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Exploring Simultaneous Equation in Univariate Unconstrained Problems: Intersection Strategies

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Abstract The Steepest Descent Intersect (SDI) is an algorithm developed to solve unconstrained optimization problems, particularly those with multiple local minima or maxima. The algorithm uses simultaneous equation techniques to find the horizontal line-objective function intersection, generating a promising initial point that converges to a local minimum. This process continues until no intersection occurs, indicating the current solution has reached the lowest position, which represents the global solution. SDI can build a "bridge" within the valley for multimodal and wavy functions, allowing it to escape from the valley bottom and increase its chances of finding the global minimum. The steepest descent is sensitive to step size, with large steps causing large errors and small steps slowing down convergence rates. SDI's characteristics help handle the sensitive issue of steepest descent regarding step size. The simulation results show the reliability of SDI in generating promising initial points and identifying the global minimum point, making it a good algorithm for solving univariate unconstrained optimization problems.

Keywords Univariate, Unconstrained Optimization, Global Minimizer, Steepest Descent, Simultaneous Equation.

Mathematics Subject Classification 65K05, 90C26.

1 Introduction

In the process of making a decision, optimization is applied to evaluate every possible solution to give the best answer to a certain problem. According to [1], unconstrained optimization can be considered to

$$\begin{aligned} &\text{minimize} f(x), \\ &x \in R^n \end{aligned} \tag{1}$$

where$f \in C^0$ on R^n , meaning f is continuous. It is assumed that f is also continuously differentiable as $f \in C^1$ holds and even $f \in C^2$ for some cases. Choosing an approach to solve the problems depends on the criteria such as the size, the availability of $\nabla f(x)$ and $\nabla^2 f(x)$, the solution requirement, convexity of properties of f and an appropriate estimation of the location of a stationary point.

Additionally, according to [2], it can be understood as finding the minimum or maximum of nonlinear functions without any restriction. Unconstrained optimization is split between local and global optimization. There are a lot of existing local optimization methods such as steepest descent method, conjugate gradient methods, Newton methods, and quasi-Newton methods. They are used to locate a local and usually are iterative methods. They will stop executing when they find one local solution or when the criterion is achieved.

Unconstrained optimization problems can have multiple local minima or maxima and the algorithms may stuck in the local and not finding the global one, so this may lead to suboptimal solutions. This happens when the algorithms fail to converge and stop executing at the last optima. The reasons behind this failure to converge may vary depends on the developed methods such as sensitive initial point [3], quality of lower bound of optimum obtained [4], and improper step size due to gradients computation depending on infinite differences [5].

Steepest descent, also known as gradient descent, is one of the known methods to solve local optimization. The method has been applied in many areas such as data analysis [6] image processing [7] and, machine learning [8]. It has several advantages. According to Wang [9], this method is simple and has fast repetition, plus, one of the notable advantages is that it is guaranteed to locate the minimum pint despite the huge number of iterations as long as the optimal point exists. Not only that, but the steepest descent also has a nice convergence theory because of the straightforward method [10]. However, it also has some weaknesses. When managing some nonlinear problems, the converging rate will become slow [10]. The reason for the slow convergence rate is the high condition number of Hessian matrix of the cost function at the optimum point [11]. Additionally, it is also sensitive toward step size as a large step size might result in large error, but a small step size slows down the convergence rate thus resulting in unending iterations to find the minimum [9]. Since steepest descent is only used to solve local optimization problems, this study proposes an extension of this method to solve univariate unconstrained optimization problems by line intersection and simultaneous equation. With the simultaneous equation technique, the extended steepest descent method is assumed to be able to converge to the exact solution and will not get stuck in one of the local solutions.

2 Steepest Descent

The steepest descent method was proposed by Cauchy in 1847. According to [12], this method relies on gradient to choose the next descent direction d . The gradient at the kth iterate $x^{(k)}$ can be defined as

$$g^{(k)} = \nabla f(x^{(k)}) \tag{2}$$

The first-order Taylor series approximation of current iteration provides the stimulation for the gradient descent

$$g^{(k)} = \nabla f(x^{(k)})$$
$$f(x^{(k)} + \alpha d) \approx f(x^{(k)}) + \alpha d^T g^{(k)} \tag{3}$$

The direction d that minimizes the first-order approximation subject to the constraint that $||d|| = 1$, hence for steepest descent, the direction that minimizes the first-order approximation would be the opposite gradient:

$$d^{(k)} = -\frac{g^{(k)}}{||g^{(k)}||} \tag{4}$$

As long as the objective function is smooth, the step size α is sufficiently small and the location is not at the points where gradient is 0, the steepest descent is confirmed to succeed. If the chosen step size maximally decreases f producing jagged search path, then the following descent direction will always be orthogonal to the current direction. Thus, optimizing the step size at each step will result in:

$$\alpha^{(k)} = \arg\min_{\alpha} f\left(x^{(k)} + \alpha d^{(k)}\right) \tag{5}$$

The optimization entails that the directional derivative equals zero.

$$\nabla f(x^{(k)} + \alpha d^{(k)})^T d^{(k)} = 0 \tag{6}$$

It is known that

$$d^{(k+1)} = -\frac{\nabla f\left(x^{(k)} + \alpha d^{(k)}\right)}{\|\nabla f\left(x^{(k)} + \alpha d^{(k)}\right)\|} \tag{7}$$

Hence

$$d^{(k+1)T} d^{(k)} = 0 \tag{8}$$

which means that $d^{(k+1)}$ and $d^{(k)}$ are orthogonal.

The steepest descent is known to have a nice convergence theory due to its simplicity. However, it can converge slowly when handling mildly nonlinear problems [10]. Since then, more studies have been conducted to propose new methods that extend the steepest descent method like [13–16]. The steepest descent method is shown to be improved by integrating it with other methods. Hence, this paper aims to extend the steepest descent method by cooperating with simultaneous equations to solve global optimization problems. With the simultaneous equation technique, the developed method, named Steepest Descent Intersect (SDI) can produce the initial point which can guarantee to converge to a minimum.

3 Steepest Descent Intersect (SDI)

Steepest Descent Intersect (SDI) is a combination of Steepest Descent and simultaneous equation. The algorithm will first run a local search by using Steepest Descent and use the upper bound of the domain as the initial point to find a local solution. Then, a horizontal line will be established by using the y-coordinate of the local solution found in the previous step.

The SDI method then implements simultaneous equation techniques to identify the intersection point between the horizontal line and the objective function. The nearest intersection point from the current local solution will become the subsequent initial point for locating further local solution. The iterative process continues until no intersection occurs, signifying that the current solution has reached the lowest position of the function. Therefore, the last solution also represents the global solution. The algorithms algorithm of SDI is presented as below

SDI Algorithm:

Let (x, y) be the domain of objective function. Use lower bound as the initial point x_0

1. For $i = 0, ..., m$
2. Run the steepest descent to get the local solution,(x_m, y_m)
3. End
4. Establish a horizontal line by using $y = y_m$
5. Do simultaneous equation to get the nearest intersection point with the current solution (x_m, y_m)
6. Use the intersection point found in step 6 as next initial point
7. Go to Step 2
8. Repeat the Step 2 to Step 8 until no intersection detected
9. 9.Output: global solution is the last (x_m, y_m) found
10. Stop

In the next section, the ability of the developed algorithm, SDI in locating the global solution will be tested by using the benchmark functions selected. The central idea of SDI is to construct a "bridge" across any valley dividing two local maxima points, permitting the transition of a certain maximizing technique from one local maximum to an improved one. Therefore, the benchmark functions selected are multi-modal, with every local minimum value being different, so that the capability of SDI in locating the global solutions can be determined.

4 Result and Discussion

Fifteen benchmark functions will be used to test the capability of SDI in locating the global minimizer. The details and results are presented in Section 4.1. Subsequently, SDI will be applied to solve the portfolio optimization problem. Due to the limitation of SDI, which can only solve univariate global optimization problems, only two assets are selected. Further details can be found in Section 4.2.

4.1 Numerical Result

The table below shows the details of 10 selected benchmark functions that taken from [17–19] and [20]. The stopping criterion applied when running the steepest descent for finding the local solution is 10^{-6} .

Table 1: Benchmark Function

No	$f(x)$	$x \in [a, b]$	No. Local minimizer	$f(x^*)$
1	$cos(x) - sin(5x) + 1$	[0.2,7]	6	-0.952897
2	$sin(x) + sin(\frac{10x}{3})$	[2.7,7.5]	3	-1.899599
3	$-xsin(x)$	[0,10]	2	-7.91673
4	$sin(x)^3 + cos(x)^3$	[0,6.28]	3	-1
5	$\frac{x^2}{20} - cos(x) + 2$	[-20,20]	7	1
6	$x + sin(5x)$	[0.2,7]	7	-0.077590
7	$-e^{-x} sin(2\pi x)$	[0,4]	4	-0.788685
8	$-x + sin(3x) + 1$	[0.2,7]	5	-5.815675
9	$-x - sin(3x) + 1.6$	[0.2,7]	4	-6.262872
10	$xsin(x) + sin(\frac{10x}{3}) + ln(x) - 0.84 + 1.3$	[0.2,7]	4	-7.047444
11	$sin(x)cos(x) - 1.5sin^2(x) + 1.2$	[0.2,7]	3	-0.451388
12	$3x^4 - 4x^3 - 1.2x^2 + 9$	[-2, 3]	2	-23
13	$-(1.4 - 3x)sin(18x)$	[0, 1.2]	4	-1.48907
14	$sin(x) + sin\frac{10x}{3} - ln(x) - 0.84x$	[2.7, 7.5]	3	-4.601308
15	$2cosx + cos2x + 5$	[0.2, 7]	3	3.5

The selected benchmark functions are multimodal, which increases the complexity and difficulty for a global search algorithm to accurately locate the global solution [17]. Multimodal functions have multiple local optima, making it challenging for the algorithm to distinguish between local and global optima [19] and [18]. In the next section, the ability of the developed algorithm, SDI will be tested by using the benchmark functions selected. The numerical experiment will be carried out by using Python, and the result is displayed in Table 2.

Table 2: Result Experiment

No	$x \in [a, b]$	i	(x_0, y_0)	(x_m, y_m)	$(x^*, f(x^*))$
1	[0.2,7]	2	(0.2,1.138596), (1.317763, 0.949450), (2.551284, -0.019675)	(0.321399, 0.949450), (1.605466, -0.019675), (2.833973, -0.952522)	(2.833973, -0.952522)
2	[2.7,7.5]	1	(2.7, 0.8394984), (4.773105, -1.199165)	(3.375485, -1.199165), (5.134125, -1.898796)	(5.134125, -1.898796)
3	[0.1,10]	1	(0.1, -0.009983), (6.563548, -1.816161)	(1.977255, -1.816161), (7.961589, -7.915538)	(7.973866, -7.916633)
4	[0.1, 6.28]	2	(0.1, 0.986082), (1.978170, 0.711844)	(0.718408, 0.711844), (3.097887, -0.997054), (4.756098, -0.997054)	(3.097887, -0.997054), (4.756098, -0.997054)
5	[-20, 20]	1	(-20, 21.5919179), -2.161568, 2.790621)	(-5.726309, 2.790621), (-0.039512, 1.000859)	(-0.039512, 1.000859)
6	[0.5, 7]	0	(0.5, 1.098472)	(0.898666, -0.077436)	(0.898666, -0.077436)
7	[0, 4]	0	(0,0)	(0.220743, -0.788411)	(0.220743, -0.788411)
8	[0.5, 7]	2	(0.5, 1.497495), (3.010103, -1.625787), (5.104513, -3.720235)	(1.668040, -1.625787), (3.762825, -3.720235), (5.856917, -5.814589)	(5.856917, -5.814589)
9	[0.2, 7]	3	(0.2, 0.835357), (1.962898, 0.021439), (4.057561, -2.073966), (6.151694, -4.167373)	(0.620637, 0.021439), (2.726062, -2.073966), (4.809584, -4.167373), (6.904571, -6.261848)	(6.904571, -6.261848)
10	[1, 7]	1	(1, 1.110903), (2.834356, 0.795065)	(1.283589, 0.795065), (5.125861, -7.046854)	(5.125861, -7.046854)
11	[0.2, 7]		(0.2, 1.335505)	(-1.264550, -0.451118)	(1.877031, -0.451118), (4.994156, -0.451118)
12	[-2, 3]	0	(-2, 41) (0.612558, 4.000263)	(-1.003812, 4.000263) (1.998088, -22.999868)	(1.998088, -22.999868)
13	[0, 1.2]	1	(0, 0), (0.927290, -1.150148)	(0.078995, -1.150148), (0.965856, -1.489059)	(0.965856, -1.489059)
14	[2.7, 7.5]	1	(2.7, -0.435250), (4.566276, -2.838303)	(3.425947, -2.838303), (5.188418, -4.600536)	(5.188418, -4.600536)
15	[0.2, 7]	0	(0.2, 7.881194)	(2.050982, 3.500313), (4.144256, 3.500313)	(2.080020, 3.502897), (4.144256, 3.502897)

From the results obtained, SDI can find all the global solutions of the selected benchmark functions. The step size used throughout the experiment is 0.00005 [21]. As previously stated, Steepest Descent is a local optimization method that is sensitive towards step size. Unsuitable step size might result in a slow convergence rate or worse since it will diverge and fail to locate the local solution. Therefore, from the outcome of the experiments, SDI can overcome the obstacles by integrating the Steepest Descent and simultaneous equations. Simultaneous equation plays an important role in SDI that allows the algorithm to generate promising initial points using the y-coordinate of the previous local solution found as a horizontal line. The horizontal line intersects with the objective function, creating an intersection point that is used as the promising initial point. With this technique, SDI is able to locate the global minimizer without identify all the local minimizer.

For example, the graph below shows the figure of the first benchmark function, $f(x) = cos(x) - sin(5x) + 1$.

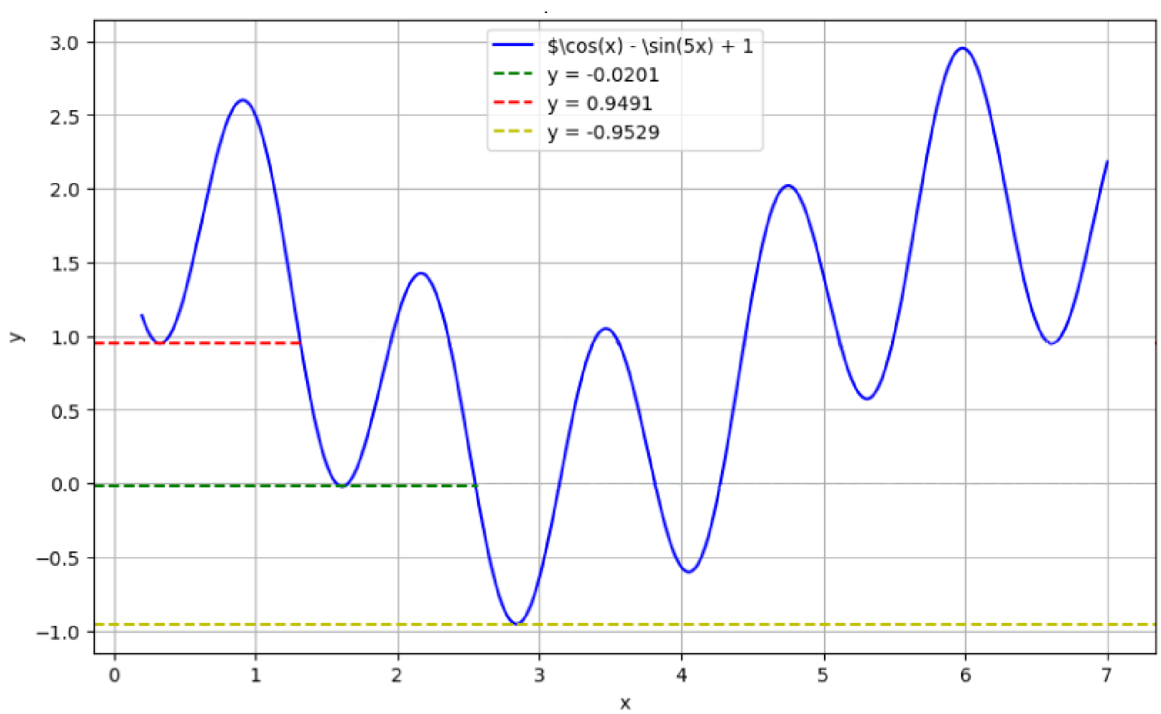

Figure 1: First benchmark function

Figure 3 above shows the graph of first benchmark function with the domain $x \in [0.2, 7]$. As illustrated, the graph has 6 local minimizers, but the Steepest Descent Intersect can only detect 3 minimizers to get the global solution. This is because there is no intersection after the third minimizers. The third local minimizer found is (2.8393, -0.9529), when $y = -0.9529$ which is represented by a yellow line, it is obvious that, it will not collide with the objective function, $f(x)$, except the current minimum point. This mechanism allows SDI to have less time complexity since less iteration needed.

From Table 3, each initial point successfully attains the local solution, and the last local solution is also the global solution. SDI is designed to stop when an intersection no longer occurs, indicating that the last local solution found is the lowest point of the function. Additionally, despite the number of intersection points generated by SDI for each function being less than the number of local solutions, SDI is still able to locate the global solution. This is due to the mechanism of SDI in identifying the global solution that does not rely on comparing the function values among all the local solutions found. Instead, it depends on the concept of simultaneous equations and the new position is assumed to be the lowest point of the objective function when the intersection no longer occurs. This is also the strength of SDI.

4.2 Application of SDI on Portfolio Optimization

Portfolio optimization can be used to find the best allocation for each asset such that one can minimize the risk while maximizing the profit. It is usually modeled in linear programming structure with many constraints. However, it can also be represented by an unconstrained minimization problem through reformulation of the problem.

Portfolio selection is important in financial mathematics, risk management and economics [22]. It describes how investors should allocate their wealth among several assets [23]. It also focuses on answering the question of how to trade off risk and return while selecting assets under certain conditions [24].

In this paper, SDI will be applied to solve unimodal portfolio optimization problems. The portfolio model will be developed based on Bartholomew-Biggs's method [25] and the data used is taken from the official website of RHB bank : `https://rhbassetmanagement.rhbgroup.com/myinvest/client/funds/daily-price`.

The table below shows the year returns of RHB Capital Fund and RHB Asean Fund for past 5 years. Based on the basic minimum risk problem model that was introduced by Bartholomew-Biggs's, the equation for minimizing the risk, V is obtained, as shown in Eq. (6.1).

Table 3: Returns of Fund Data

Fund	**Years Returns (%)**				
	2019	**2020**	**2021**	**2022**	2023
RHB Capital Fund	1.08	42.8	-14.45	-2.93	6.96
RHB Asean Fund	4.59	-9.42	-2.71	0.12	3.22

$$V = 523.34x^2 - 173.16x + 24.8 \tag{9}$$

Then, by employing SDI, the minimum risk portfolio for the funds in Table 6.3 is x=0.1654. This implies that the less risky strategy is to invest approximately 16.54% of the capital in Fund A and 83.46% in Fund B, as the total allocation must sum to 100%.

5 Conclusion

In this paper, a new method is introduced to find global minimum points for univariate unconstrained optimization. The new method combines the existing local search which is the Steepest Descent with simultaneous equations, named Steepest Descent Intersect (SDI). The result of the experiment implies that the extension is a success and shows the reliability of the new algorithm in generating promising initial points and in identifying the global minimum point.

SDI had some significant advantages. First, SDI can attain the minimizer regardless of the step size, which is a notable improvement over the traditional steepest descent method, highlighting its robustness and reliability in various optimization scenarios. Additionally, the initial points generated using simultaneous equations always converge to the solution because they are always within the basin of attraction. When the initial point chosen falls inside a basin of attraction, it will surely converge to the minimizer. This ensures that the SDI does not have divergence problems.

Another distinctive feature of SDI is its ability to construct a "bridge" within the valley when dealing with multimodal and wavy functions. This capability allows it to escape from the bottom of a valley, preventing it from getting stuck in a local minimum and thus increasing the chances of finding the global minimum.

Furthermore, while the majority of existing global optimization methods like the Homotopy Optimization Method (HOM) [26], Homotopy with 2-Step Predictor-Corrector Method (HSPM) [27] and Kerk and Rohanin Trusted's Interval (KRTI) [28] focus on identifying the global minimum among the local minima found, SDI employs a different mechanism. Comparing the function values of each and every local solution discovered was not necessary. Rather, in the absence of a collision, the present location is taken to be at the bottom of the objective function using the idea of the simultaneous equation, suggesting that the final local solution discovered by SDI is also the global solution. These significant points underscore the potential of SDI as a powerful tool in the field of optimization.

Furthermore, the capability of SDI in solving portfolio optimization has been demonstrated. However, since SDI is designed to handle only unimodal optimization problems, the number of assets involved in modeling the objective function for portfolio optimization is limited. Therefore, as a consideration for future work, SDI can be extended to address large-scale optimization problems.

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