(\frac{2u-(a+b)}{b-a}\right)C_m^{(\alpha)*}(u) - (m+2\alpha-1)C_{m-1}^{(\alpha)*}(u)\right], m \geq 1 \tag{5}$$

where $C_0^{(\alpha)*}(u) = 1$, $C_1^{(\alpha)*}(u) = 2\alpha\left(\frac{2u-(a+b)}{b-a}\right)$.

The analytic form corresponding to Eq. (5) in the interval $u \in [0,1]$ is given as:

$$C_m^{(\alpha)*}(u) = \sum_{n=0}^{m} \frac{(-1)^n \Gamma(2\alpha+2m-n)\Gamma(\alpha+\frac{1}{2})}{(m-n)!\Gamma(n+1)\Gamma(m-n+\alpha+\frac{1}{2})\Gamma(2\alpha)} u^{m-n}, \tag{6}$$

with the following orthogonality condition:

$$\langle C_m^{(\alpha)*}(u),\ C_n^{(\alpha)*}(u)\rangle = \int_0^1 \left(u-u^2\right)^{\left(\alpha-\frac{1}{2}\right)} C_m^{(\alpha)*}(u)C_n^{(\alpha)*}(u)du = \begin{cases} 0, & \text{for } m \neq n \\ \frac{\pi 2^{1-4\alpha}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2(n+\alpha)}, & \text{for } m = n \end{cases}. \tag{7}$$

See [22, 23] for more details.

2.3 Caputo fractional differentiation operator

The Caputo fractional differentiation operator D^δ, of order δ is defined as follows:

$$D^\delta g(u) = \frac{1}{\Gamma(n-\delta)}\int_0^u \frac{g^{(i)}(u)}{(u-t)^{\delta+1-i}}dt, \delta > 0,\ i-1 < \delta < i, i \in \mathbb{N}, \tag{8}$$

with the linearity property:

$$D^\delta(\sigma f(u) + \varsigma g(u)) = \sigma D^\delta f(u) + \varsigma D^\delta g(u), \tag{9}$$

where, σ and ς are constants. The following results are obtained:

$$D^\delta u^n = \begin{cases} 0,\ \text{for } \ n \in \mathbb{N}_0\ ,\ n < \lceil\delta\rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}u^{n-\delta},\ \text{for } n \in \mathbb{N}_0\ ,\ n \geq \lceil\delta\rceil \end{cases}, \tag{10}$$

where $\lceil\delta\rceil$ is the smallest integer greater than or equal to δ.

Theorem 2.1 *Let $C_m^{(\alpha)*}(u), u \in [0,\ 1]$ be a Gegenbauer polynomial of order m, then the Caputo fractional derivative of $C_m^{(\alpha)*}(u)$ in terms of Gegenbauer polynomials is:*

$$D^\delta\left(C_m^{(\alpha)*}(u)\right) = \sum_{n=0}^{m-\lceil\delta\rceil} \frac{(-1)^n\Gamma(2\alpha+2m-n)\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+1)\Gamma(m-n+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(m+1-n-\delta)}u^{m-n-\delta}. \tag{11}$$

See [23, 24] for the proof.

Theorem 2.2 *Let the fractional derivative of $S(u)$ of order N be expressed in terms of Gegenbauer polynomials, that's*

$$D^\delta\left(S_N(u)\right) = \sum_{m=0}^{N} \mu_m D^\delta\left(C_m^{(\alpha)*}(u)\right), \tag{12}$$

then

$$D^\delta\left(S_N(u)\right) = \sum_{m=\lceil\delta\rceil}^{N}\sum_{n=0}^{m-\lceil\delta\rceil} \mu_m H_{m,n}u^{m-n-\delta}, \tag{13}$$

where

$$H_{m,n} = \frac{(-1)^n\Gamma(2\alpha+2m-n)\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+1)\Gamma(m-n+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(m+1-n-\delta)}. \tag{14}$$

Proof

Applying theorem 2.1 to Eq. (12), we obtain

$$D^{\delta}(S_N(u)) = \sum_{m=\lceil\delta\rceil}^{N} \sum_{n=0}^{m-\lceil\delta\rceil} \mu_m \frac{(-1)^n \Gamma(2\alpha+2m-n)\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+1)\Gamma(m-n+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(m+1-n-\delta)} u^{m-n-\delta}. \quad (15)$$

After simplification, Eq. (15) becomes:

$$D^{\delta}(S_N(u)) = \sum_{m=\lceil\delta\rceil}^{N} \sum_{n=0}^{m-\lceil\delta\rceil} \mu_m H_{m,n} u^{m-n-\delta}, \quad (16)$$

where

$$H_{m,n} = \frac{(-1)^n \Gamma(2\alpha+2m-n)\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+1)\Gamma(m-n+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(m+1-n-\delta)}. \quad (17)$$

3 Formulation of the Scheme

In this section, we give the technique involved in the formulation of the proposed methods for both the Gegenbauer-Galerkin method and Gegenbauer-collocation method as follows:

3.1 Gegenbauer-Galerkin method

Considering Eq. (1) together with the non-local boundary conditions given in Eq. (2), the approximate solution $y_N(x)$ corresponding to the exact solution $y(x)$ using Galerkin's method are derived as follows:

$$y_N(x) = \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(x). \quad (18)$$

Substituting Eq. (18) in Eq. (1), then apply theorem 2.1 to the fractional part, we obtain

$$\begin{aligned}\sum_{m=\lceil\delta\rceil}^{N} \sum_{n=0}^{m-\lceil\delta\rceil} \mu_m H_{m,n} x^{m-n-\delta} =& G(x) \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(x) + f(x) + \nu_1 \int_0^x k_1(x,t) \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t) dt \\ &+ \nu_2 \int_0^1 k_2(x,t) \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t) dt.\end{aligned} \quad (19)$$

Multiply both sides of Eq. (19) by the shifted Gegenbauer polynomials $C_j^{(\alpha)*}(x), j = \lceil\delta\rceil, \lceil\delta\rceil + 1, \ldots, N$, then integrate the resulting equation in the interval $[a, b]$, we obtain:

$$\begin{aligned}&\int_a^b \Bigg[\sum_{m=\lceil\delta\rceil}^{N} \sum_{n=0}^{m-\lceil\delta\rceil} \mu_m H_{m,n} x^{m-n-\delta} - G(x) \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(x) - \nu_1 \int_0^x k_1(x,t) \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t) dt \\ &-\nu_2 \int_0^1 k_2(x,t) \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t) dt \Bigg] C_j^{(\alpha)*}(x) dx = \int_a^b f(x) C_j^{(\alpha)*}(x) dx.\end{aligned} \quad (20)$$

Putting Eq. (20) in matrix form, we have:

$$\Upsilon = \mu\chi, \tag{21}$$

where Υ is a $(N - \lceil\delta\rceil + 1) \times (N+1)$ matrix, μ and χ are column matrices of $(N+1) \times 1$. The remaining equations are derived from the non-local boundary conditions, that's:

$$\begin{aligned}&\sum_{j=1}^{k}\left(\alpha_{ij}\left[\sum_{m=0}^{N}\mu_m \frac{d^{j-1}}{dx^{j-1}}C_m^{(\alpha)*}(x)\right]_{x=a} + \gamma_{ij}\left[\sum_{m=0}^{N}\mu_m \frac{d^{j-1}}{dx^{j-1}}C_m^{(\alpha)*}(x)\right]_{x=b}\right)\\ &+ \rho_i \int_a^b S_i(x)\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(x)dx = k_i,\ \ i = 1, 2, \ldots, k,\end{aligned} \tag{22}$$

to have $(N + 1)$ equations.

3.2 Gegenbauer-collocation method

Collocating Eq. (19) at $x = x_j, j = 0, 1, \ldots, (N+1-\lceil\delta\rceil)$ using the zeros of shifted Gegenbauer polynomial $C_N + 1 - \lceil\delta\rceil^{(\alpha)*}(x)$, we obtain:

$$\begin{aligned}\sum_{m=\lceil\delta\rceil}^{N}\sum_{n=0}^{m-\lceil\delta\rceil}\mu_m H_{m,n} x_j^{m-n-\delta} =& G(x_j)\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(x_j) + f(x_j) + \nu_1\int_0^{x_j} k_1(x,t)\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(t)dt\\ &+ \nu_2\int_0^1 k_2(x_j,t)\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(t)dt,\end{aligned} \tag{23}$$

with non-local boundary conditions giving in Eq. (22) we obtain $(N + 1)$ algebraic equations with $(N + 1)$ unknowns, which will be then be solved to obtain the $\mu_m, m = 0, 1, \ldots, N$ and subsequently the approximate solution $y_N(x)$.

3.3 Convergence and Stability Analysis

Theorem 3.1 *Assume y,v* $\in$ *s if* $(s, \parallel . \parallel)$ *is a Banach Space* Ψ*:* $s \to s$ *satisfying*

$$\parallel \Psi y - \Psi v \leq L \parallel y - \Psi v \parallel + \chi \parallel y - v \parallel \tag{24}$$

where $L \geq 0$*,* $0 \leq \chi \leq 1$*. Suppose that* Ψ *has fixed point. Then* Ψ *is Picard* Ψ*-stable*

Lemma 3.2 *If the function f(x) is continuous function then, a FOIE Eq.(1) is equivalent to the integral equation*

$$\begin{aligned}y(x) = k_i + \frac{1}{\Gamma(\delta)}\int_0^x (x-s)^{\delta-1}\left[G(s)y(s) + f(s)\right]ds + \frac{\nu_1}{\Gamma(\delta)}\int_0^x (x-s)^{\delta-1}\left(\int_0^s k_1(x,w)y(w)dw\right)ds\\ + \frac{\nu_2}{\Gamma(\delta)}\int_0^1 (x-s)^{\delta-1}\left(\int_0^s k_2(x,w)y(w)dw\right)ds, \quad for \ \ x \in [0,1].\end{aligned} \tag{25}$$

For $m,\ n \in \mathbb{N}$ if

$$\begin{aligned}\Psi y_m &= \nu_1 \int_0^x k_1(x,t) y_m(t)dt + \nu_2 \int_0^1 k_2(x,t) y_m(t)dt + G(x) y_m(x) \text{and}\\ \Psi y_n &= \nu_1 \int_0^x k_1(x,t) y_n(t)dt + \nu_2 \int_0^1 k_2(x,t) y_n(t)dt + G(x) y_n(x),\end{aligned} \tag{26}$$

then,

$$\begin{aligned}\parallel \Psi y_m - \Psi y_n \parallel =& \parallel \nu_1 \int_0^x k_1(x,t) y_m(t)dt + \nu_2 \int_0^1 k_2(x,t) y_m(t)dt + G(x) y_m(x)\\ &- \nu_1 \int_0^x k_1(x,t) y_n(t)dt - \nu_2 \int_0^1 k_2(x,t) y_n(t)dt - G(x) y_m(x) \parallel\\ &\leq |\nu_1| \left[\int_0^x \int_0^x k_1^2(x,s)dsdx\right]^{\frac{1}{2}} \parallel y_m(x) - y_n(x) \parallel\\ &\leq |\nu_2| \left[\int_0^1 \int_0^1 k_2^2(x,s)dsdx\right]^{\frac{1}{2}} \parallel y_m(x) - y_n(x) \parallel + G(x) \parallel y_m(x) - y_n(x) \parallel\end{aligned} \tag{27}$$

Therefore if $|\nu_1| < \frac{1}{\xi_1}$ and $|\nu_2| < \frac{1}{\xi_2}$, where

$$\begin{aligned}\xi_1 &= \left[\int_0^x \int_0^x k_1^2(x,s)dsdx\right]^{\frac{1}{2}}, \quad \text{and}\\ \xi_2 &= \left[\int_0^1 \int_0^1 k_2^2(x,s)dsdx\right]^{\frac{1}{2}},\end{aligned} \tag{28}$$

then, the mapping Ψ has a fixed point. In particular , for $L = 0$ and

$$\chi = |\nu_1| \left[\int_0^x \int_0^x k_1^2(x,s)dsdx\right]^{\frac{1}{2}} + |\nu_2| \left[\int_0^1 \int_0^1 k_2^2(x,s)dsdx\right]^{\frac{1}{2}} + G(x).$$

Then, Theorem 3.1 hold for mapping Ψ and hence it is Ψ- stable

Definition 1 *The function $y,\ v \in C^1[0,1], \mathbb{R}$ satisfying the lipschitz conditions if there exist real constant $\eta > 0$ such that*

$$|y(x) - v(x)| \leq \eta |y - v| \tag{29}$$

Theorem 3.3 *Assume*

$$\begin{aligned}Ty(x) &= G(x)y(x) + f(x) + \nu_1 \int_0^x k_1(x,t)y(t)dt, \quad \textit{and}\\ Ty(x) &= G(x)y(x) + f(x) + \nu_2 \int_0^1 k_2(x,t)y(t)dt,\end{aligned} \tag{30}$$

then Eq. (1) converges if and only if the two hypothesis are satisfied *(i)* $\langle Ty(x) - Tv(x), y(t) - v(t)\rangle \leq \alpha|y(x) - v(x)|^2$

(ii) *for* $\alpha > 0 \forall\ y,\ v \in C[0,1]$ *there exist a constant* $C(\varphi) > 0$ *such that*

$$\| y \| \leq \varphi \| v \| \leq \varphi,$$

we have

$$\langle T(y) - T(v), w\rangle \leq C\varphi \| y - v \| \| w \|,$$

for every $w \in [0,1]$

Proof

Applying to Eq. (1), we obtain:

$$\begin{aligned} T(y) - T(v) =& G(x)y(x) - G(x)v(x) + \nu_1 \int_0^x k_1(x,t)[y(t) - v(t)]dt \\ &+ \nu_2 \int_0^1 k_2(x,t)[y(t) - v(t)]dt. \end{aligned} \tag{31}$$

The inner product corresponding to Eq. (31) is given by:

$$\begin{aligned} \langle T(y) - T(v), (y - v)\rangle =& \langle G(x)y(x) - G(x)v(x) + \nu_1 \int_0^x k_1(x,t)[y(x) - v(x)]dt \\ &+ \nu_2 \int_0^1 k_2(x,t)[y(x) - v(x)]dt, (y - v)\rangle. \end{aligned} \tag{32}$$

Hence, by using the Schwartz inequality and Eq.(29), we have

$$\begin{aligned} \langle T(y) - T(v)\rangle \leq& (\xi \| y - v \| + \sup_{0\leq x\leq 1} k_1(x,t)\eta \| y - v \|)(\| y - v \|) \\ &+ |\nu_2| \sup_{0\leq x\leq 1} k_2(x,t)\eta \| y - v \| (\| y - v \|). \end{aligned}$$

This implies,

$$(T(y) - T(v), (y - v)) \leq k \| y - v \|^2 .$$

On the other hand,

$$\begin{aligned} \langle T(y) - T(v), w\rangle =& \langle G(x)y(x) - G(x)v(x) + \nu_1 \int_0^x k_1(x,t)[y(t) - v(t)]dt \\ &+ \nu_2 \int_0^1 k_2(x,t)[y(t) - v(t)]dt, w\rangle. \end{aligned} \tag{33}$$

Again we have,

$$\langle T(y) - T(v), w\rangle \leq C(\varphi) \| y - v \| \| w \| .$$

4 Numerical Examples

The implementation of the proposed method is presented here, by solving some selected examples from the literature with polynomial and non-polynomial exact solutions. We compute the maximum absolute error χ_N for each problem considered and compare with the existing one in the literature, where:

$$\chi_N = \max_{0\leq i\leq 100} |y(x_i) - y_N(x_i)|, \quad x_i = a + ih. \tag{34}$$

Example 4.1

Consider the following FOIE [21, 25].

$$D^{\frac{1}{2}}y(x) + \frac{x^2}{3}e^x y(x) = \frac{\sqrt{x}}{\Gamma(1.5)} - \frac{1}{2}x^2 + e^x \int_0^x ty(t)dt + \int_0^1 x^2 y(t)dt, \tag{35}$$

with non-local condition:

$$y(0) + y(1) - 3\int_0^1 ty(t)dt = 0, \tag{36}$$

and exact solution $y(x) = x$.

Applying the procedure explained in section 3, by seeking an approximate solution of the form (18), then substitute in Eq. (35), we obtain equation of the form (19), that's:

$$\begin{aligned}\sum_{m=\lceil\delta\rceil}^{N}\sum_{n=0}^{m-\lceil\delta\rceil} \mu_m H_{m,n} x^{m-n-\delta} + \frac{x^2}{3}e^x\left(\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(x)\right) = \frac{\sqrt{x}}{\Gamma(1.5)} - \frac{1}{2}x^2 \\ + e^x\int_0^x t\left(\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(t)\right)dt + \int_0^1 x^2\left(\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(t)\right)dt\end{aligned} \tag{37}$$

where $\delta = \frac{1}{2}, \mu_m,\ m = 0, 1, \ldots, N$ are unknowns to be determined, N is the degree of approximation. $H_{m,n}$ is given in Eq. (17).

Now for the Gegengauer-Galerkin method, we multiply Eq. (37) by $C_j^{(\alpha)*}(x), j = \lceil\delta\rceil, \lceil\delta\rceil + 1, \ldots, N$, then integrate the resulting equation in the interval $[0, 1]$, we obtain:

$$\begin{aligned}&\int_0^1\left[\sum_{m=\lceil\delta\rceil}^{N}\sum_{n=0}^{m-\lceil\delta\rceil}\mu_m H_{m,n}x^{m-n-\delta} + \frac{x^2}{3}e^x\left(\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(x)\right) - e^x\int_0^x t\left(\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(t)\right)dt\right. \\ &\left. - \int_0^1 x^2\left(\sum_{m=0}^{N}\mu_m C_m^{(\alpha)*}(t)\right)dt\right]C_j^{(\alpha)*}(x)dx = \int_0^1\left[\frac{\sqrt{x}}{\Gamma(1.5)} - \frac{1}{2}x^2\right]C_j^{(\alpha)*}(x)dx.\end{aligned} \tag{38}$$

To obtain the $\mu_m, m = 0, 1, \ldots, N$, we solve Eq. (38) together with the attached condition (36), that's:

$$y(0) + y(1) - 3\int_0^1 ty(t)dt = 0 = \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(0) + \sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(1) - 3\int_0^1 t\left(\sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t)\right) dt$$
$$\Rightarrow \sum_{m=0}^{N} \mu_m \frac{(-1)^m \Gamma(m+2\alpha)}{m!\Gamma(2\alpha)} + \sum_{m=0}^{N} \mu_m \frac{\Gamma(m+2\alpha)}{m!\Gamma(2\alpha)} - 3\int_0^1 t\left(\sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t)\right) dt = 0. \tag{39}$$

Again, for the Gegenbauer-collocation method, we collocate Eq. (37) at $x = x_j, j = 0, 1, \ldots (N + 1 - \lceil\delta\rceil)$ using the roots of shifted Gegenbauer polynomial $C_{N+1-\lceil\delta\rceil}^{(\alpha)*}(x)$ and we obtain:

$$\sum_{m=\lceil\delta\rceil}^{N} \sum_{n=0}^{m-\lceil\delta\rceil} \mu_m H_{m,n} x_j^{m-n-\delta} + \frac{x^2}{3} e^{x_j} \left(\sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(x_j)\right) = \frac{\sqrt{x_j}}{\Gamma(1.5)} - \frac{1}{2}x_j^2$$
$$+ e^{x_j} \int_0^{x_j} t\left(\sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t)\right) dt + \int_0^1 x_j^2 \left(\sum_{m=0}^{N} \mu_m C_m^{(\alpha)*}(t)\right) dt. \tag{40}$$

Eq. (40) together with the non-local boundary condition (39) gives $(N + 1)$ unique equations, which we then solved to obtain the unknowns $\mu_m, m = 0, 1, \ldots, N$, then substituted in Eq. (18) to obtain the approximate solution.

This example was solved in [21] and obtained their approximate solution using Pade approximations with maximum absolute error of 8.69×10^{-5}. Also [25] used Bernstein polynomials as an approximating polynomial and obtained 4.90×10^{-11} as their maximum absolute error but in our proposed method, we obtain exact solution using both Gegenbauer-Galerkin method and Gegenbauer-collocation method. Figure 1 shows the exact solution and its corresponding approximate at $N = 4$.

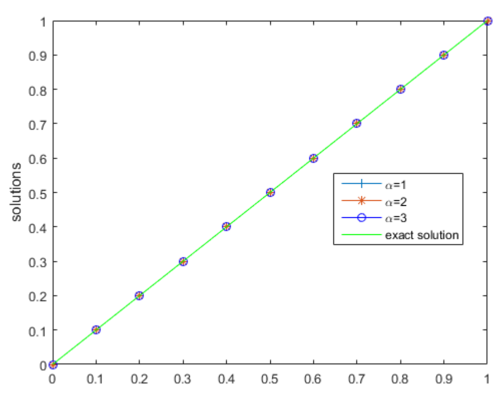

Figure 1: The exact and approximate solutions at various values of α for Example 4.1

Example 4.2

Consider the following problem [25].

$$D^{\frac{1}{3}}y(x) - \int_0^x x^2 \exp(xt)y(t)dt = \frac{3}{2}\frac{x^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} - 1 + \exp(x^2) - x^2 \exp(x^2), \tag{41}$$

subject to non-local boundary condition

$$y(0) + 2y(1) + 3\int_0^1 ty(t)dt = 3, \tag{42}$$

with the exact solution $y(x) = x$.

The problem here was solved using the approach explained in section 3. The same example was solved by [25] using Bernstein approximation and obtained 3.31×10^{-7} as maximum absolute error while we obtain exact solution using both Gegenbauer-Galerkin method and Gegenbauer-collocation method. Figure 2 shows the relationship between proposed method and the exact solution.

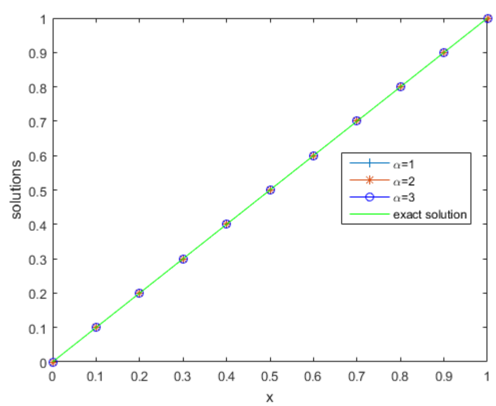

Figure 2: The exact and approximate solutions at various values of α for Example 4.2

Example 4.3

Consider the following FOIE [25].

$$\begin{aligned} &D^{\frac{1}{2}}y(x) + \int_0^x ty(t)dt + \int_0^1 t^2y(t)dt = (\operatorname{erf}(\sqrt{x}) + x - 1)\exp(x) + \exp(1) - 1, \\ &y(0) - \int_0^1 ty(t)dt = 0, \end{aligned} \tag{43}$$

The exact solution is $y(x) = \exp(x)$.

Table 1: Absolute errors for Example 4.3 using $\alpha = 2$

x	PM ($\alpha = 2$)	[25]
0	4.67×10^{-6}	1.82×10^{-5}
0.2	3.72×10^{-4}	1.05×10^{-4}
0.4	3.36×10^{-5}	2.91×10^{-5}
0.6	4.85×10^{-4}	2.85×10^{-5}
0.8	3.21×10^{-5}	1.50×10^{-5}
1.0	2.79×10^{-6}	4.20×10^{-6}

The results obtained in Example 4.3 using the proposed methods (PM) are in good agreement with the results obtained in [25], as the two give the same degree of accuracy in the their maximum absolute errors, as shown in Table 1. Figure 3 shows the exact and its corresponding approximate solution.

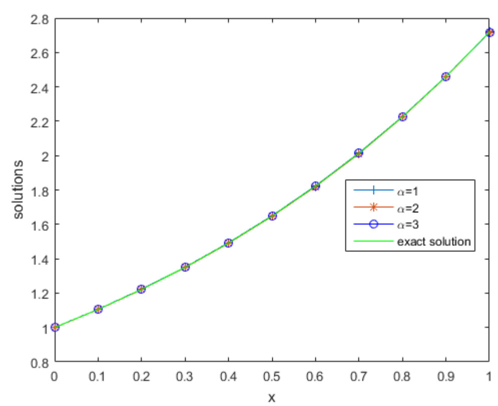

Figure 3: The exact and approximate solutions at various values of α for Example 4.3

Example 4.4

Consider the following FOIE: (see [7, 26])

$$\begin{aligned} &D^{\frac{1}{2}} y(x) = f(x) - \int_0^1 xty(t)dt, \\ &y(0) = 0, \end{aligned} \tag{44}$$

where

$$f(x) = \frac{\frac{8x^{3/2}}{3} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12}.$$

The exact solution is $y(x) = x^2 - x$.

[26] got an approximate solution with the maximum absolute error of 4.10×10^{-5} and 1.10×10^{-4} using standard least squares method and the perturbed least squares method, respectively. In our proposed method, we obtain exact solution and [7] obtained exact solution. Figure 4 display the graph of exact solution and its corresponding approximant.

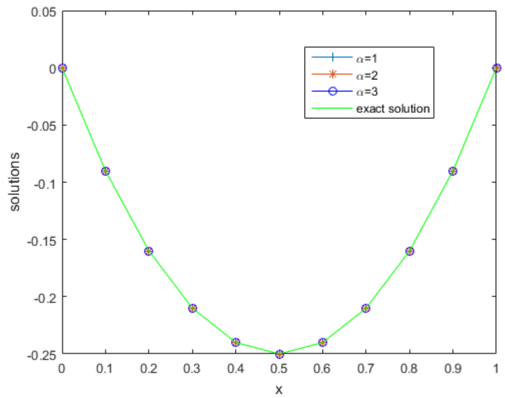

Figure 4: Relationship between the exact and approximate solutions at various values of α for Example 4.4

Example 4.5

Consider the following FOIE (see [6]).

$$D^{\sqrt{3}}y(x) = \frac{2}{\Gamma(3-\sqrt{3})}x^{2-\sqrt{3}} + 2\sin(x) - 2x + \int_0^x \cos(x-t)y(t)dt,$$
$$\text{subject to} \;\; y(0) = 0, \;\; y'(0) = 0,$$

with the exact solution $y(x) = x^2$.

The exact solution was obtained using the proposed method (PM) which is in good agreement with the result reported in [6]. Figure 5 displays the exact and its corresponding approximate at various values of α.

Example 4.6

Consider the following FOIE with weakly singular kernel [27]:

$$D^{\frac{1}{4}}y(x) - \frac{1}{2}\int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}}dt - \frac{1}{3}\int_0^1 (x-t)y(t)dt = f(x)$$
$$y(0) = 0, \;\; \text{with exact solution} \;\; y(x) = x^2 + x^3$$
$$\text{and } f(x) = \frac{\Gamma(3)}{\Gamma\left(\frac{11}{4}\right)}x^{\frac{7}{4}} + \frac{\Gamma(4)}{\Gamma\left(\frac{15}{4}\right)}x^{\frac{11}{4}} - \frac{\sqrt{\pi}\Gamma(3)}{2\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}} - \frac{7}{36}x + \frac{3}{20}$$

Table 2 is the absolute errors at $\alpha = 1$ and $\alpha = 2$ as it compares with the results obtained in [27]. Figure 6 is the corresponding figure.

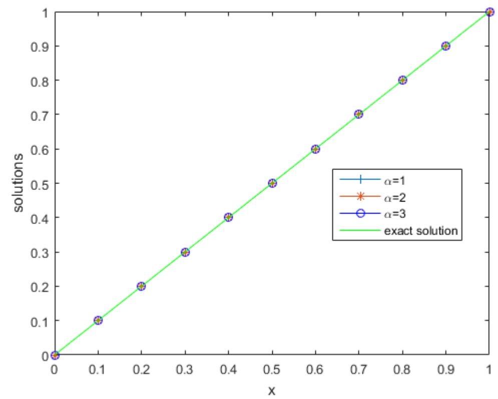

Figure 5: The exact and approximate solutions at various values of α for Example 4.5

Table 2: Absolute errors for Example 4.3 using $\alpha = 2$

x	$\alpha = 1$	$\alpha = 2$	[27]
0	0	0	1.44×10^{-4}
$\frac{1}{6}$	4.13×10^{-5}	4.10×10^{-5}	2.26×10^{-4}
$\frac{1}{3}$	2.71×10^{-4}	2.67×10^{-4}	5.98×10^{-4}
$\frac{1}{2}$	6.89×10^{-4}	6.81×10^{-4}	1.13×10^{-3}
$\frac{2}{3}$	1.13×10^{-3}	1.10×10^{-3}	1.50×10^{-3}
$\frac{5}{6}$	1.21×10^{-3}	1.17×10^{-3}	2.21×10^{-3}

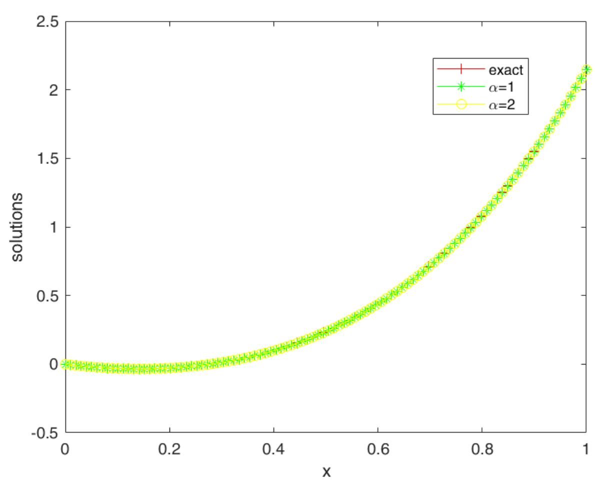

Figure 6: The exact and approximate solutions at various values of α for Example 4.6

5 Discussion of Results

Table 1 and Table 2 depict the maximum absolute errors obtained, for Example 4.3 and Example 4.6, respectively. We observed that the results obtained from this example are in good agreement with the results obtained in the literature (such as Laguerre polynomials and Bernstein polynomials as approximations), as both give the same degree of accuracy in their maximum absolute error. In Examples $4.1, 4.2, 4.4$, and 4.5, we obtained an exact solution that either gave better results or was in good agreement with the results obtained in the literature as explained in each example. We observed that the approximate solutions to all the examples with polynomial exact solutions produced the exact results except Example 4.6 with weakly singular kernel, although good approximation was still produced. Figures 1 until Figure 6 are the plots of the exact solution and their corresponding approximate solutions.

6 Conclusion

In this paper, we proposed Gegenbauer-Galerkin method and Gegenbauer-collocation method for solving fractional integro-differential equations using shifted Gegenbauer polynomial as an approximation. We used the two methods separately to transform the integro-differential equations into a system of algebraic linear equations and the fractional part of the integro-differential equation was removed using Caputo properties and the derived theorems. The equations were solved together with the non-local boundary conditions to obtain the unknown coefficients $\mu_m, m = 0, 1, \ldots, N$ and subsequently, the approximate solutions at various values of α reflecting in the Gegenbauer polynomials. For experiment, we used $\alpha = 1,\ 2,\ 3$ for the shifted Gegenbauer polynomials $C_i^{(\alpha)}(x), i \geq 0$. The methods are implemented on some selected examples from the literature. We obtained the exact solutions in examples with polynomial exact solution and better results to the example with non polynomial exact solution. Obviously, from the numerical results, the proposed methods is effective, robust because it generate results of some other orthogonal polynomials like Legendre polynomial $P_i(x)$ when $\alpha = \frac{1}{2}$, second kind Chebyshev polynomial $U_i(x)$ when $\alpha = 1$ and so on. From the tables of results and figures, the proposed methods are effective and accurate.

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