MATEMATIKA, MJIAM, Volume 41, Number 2, 179–192

Investigation of the Asymptotics of Quadratic Lotka-Volterra Mappings and Their Connections with Elements of Graph Theory

[1,2]Dilfuza Eshmamatova*, **[1]Marjona Islomova and [1]Axror Bozorov**

[1]Department of Higher Mathematics, Tashkent State Transport University
100167, Tashkent, Uzbekistan

[2]V. I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan
100174, Tashkent, Uzbekistan

*Corresponding author: 24dil@mail.ru

Article history
Received: 19 October 2024
Received in revised form: 23 March 2025
Accepted: 14 May 2025
Published on line: 1 August 2025

Abstract The relevance of studying discrete Lotka-Volterra mappings lies in their applicability to modeling epidemiological and environmental problems. In this regard, this work focuses on the asymptotic behavior of these mappings and their connection with graph theory. The paper demonstrates that, when these mappings are in a general position, they can be associated with complete oriented graphs. However, in degenerate cases, it may be associated with partially oriented graphs. Furthermore, the paper proves that, if the skew-symmetric matrix corresponding to a Lotka-Volterra mapping is not in general position, then the set of fixed points becomes infinite. In addition, sets of fixed points are constructed for systems that are discrete analogues of continuous compartmental models, SIR, $SIRD$, and the characteristics of these fixed points are studied by analyzing the spectrum of the Jacobian matrix and constructing a phase portrait of the trajectories of the interior points.

Keywords Lotka-Volterra mapping, homeomorphism, skew-symmetric matrix, tournament, partially oriented graph, fixed point.

Mathematics Subject Classification 37B25, 37C25, 37C27.

1 Introduction

Interest in nonlinear dynamical systems began in the last century, and the contribution to the application of these systems in epidemiology can be seen in the work of Kermack and McKendrick [1]. In this paper, they present an autonomous system of differential equations, which forms the basis for the SIR compartmental model. These models are still relevant today, and on their basis, a whole hierarchy of compartment models has been developed, including $SIR, SEIR, SIRS, MSEIR, SIRD$ and others [2–9].

It should be noted that many physical systems can be reduced to the study of Markov processes. However, as it turns out, not all systems can be described using them. An example of a system that cannot be described by Markov processes is a process described by quadratic stochastic mappings. The interest in quadratic stochastic mappings began with the work of Bernstein [10]. Later, this theory was further developed by scientists such as Ulam, Pasta, and Zimakov [11–13].

The proposed paper focuses on the study of the asymptotic behavior of quadratic stochastic Lotka-Volterra operators, which were introduced in the works of Ganikhodzhaev R.N. He has made a significant contribution to the research on this class of mappings and their connections to directed graphs [13]. It has been found that if the skew-symmetric matrices associated with these mappings are not in a general position, they can be related to partially oriented graphs.

The relevance of studying discrete Lotka-Volterra mappings lies in their applicability to modeling epidemiological and environmental problems. In this regard, this paper is devoted to the asymptotic behavior of these mappings and their relation to graph theory. It is noted in the paper that when these mappings are in a common position, they can be associated with complete directed graphs. However, in degenerate cases, they can be associated with partially oriented graphs. In addition, the paper shows that if the skew-symmetric matrix associated with the Lotka-Volterra system is not in a general position, then there are an infinite number of fixed points. It is also proven that, in the case of a general position, all the fixed points of the system are hyperbolic. Sets of fixed points are constructed for Lotka-Volterra systems, which are discrete analogues of continuous $SIR, SIRD$ compartmental models. The characteristics of these fixed points are explored by analyzing the spectrum of the Jacobian matrix and constructing phase portraits of the trajectories of interior points.

2 Preliminary information

Let $x = (x_1, x_2, ..., x_m)$ be a certain probability distribution of individuals according to characteristics. Assuming that crosses occur randomly (panmixia) and there are no crosses between different generations, to describe the law of transition from the probability distribution in the next generation, we obtain the following dynamic system:

$$x_k^{(n+1)} = x_k^{(n)} \left(1 + \sum_{i=1}^{m} a_{ki} x_i^{(n)} \right),$$

where $k = 1, 2, ..., m, \quad n = 0, 1, ..., \ x^0 = (x_1^0, ..., x_m^0)$ is the probability distribution in the initial generation, $a_{ki} = -a_{ik}, \ |a_{ki}| \leq 1$ are inheritance coefficients [14–18]. Consider the quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ introduced in the work [14]:

$$(Vx)_k = x_k^{'} = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1, m},$$

where $x = (x_1, x_2, ..., x_m) \in S^{m-1}$, $(Vx)_k$ is the k-th coordinate of the point $Vx \in S^{m-1}$ and the coefficients $P_{ij,k}$ satisfy the conditions

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1. \tag{1}$$

It is easy to see that conditions (1) ensure the invariance of the simplex

$$S^{m-1} = \left\{ x = (x_1, ..., x_m) : \; x_i \geq 0; \; \sum_{i=1}^{m} x_i = 1 \right\}$$

with respect to the mapping V.

Definition 1 *[14] If*

$$P_{ij,k} = 0, \quad by \;\; k \notin \{i, j\}, \tag{2}$$

then the quadratic stochastic operator V is called the Lotka-Volterra mapping on the simplex S^{m-1}.

(2) implies that $P_{ik,k} + P_{ik,i} = 1$ for any $i, k = \overline{1, m}$. Therefore, assuming

$$a_{ki} = \begin{cases} 2P_{ik,k} - 1, & i \neq k, \\ 0, & i = k, \end{cases} \tag{3}$$

the Lotka-Volterra mapping V can be rewritten as [14]:

$$x = \left(x_1 \left(1 + \sum_{i=1}^{m} a_{1i} x_i \right), \; x_2 \left(1 + \sum_{i=1}^{m} a_{2i} x_i \right), ..., x_m \left(1 + \sum_{i=1}^{m} a_{mi} x_i \right) \right).$$

Let $Vx = (x_1^{'}, x_2^{'}, ..., x_m^{'})$. Then,

$$x_k^{'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right), \quad k = \overline{1, m}. \tag{4}$$

Relation (4) is called the canonical form of the Lotka-Volterra mapping on the simplex S^{m-1}.

Note that (3) implies

$$a_{ki} = -a_{ik} \text{ and } |a_{ki}| \leq 1. \tag{5}$$

Since the Lotka-Volterra mapping V is uniquely determined by specifying the skew-symmetric matrix $A = (a_{ki})$ with the additional condition $|a_{ki}| \leq 1$, V can be identified with a point in the space $\mathbb{R}^{\frac{m(m-1)}{2}}$, because the skew-symmetric matrix has $\frac{m(m-1)}{2}$ free parameters. Therefore, the totality of all Lotka-Volterra mappings is a polyhedron, more precisely, a cube of the dimension $\frac{m(m-1)}{2}$ in the space $\mathbb{R}^{\frac{m(m-1)}{2}}$.

Recall that a minor is called a major if it is composed of rows and columns with the same numbers [19]. Recall the following definition from [20] and [21].

Definition 2 *[20] A skew-symmetric $A = (a_{ki})$ matrix is called a matrix of general position if all its major minors of even order are nonzero.*

If a skew-symmetric matrix is a general position matrix, then the corresponding Lotka-Volterra mapping V with coefficients a_{ki} is called an operator of general position.

In [20–24], it is proved that skew-symmetric matrices of general position form an open and everywhere dense subset in the set of all skew-symmetric matrices.

For example, consider the mapping $V : S^3 \to S^3$, which has the form:

$$\begin{cases} x_1' = x_1(1 + ax_2 - bx_3 + cx_4), \\ x_2' = x_2(1 - ax_1 + dx_3 - ex_4), \\ x_3' = x_3(1 + bx_1 - dx_2 + fx_4), \\ x_4' = x_4(1 - cx_1 + ex_2 - fx_3), \end{cases}$$

where $a, b, ..., f \in [-1; 1]$. This operator is an operator of general position if and only if the coefficients $a, b, ..., f \in [-1; 1]$ satisfy the following conditions:

$$a, b, ..., f \neq 0 \quad and \quad af - be + cd \neq 0.$$

Lemma 1 *Let $A = (a_{ki})$ be a skew-symmetric matrix and $b_1, ..., b_m > 0$. Then*

$$\begin{vmatrix} b_1 & a_{12} & \dots & a_{1m} \\ a_{21} & b_2 & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & b_m \end{vmatrix} \geq b_1 \cdot b_2 \cdot ... \cdot b_m.$$

The proof of Lemma 1 can easily be obtained by induction with respect to m.

The point $x = (x_1, ..., x_m)$ is called a relative interior point of the simplex S^{m-1} if $x_i > 0,\ i = \overline{1, m}$. Using Lemma 1, it is obtained that the Jacobian of mapping (4) is positive at all relatively interior points of the simplex. Hence, the quadratic Lotka-Volterra mapping is a local homeomorphism in a neighborhood of any relatively interior point of S^{m-1}. Since S^{m-1} is compact, the local homeomorphism of V implies its homeomorphism [20, 24].

In the case where the conditions of Definition 2 are not met, the Lotka-Volterra mapping and the corresponding skew-symmetric matrix are not in general position. This is a case where the mapping and its matrix are degenerate.

In [24], it was proved that the quadratic Lotka-Volterra mapping is a homeomorphism of the S^{m-1} simplex.

In problems of population genetics, one needs to study the evolution of a biological system over time. In many cases, the evolution of the system is described by quadratic mappings of the simplex into itself. From a biological point of view, the homeomorphism of the evolution operator means the possibility of restoring the prehistory of a biological system according to the known state of the system at the moment. The quadratic version of the Lotka-Volterra simplex mapping is a special case of quadratic homeomorphisms.

3 Connection of Graph Theory Elements in the Study of Lotka-Volterra Systems

As it is known [24] each quadratic Lotka-Volterra mapping, given on a finite-dimensional simplex, defines a certain tournament, the properties of which allow us to study the asymptotic behavior of the trajectories of this mapping. In this paper, the concepts of a tournament, a homogeneous tournament, and a partially oriented graph are introduced. These concepts are used to study the Lotka-Volterra systems. Before proceeding to this part of the work, it is necessary to recall the basic concepts of graph theory presented in [25–27] and [28], as well as in [29].

Definition 3 *[25] A graph G is a finite nonempty set Y containing p vertices and a given set E containing q disordered pairs of different vertices from Y.*

Each pair of vertices $y = \{u, v\}$ in E is called an edge of the graph G. This means that y connects u and v. Here, E is a set of unordered pairs of different vertices from the set Y. The vertex u is incident with the edge y, as is v with y.
If two different edges y and z are incident to the same vertex, then they are called adjacent.
A graph with p vertices and q edges is called a (p, q) graph.

From the above information, it is clear that there can be no loops in the graph, that is, edges connecting vertices to each other.

Definition 4 *[27] A directed graph or digraph D is a finite nonempty set of vertices and a given set of ordered pairs of different vertices.*

The elements of the set E are called oriented edges or arcs.

Definition 5 *[27] Pairs of vertices that are connected by more than one edge are called multiples.*

There are no loops or multiple arcs in a digraph.

Definition 6 *[29] A directed graph is a digraph in which no pair of vertices is connected by a symmetric pair of arcs.*

Definition 6 implies that each orientation of a graph generates a directed graph.

It is known from [23] and [24] that if the skew-symmetric matrix $A = (a_{ki})$ is in general position, then it corresponds to a tournament-complete oriented graph. Recall the definition of a tournament from [24].

Let $A = (a_{ki})$ be a general position matrix of the Lotka-Volterra operator (4) satisfying conditions (5). Suppose that $a_{ki} \neq 0$ for $k \neq i$. On the plane, m points are taken and numbered with numbers $1, 2, ..., m$. Then the point k is connected to the point i by an arrow directed from k to i if $a_{ki} < 0$, and vice versa if $a_{ki} > 0$. The graph constructed in this way is called the tournament of dynamic system (4) with a skew-symmetric matrix $A = (a_{ki})$ and is denoted by T_m.

In the case of $m = 3$, according to the definition of tournaments, skew-symmetric matrices

$$A = \begin{pmatrix} 0 & a_{12} & -a_{13} \\ -a_{12} & 0 & a_{23} \\ a_{13} & -a_{23} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}$$

correspond to the tournaments shown in Figure 1 [20].

A tournament that has no cycles is called transitive. The concept of a subtournament is naturally defined. In the definitions, we follow the terminology adopted in [24–26].

Let x_1, x_2 be the vertices of a tournament. The entry $x_1 \to x_2$ means that the edge connecting x_1 and x_2 is directed from x_1 to x_2. A finite sequence of vertices $x_1 \to x_2 \to ... \to x_p$ is called a route if $x_i \neq x_j$ at $i \neq j$. A cycle is a closed route, that is, $x_p = x_1$.

The tournament is strong if there is a route with the beginning of $x, y \in Y$ and the end of x for any peaks of y.

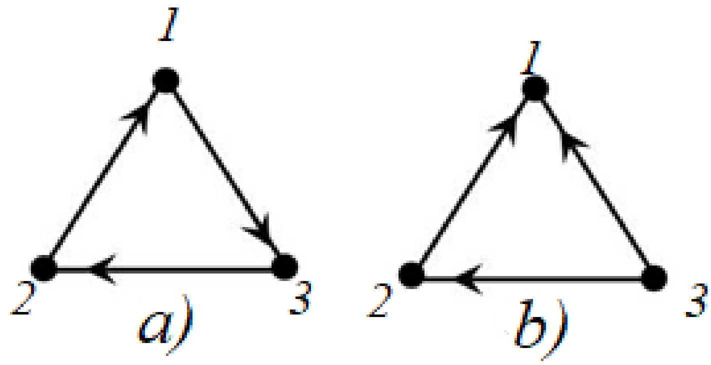

Figure 1: A tournament is called a) a cyclic triple, b) a transitive triple.

It has been proven [21] that a tournament is considered strong if and only if it contains a cycle with a length equal to $|Y|$, where $|Y|$ represents the number of elements in the set Y.

Definition 7 *[21] A tournament is called homogeneous if any of its subtournaments is either strong or transitive.*

Obviously, for with $|Y| \leq 3$, any tournament is homogeneous. It is shown in [27] that for $|Y| = 4$, there are four pairwise non-isomorphic homogeneous tournaments, and in [21], the criterion of tournament uniformity is proved.

Table 1 below shows the number of tournaments, the number of strong tournaments, as well as the number of homogeneous tournaments at $m \leq 6$.

Table 1: Number of tournaments, strong tournaments and homogeneous tournaments for $m \leq 6$.

m	*Number of tournaments*	*Number of strong tournaments*	*Number of homogeneous tournaments*
2	1	0	1
3	2	1	2
4	4	1	2
5	12	6	4
6	56	32	10

If the corresponding skew-symmetric matrix of the Lotka-Volterra system is not in general position, i.e. if there is a major minor equal to zero, then a matrix of this kind is called degenerate. This case is possible only if some coefficients of the skew-symmetric matrix are zero, i.e. $a_{ki} = 0$.

An undirected graph is a graph that has no oriented edges, and, generally speaking, it can be included in partially oriented graphs [27]. A digraph can be considered as a partially oriented graph in which each symmetric pair of oriented edges is replaced by an undirected edge. If the skew-symmetric matrix corresponding to system (4) is degenerate, then a partially oriented graph can be matched to it.

Definition 8 *A graph is called partially oriented if it contains both oriented and undirected edges.*

For example, consider the mapping

$$V_1 : \begin{cases} x_1^{'} = x_1(1 + a_{12}x_2), \\ x_2^{'} = x_2(1 - a_{12}x_1), \\ x_3^{'} = x_3, \end{cases}$$

with the matrix

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ -a_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The partially oriented graph of this matrix has the form as shown in Figure 2.

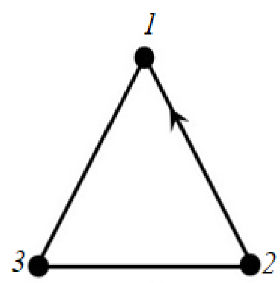

Figure 2: A partially oriented graph with one directed edge.

The total number of partially oriented graphs with three vertices is given in [27]. In this paper, the relationship of partially oriented graphs with degenerate Lotka-Volterra mappings is shown. This contributes to a more understandable presentation of the phase portrait of the trajectory of the internal points of these mappings.

4 Fixed Points of Degenerate and Nondegenerate Lotka-Volterra Mappings

The main problem in researching the dynamics of the trajectories of the interior points of mappings of the simplex is to find their fixed points and study their characters.

Let $X = \{x \in S^{m-1} : Vx = x\}$ be the set of fixed points of the Lotka-Volterra mapping V. Since $V : S^{m-1} \to S^{m-1}$ is continuous and S^{m-1} is a convex compact, then according to the Bohl-Brauer theorem, the set of fixed points V is not empty.

In general, it is known that Lotka-Volterra mappings on a simplex can have a continuum of fixed points, which significantly complicates the analysis of the asymptotic behavior of trajectories [21, 24].

According to [21–24], when the Lotka-Volterra mapping is in a general position, the number of fixed points is finite and the vertices of the simplex are fixed points. In contrast to these works, it is considered the degenerate case here. For example, let the Lotka-Volterra mapping $V : S^2 \to S^2$ have the form:

$$V : \begin{cases} x_1^{'} = x_1(1 - x_2), \\ x_2^{'} = x_2(1 + x_1 - x_3), \\ x_3^{'} = x_3(1 + x_2). \end{cases}$$

This mapping has infinitely many fixed points. The fixed points of this mapping V are all points of the segment connecting the vertices e_1 and e_3, as well as the vertex e_2 (see Figure 3).

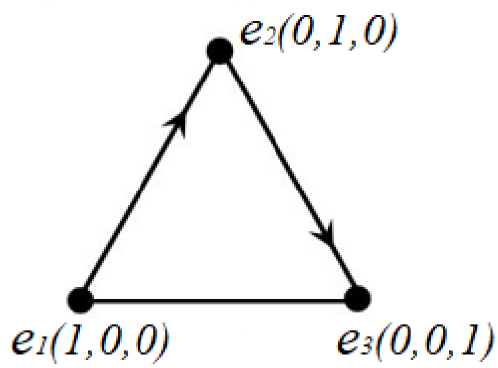

Figure 3: A partially oriented graph with two directed edges.

The definitions of [30–32] and [33] that describe the characters of fixed points are given below.

Definition 9 *[30] A fixed point x is called repulsive (a repeller) if there is such a neighborhood that the trajectory of any point from this neighborhood, with the exception of x itself, leaves this neighborhood in a finite number of steps.*

Definition 10 *[30] A fixed point x is called an attractor if there exists such a neighborhood that the trajectory of any point from this neighborhood remains in this neighborhood for a finite number of steps.*

The characters of the fixed points are investigated by analyzing the spectrum of the Jacobian matrix. For this purpose, it is convenient to consider the Jacobian not only as a determinant, but also as a linear operator, i.e., the differential of the mapping. Usually, the spectrum of the Jacobi matrix at the point x is denoted by $\sigma(J(x))$.

Let $B = \{z \in \mathbb{C} : |z| < 1\}$, the unit open circle on the complex plane $\mathbb{C}$ and $\overline{B}$ be its closure.

As it is known [31], for differentiable mappings, a fixed point is attractive if $\sigma(J(x)) \subset B$, and repulsive if $\sigma(J(x)) \cap \overline{B} = \varnothing$.

Definition 11 *A fixed point of a differentiable mapping is called hyperbolic if* $\sigma(J(x)) \cap \partial B = \varnothing$, *where* $\partial B = \{z \in \mathbb{C} : |z| = 1\}$ *is the boundary of* B.

Theorem 1 *All fixed points of a general position of the Lotka-Volterra mapping are hyperbolic.*

Proof The mapping V is a general position mapping, this implies finiteness of the number of fixed points and all fixed points are isolated. Further, the mapping V is in a general position, this implies that its narrowing to the edge Γ_α will also be a Lotka-Volterra mapping in a general position [22, 23]. Then it is enough to assume that an arbitrary fixed point is interior, i.e. $x = (x_1, ..., x_m)$, where $x_i > 0, \quad i = \overline{1, m}$. In this case $Vx = x$ implies

$$x_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k = \overline{1, m}.$$

Since $x_k > 0, \ \sum_{i=1}^{m} a_{ki} x_i = 0.$

Therefore, the Jacobian has the form:

$$J(x) = \begin{pmatrix} 1 & a_{12}x_1 & \dots & a_{1m}x_1 \\ a_{21}x_2 & 1 & \dots & a_{2m}x_2 \\ \dots & \dots & \dots & \dots \\ a_{m1}x_m & a_{m2}x_m & \dots & 1 \end{pmatrix} = I + x_1 \cdot x_2 \cdot ... \cdot x_m \cdot A,$$

where I is a unit matrix and $A = (a_{ki})$ is a skew-symmetric matrix.

According to [34], the eigenvalues of the matrix $I + A$, where A is a skew-symmetric matrix in a general position, are complex numbers modulo one, i.e.

$$\sigma(J(x)) \cap \overline{B} = \varnothing.$$

Therefore, x is a hyperbolic fixed point.

□

As mentioned above, according to [21–24] and [25], in the case where the Lotka-Volterra mapping is in a general position, the number of fixed points is finite. For degenerate Lotka-Volterra mappings, this fact turned out to be different. As a result, the following theorem was obtained for degenerate mappings.

Theorem 2 *If the skew-symmetric matrix corresponding to mapping (4) is degenerate, then the number of fixed points of this mapping is infinite.*

Proof To prove Theorem 2 consider the following case:

$$V_2 : \begin{cases} x_1' = x_1(1 - a_{12}x_2 + a_{13}x_3), \\ x_2' = x_2(1 + a_{12}x_1), \\ x_3' = x_3(1 - a_{13}x_1). \end{cases}$$

By solving the equation $Vx = x$, the fixed points of the mapping are found. This is the vertex $e_1(1, 0, 0)$ and all the points belonging to the edge of $\Gamma_{13} = \{\alpha, 0, 1 - \alpha\}, \ 0 \leq \alpha \leq 1$.

The characters of the fixed points are revealed by analyzing the spectrum of the Jacobian:

$$(1-\lambda)^3 + [(x_3 - x_1)a_{13} + (x_1 - x_2)a_{12})](1-\lambda)^2 + [a_{12}a_{13}x_1(x_2 + x_3 - x_1)](1-\lambda) = 0.$$

By solving the last equation, the eigenvalues of the mapping V are obtained

$$\lambda_1 = 1, \ \lambda_{2,3} = \frac{2 + [(x_3 - x_1)a_{13} + (x_1 - x_2)a_{12}] \mp \sqrt{D}}{2},$$

$$D = [(x_3 - x_1)a_{13} + (x_1 - x_2)a_{12}]^2 - 4a_{12}a_{13}x_1(x_2 + x_3 - x_1).$$

According to the definitions of fixed points, the following results are obtained and presented in Table 2.

Table 2: Description of the characters of fixed points of the mapping V_2.

Fixed points	Eigenvalues	The type of the fixed point
$e_1(1,0,0)$	$\lambda_1 = 1; \lambda_2 = 1 + a_{12}; \lambda_3 = 1 - a_{13}$	saddle
$e_2(0,1,0)$	$\lambda_1 = \lambda_2 = 1; \lambda_3 = 1 - a_{12}$	attracting (an attractor)
$e_3(0,0,1)$	$\lambda_1 = \lambda_2 = 1; \lambda_3 = 1 + a_{13}$	a repeller
$\forall x^* \epsilon (e_3, N)$	$\lambda_1 = \lambda_2 = 1; \lambda_3 = 1 - \alpha a_{12} + (1-\alpha)a_{13}$	a repeller
$\forall x^* \epsilon (N, e_2)$	$\lambda_1 = \lambda_2 = 1; \lambda_3 = 1 - \alpha a_{12} + (1-\alpha)a_{13}$	attracting (an attractor)

□

The phase portraits of the trajectories of the interior points, according to the character of the fixed points, are shown in Figure 4.

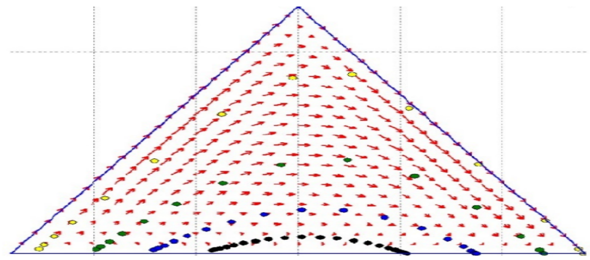

Figure 4: Phase portrait of trajectories of interior points of the degenerate mapping V_2.

This case was given for a reason, as this special case can be represented as a discrete model of the compartmental models SIR, $SIRD$ [36–39]. As it turned out, the degenerate cases of quadratic Lotka-Volterra mappings can act as a mathematical model in problems of epidemiology and economics. This justifies the relevance of these discrete mappings [37–41].

5 Conclusion

In the paper, the class of quadratic Lotka-Volterra mappings with matrices in a general position and degenerate matrices is considered. As it turned out [14,15,20,21], in the case when the skew-symmetric matrix corresponding to the discrete Lotka-Volterra mapping (4), introduced in [14] and [15] is a matrix in a general position, the mapping can be associated with a tournament. But in the case when the skew-symmetric matrix is degenerate, then the mapping is associated with a partially directed graph. The study of Lotka-Volterra mappings (4) using elements of the graph theory helps to clearly see the phase portrait of the flow of trajectories of interior points.

In [20] and [24], it was proved that if the Lotka-Volterra mapping is a mapping in general position, then its set of fixed points is always finite. In this paper, it is proved that, in the degenerate case, the set of fixed points is infinite. It was also possible to prove that the fixed points of the Lotka-Volterra mapping in a general position are always hyperbolic. The elements of the set are called oriented edges or arcs. In the problems of population genetics, there is a need to study the evolution of a biological system over time. In many cases, the evolution of the system is described by quadratic mappings of the simplex into itself. From a biological point of view, the homeomorphism of the evolution operator means the possibility of restoring the prehistory of a biological system according to the known state of the system at the moment. In addition, it was possible to prove that the fixed points of the Lotka-Volterra mapping in a general position are always hyperbolic.

Acknowledgments

The authors would like to thank Tashkent State Transport University, as well as Universiti Teknologi MARA for the opportunity to publish the work.

References

[1] Kermack, W. O. and McKendrick, A. G. Contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*. 1927. 115(772): 700-721.

[2] Murray, J. D. *Mathematical Biology*. 3th Edition. Berlin: Springer. 2002.

[3] Tessema, F. S., Bole, B. K. and Koya, P. K. Dynamics of a Stochastic and Deterministic SVIQRS Cholera Epidemic Model. *Applied Mathematics and Information Sciences*. 2022. 16(6): 961-971.

[4] Liu, L., Wei, X. and Zhang, N. Global stability of a network-based SIRS epidemic model with nonmonotone incidence rate. *Physica A: Statistical Mechanics and its Applications.*. 2019. 515(1): 587-599.

[5] Liu, Q. and Jiang, D. Dynamical behavior of a stochastic multigroup SIR epidemic model. *Physica A: Statistical Mechanics and its Applications*. 2019. 526(120975).

[6] Li, Y., Wang, B., Peng, R., Zhou, C., Zhan, Y., Liu, Z., Jiang, X. and Zhao, B. Mathematical Modeling and Epidemic Prediction of COVID-19 and Its Significance to Epidemic Prevention and Control Measures. *Annals of Infectious Disease and Epidemiology*. 2020. 5(1): 1052.

[7] Ojo, M. M. and Akinpelu, F. O. Lyapunov Functions and Global Properties of $SEIR$ Epidemic Model. *International journal of Chemistry, Mathematics and Physics*. 2017. 1(1): 1116.

[8] Radulescu, A., Williams, C. and Cavanagh, K. Management strategies in a $SEIR$-type model of COVID 19 community spread. *International journal of Chemistry, Mathematics and Physics*. 2020. 10(1): 21256.

[9] Rost, G. and Wu, J. SEIR epidemiological model with varying infectivity and infinite delay. *Mathematical biosciences and engineering*. 2008. 5(2): 389402.

[10] Bernstein, S. N. Mathematical problems of modern biology. *Science in Ukraine*. 1922. 1: 14-19.

[11] Ulam, S. M. A collection of mathematical problems. *Los Alamos scientific laboratories New Mexico*. 1964.

[12] Fermi, E., Pasta, J. and Ulam, S. Studies of Non-Linear Problems. *Los Alamos Scient. Lab. Raport LA*. 1955.

[13] Zimakov, N. P. Finite-dimensional discrete linear stochastic systems of accelerating time and their application to quadratic stochastic dynamical systems. *Mathematical Notes*. 1996. 59(5): 511517.

[14] Ganikhodzhaev, R. N. Quadratic stochastic operators, Lyapunov function and tournaments. *Acad. Sci. Sb. Math,*. 1993. 76(2): 489-506.

[15] Eshmamatova D. B. and Yusupov F. A. Dynamics of compositions of some Lotka-Volterra mappings operating in a two-dimensional simplex. *Turkish Journal of Mathematics*. 2024. 48(3): 391-406.

[16] Ewens, W. J. Mathematical population genetics. 2th Edition. *Springer-Verlag, New York*. 2004.

[17] Carrasco-Gutierrez, E. C. and Sosa, W. A discrete dynamical system and its applications. *Pesquisa Operacional*. 2019. 39(3): 457–469.

[18] Brauer, F. and Castillo-Chavez, C. Mathematical models in population biology and epidemiology. 2th Edition. *Springer Science+Business Media, LLC 2012*. 2012.

[19] James E. Gentle Matrix Algebra. Theory, Computations and Applications in Statistics *Springer*. 2017.

[20] Tadzhieva, M. A., Eshmamatova, D. B. and Ganikhodzhaev, R. N. Volterra-type quadratic stochastic operators with a homogeneous tournament., *Journal of Mathematical Sciences,* 2024. 278(3): 546-556.

[21] Eshmamatova, D. B., Tadhzieva, M. A. and Ganikhodhzaev, R. N. Criteria for the existence of internal fixed points of lotka-volterra quadratic stochastic mappings with homogeneous tournaments acting in an $(m-1)$ - dimensional simplex. *Journal of Applied Nonlinear Dynamics,*. 2023. 12(4): 679-688.

[22] Eshmamatova, D. B. and Ganikhodzhaev, R. N. Asymptotic behavior of volterra type discrete dynamical systems. *In Aip conference proceedings*. 2024. 3085(1): 020009.

[23] Eshmamatova, D. B., Tadhzieva, M. A. and Ganikhodhzaev, R. N. Invariant sets of Lotka – Volterra mappings acting in a four-dimensional simplex. *Aip conference proceedings*. 2024. 326.

[24] Eshmamatova, D. B. and Ganikhodhzaev, R. N. Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories. *Vladikavkaz. Mat. Zh.,*. 2006. 8(2): 12-28.

[25] Harary, F. und Palmer, E. M. Graph theory. *Addison-Wesley*. 1964.

[26] Topics on tournaments. *In Academic press.p*. 2013. 132.

[27] Harary, F. Graphical enumeration. *Academic Press*. 1973.

[28] Chartrand, G., Jordon, H., Vatter, V. and Zhang, P. Graphs and digraphs. *CRC Press*. 2024.

[29] Koh, K., Dong, F. and Tay, E. Introduction to graph theory. *World Scientific*. 2023.

[30] Devaney, R. L. A first course in chaotic dynamical systems: Theory and experiment. *Taylor and Francis, LLC, USA*. 2020.

[31] Galor, O. Discrete dynamical systems. *In Springer. Berlin.* 2007.

[32] Devaney, R. L. An introduction to chaotic dynamical systems. *Berlin. Springer.* 2022.

[33] Chen, G. and Huang, Y. Chaotic maps. dynamics, fractals, and rapid fluctuations. *Washington: Springer.* 2011.

[34] Bellman, R. Introduction to the theory of matrices. *M. Nauka.* 1976.

[35] Eshmamatova, D. B. and Bozorov, A. Homeomorphisms Operating in a One-Dimensional Simplex. *AIP Conference Proceedings.* 2024. 3244(1): 020034 (1-9).

[36] Eshmamatova, D. B. Discrete analog of the SIR model. *In Aip conference proceedings.* 2023. 2781. 020024(1-10).

[37] Liu, Q. and Jiang, D. Dynamical behavior of a stochastic multigroup SIR epidemic model. *Physica A: Statistical Mechanics and its Applications,.* 2019. 3085(1).

[38] Eshmamatova, D. B. Dynamics of a discrete sird model based on Lotka – Volterra mappings. *In Aip conference proceedings.* 2024. 3004(1). 020005(1-6).

[39] Eshmamatova, D. B. Compositions of Lotka – Volterra mappings as a model for the study of viral diseases. *In Aip conference proceedings.* 2024. 3085(1). 020008(1-8)

[40] Eshmamatova, D. B., Ganikhodhzaev, R. N. and Tadhzieva, M. A. Degenerate cases in Lotka – Volterra systems. *In Aip conference proceedings.* 2024. 2781. 020034(1-7).

[41] Ganikhodzhaev R. N., Eshmamatova, D. B., Tadzhieva, M. A. and Zakirov, B. S Some degenerate cases of discrete lotkavolterra dynamical systems and their applications in epidemiology. *Journal of Mathematical Sciences (United States).* 2024.