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Efficacy of Three Semi-Analytical Techniques for Solving PDEs Arising in Turbulent Flow Motion

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Abstract This research explores the solution of coupled 2-D and 3-D Burgers' equations, a partial differential equation arising in turbulent flow motion through three accurate and efficient techniques. The proposed schemes are the mixture of Sumudu transform with three classical techniques such as homotopy perturbation method (STHPM), Adomian decomposition method (STADM), and variational iteration method (STVIM). The research aims to compare these methods, evaluating their accuracy and efficacy in solving these systems of equations. The analysis seeks to uncover the strengths and limitations of each approach, contributing to the progress of numerical techniques for addressing coupled Burgers' equations. A comparison with the Finite Difference Method (FDM) is also performed to assess the efficiency of these hybrid techniques.

Keywords Burgers' equations, Sumudu transform, Homotopy perturbation method, Adomian decomposition method, variational iteration method, Finite difference method, numerical examples.

Mathematics Subject Classification 35Q35, 65M99, 35A08, 35A25, 65H10

1 Introduction

Turbulent flow is a complex and challenging area in fluid dynamics. The Burgers' equation describes turbulent flow motion in partial differential equation (PDE). It was initially introduced by Bateman in 1915, and later applied to fluid mechanics by Burgers', who used it to model turbulent flow. The Burgers' equations in two- and three- dimensions are important mathematical models that are used extensively in many scientific fields.

In this study, we explore the solution of 2-D and 3-D coupled Burgers' equations via combination of Sumudu transform with HPM, ADM and VIM, the most modern techniques. Applied sciences are highly invested in these three approaches because they compute the solutions either exactly or in a series form with exceptional accuracy. The primary benefit of these approaches is their straightforward application to every kind of differential and integral equation, regardless of whether they are linear or non-linear. A further noteworthy benefit of the approaches is their ability to significantly minimize the computing workload while preserving a high level of numerical solution accuracy. The primary intent of this study is to compare and contrast the three approaches in order to determine how well they work for tackling coupled 2-D and 3-D Burgers' equations. A comparison with the Finite Difference Method (FDM) is also performed to assess the efficiency of these hybrid techniques. By doing the comparison analysis, we want to identify the advantages and disadvantages of each approach, offering important new perspectives on semi-analytical methods for solving coupled nonlinear PDEs. Through analysis and comparison, our aim is to contribute to the ongoing discussions surrounding numerical approaches for addressing coupled PDEs, promoting advancements in computational techniques for diverse scientific and engineering applications. Consider the (3+1)-D Burgers' equation

$$\begin{cases} \zeta_{\breve{t}} + \zeta\zeta_{\breve{x}} + \eta\zeta_y + \mu\zeta_z = \dfrac{1}{Re}\left(\zeta_{\breve{x}\breve{x}} + \zeta_{yy} + \zeta_{zz}\right), \\ \eta_{\breve{t}} + \zeta\eta_{\breve{x}} + \eta\eta_y + \mu\eta_z = \dfrac{1}{Re}\left(\eta_{\breve{x}\breve{x}} + \eta_{yy} + \eta_{zz}\right), \qquad \breve{x}, y, z \in \Delta, \quad \breve{t} > 0 \\ \mu_{\breve{t}} + \zeta\mu_{\breve{x}} + \eta\mu_y + \mu\mu_z = \dfrac{1}{Re}\left(\mu_{\breve{x}\breve{x}} + \mu_{yy} + \mu_{zz}\right), \end{cases}$$

with initial conditions

$$\begin{cases} \zeta(\breve{x}, y, z, 0) = f(\breve{x}, y, z), \\ \eta(\breve{x}, y, z, 0) = g(\breve{x}, y, z), \\ \mu(\breve{x}, y, z, 0) = h(\breve{x}, y, z). \end{cases}$$

In the given mathematical context, let Δ represents the region $\Delta = \{(\breve{x}, y, z) \mid a \leq \breve{x} \leq b,\ a \leq y \leq b,\ a \leq z \leq b\}$, the velocity components to be determined are $\zeta(\breve{x}, y, z, \breve{t})$, $\eta(\breve{x}, y, z, \breve{t})$, and $\mu(\breve{x}, y, z, \breve{t})$. Re denotes the Reynolds number, and the functions f, g, and h are known functions.

Nonlinear PDEs are extensively used in various domains of sciences. Numerous techniques have been created for addressing Burgers' equation in 2D and 3D. Finding exact solution for PDEs is still crucial in these fields, driving the search for new methods to get precise and close solutions.

In the recent years, numerous hybrid approaches have been suggested to solve coupled Burgers' equation in [1, 5, 6, 18, 20]. Variational iteration method has been suggested to handle Burgers' equations in [2, 15]. Numerical solutions of Burgers' equations have been presented in [17]. A comparative study of different schemes namely Sumudu transform HPM and Elzaki transform HPM have been presented in [16]. Homotopy perturbation method and its applications have been discussed in [10, 11, 13, 14], whereas in [19] convergence and error estimation of homotopy perturbation method has been presented. In [9] homotopy perturbation method has been applied to handle fractional order PDEs. Variational iteration method and its combined form with Sumudu transform have been explained in [7, 12]. Properties of Adomian decomposition method have been elaborated in [3, 24] , whereas in [4, 6, 8] authors explored the fusion of Sumudu transform and Adomian decomposition method. The applications of Sumudu transform decomposition method to handle various types of PDEs in [21, 22]. In [23] authors tackled the Burgers' equations via finite difference method.

2 Definition and Properties of Sumudu Transform

In this section, we present the concept of the novel Sumudu transform along with its inherent characteristics [6]. We introduce the Sumudu transform method as originally proposed by G.K. Watugala in 1993. Let $E = \left\{ f(\breve{t}) : \exists M, \tau_1, \tau_2 > 0,\ |f(\breve{t})| < M e^{\frac{\breve{t}}{\tau}},\ \text{if } \breve{t} \in (-1)^j \times [0,\infty) \right\}$ be a set of functions. Then the Sumudu transform of $f(\breve{t})$, denoted by $\dot{F}(u)$ or $S[f(\breve{t}), u]$, is defined as

$$\dot{F}(u) = S[f(\breve{t}), u] = \frac{1}{u}\int_0^\infty f(\breve{t}) e^{\frac{\breve{t}}{\tau}}\, d\breve{t}, \quad u \in (-\tau_1, -\tau_2),\ 0 < \breve{t} < \infty,$$

or

$$\dot{F}(u) = S[f(\breve{t}), u] = \frac{1}{u}\int_0^\infty f(u\breve{t}) e^{-\breve{t}}\, d\breve{t},$$

where u is a parameter that may be real or complex.

Properties:

1. Sumudu transform of some standard functions is listed below

$$S(1) = 1, \quad S(\breve{t}) = u, \quad S(e^{a\breve{t}}) = \frac{1}{1-au}, \quad S(\breve{t}^k) = k!\, u^k,$$

$$S(\sin a\breve{t}) = \frac{au}{1+a^2u^2}, \quad S(\cos a\breve{t}) = \frac{1}{1+a^2u^2}.$$

2. Sumudu transform of derivatives
   If $S[f(\breve{t})] = \dot{F}(u)$, then derivatives are defined as follows
   $S[f^{(n)}(\breve{t})] = \frac{1}{u^n}\left[\dot{F}(u) - \sum_{k=0}^{n-1} u^k f^k(0)\right].$

3 Proposed Methodologies

Consider the general nonlinear non-homogeneous partial differential equation

$$D\{\zeta(\omega, \breve{t})\} + N\{\zeta(\omega, \breve{t})\} = g(\omega, \breve{t}) \tag{1}$$

with initial conditions $\zeta(\omega, 0) = f_1(\omega)$, $\zeta_{\breve{t}}(\omega, 0) = f_2(\omega)$, where ω may represent $\{\breve{x}, y\}$ or $\{\breve{x}, y, z\}$. Here, D is a linear differential operator, N is a nonlinear differential operator, and $g(\breve{x}, y, \breve{t})$ is the source term. Let S denote the Sumudu transform. Taking the Sumudu transform on both sides of Eq. (1), we obtain

$$S[D\{\zeta(\omega, \breve{t})\}] + S[N\{\zeta(\omega, \breve{t})\}] = S[g(\omega, \breve{t})]. \tag{2}$$

Incorporating the differential property of the Sumudu transform and applying the initial conditions $S[\zeta(\omega, \breve{t})] = uS[g(\omega, \breve{t})] + f_1(\omega) - uS[N\{\zeta(\omega, \breve{t})\}]$.

Applying the inverse Sumudu transform gives

$$\zeta(\omega, \breve{t}) = G(\omega, \breve{t}) - S^{-1}\left[uS\{N\{\zeta(\omega, \breve{t})\}\}\right]. \tag{3}$$

Here $G(\omega, \breve{t})$ denotes the term resulting from the source term and the specified initial conditions.

3.1 Sumudu Transform based Homotopy Perturbation Method

In STHPM, the solution is expressed as an infinite series

$$\zeta(\omega, \breve{t}) = \sum_{n=0}^{\infty} p^n \zeta_n(\omega, \breve{t}). \tag{4}$$

The nonlinear term is expanded as

$$N[\zeta(\omega, \breve{t})] = \sum_{n=0}^{\infty} p^n H_n(\zeta), \tag{5}$$

where $H_n(\zeta)$ is He's polynomial and is given by

$$H_n(\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i \zeta_i\right)\right]_{p=0}, \quad n = 0, 1, 2, 3, \ldots \tag{6}$$

Substituting Eqs. (4) and (5) into Eq. (3), we obtain

$$\sum_{n=0}^{\infty} p^n \zeta_n(\omega, \breve{t}) = G(\omega, \breve{t}) - S^{-1}\left[uS\left\{\sum_{n=0}^{\infty} p^n H_n(\zeta)\right\}\right]. \tag{7}$$

By comparing coefficients of the same powers of p,

$$\begin{aligned}
p^0 &: \quad \zeta_0(\omega, \breve{t}) = G(\omega, \breve{t}),\\
p^1 &: \quad \zeta_1(\omega, \breve{t}) = -S^{-1}\{uS[H_0(\zeta)]\},\\
p^2 &: \quad \zeta_2(\omega, \breve{t}) = -S^{-1}\{uS[H_1(\zeta)]\},\\
p^3 &: \quad \zeta_3(\omega, \breve{t}) = -S^{-1}\{uS[H_2(\zeta)]\},
\end{aligned}$$

and so on. Thus, the series solution is given by

$$\zeta(\omega, \breve{t}) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n \zeta_n(\omega, \breve{t}),$$

$$\zeta(\omega, \breve{t}) = \zeta_0(\omega, \breve{t}) + \zeta_1(\omega, \breve{t}) + \zeta_2(\omega, \breve{t}) + \cdots$$

The convergence of Sumudu transform based HPM is discussed in [14].

3.2 Sumudu Transform based Adomian Decomposition Method (STADM)

The fundamental principle of this technique was elaborated by Eltayeb and Kilicman [8], take a general PDE and its solution in infinite series as

$$\zeta(\omega, \breve{t}) = \sum_{n=0}^{\infty} p^n \zeta_n(\omega, \breve{t}). \tag{8}$$

Nonlinear term in Adomian decomposition method is solved as

$$N\{\zeta(\omega,\breve{t})\} = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where A_n is Adomian polynomial [23], is given below

$$A_n(\zeta_0,\zeta_1,\zeta_2,\ldots,\zeta_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\zeta_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,3,\ldots$$

Using values from Eq. (8) and Eq. (9) in Eq. (3), then we obtain

$$\sum_{n=0}^{\infty}\zeta_n(\omega,\breve{t}) = G(\omega,\breve{t}) - S^{-1}\left[uS\left\{\sum_{n=0}^{\infty}A_n(\zeta)\right\}\right]. \tag{10}$$

Comparing both sides of Eq. (10),

$$\begin{aligned}
\zeta_0(\omega,\breve{t}) &= G(\omega,\breve{t}),\\
\zeta_1(\omega,\breve{t}) &= -S^{-1}\{uS[A_0(\zeta)]\},\\
\zeta_2(\omega,\breve{t}) &= -S^{-1}\{uS[A_1(\zeta)]\},\\
\zeta_3(\omega,\breve{t}) &= -S^{-1}\{uS[A_2(\zeta)]\},
\end{aligned}$$

and so on. Therefore

$$\zeta_0(\omega,\breve{t}) = S(\omega,\breve{t}), \quad \text{and} \quad \zeta_{n+1}(\omega,\breve{t}) = -S^{-1}\{uS[A_n(\zeta)]\}.$$

3.3 Sumudu Transform based Variational Iteration Method (STVIM)

This technique is discussed by Singh G. and Singh I. in [8]. Take partial derivative of Eq. (3) with respect to $\breve{t}$

$$\frac{\partial}{\partial\breve{t}}\zeta(\omega,\breve{t}) - \frac{\partial}{\partial\breve{t}}G(\omega,\breve{t}) + \frac{\partial}{\partial\breve{t}}S^{-1}[uS\{N\{\zeta(\omega,\breve{t})\}\}] = 0.$$

From variational iteration method, the correctional functional is

$$\zeta_{n+1}(\omega,\breve{t}) = \zeta_n - \int_0^{\breve{t}}\lambda\left[D\zeta_n + N(\zeta_n) - \tilde{g}\right]dp,$$

where λ is known as the Lagrange's multiplier. If $D = \frac{\partial}{\partial\breve{t}}$, then $\lambda(\xi) = -1$.

$$\zeta_{n+1}(\omega,\breve{t}) = \zeta_n - \int_0^{\breve{t}}\left[(\zeta_n)(\omega,\xi) - \frac{\partial}{\partial\xi}G(\omega,\xi) + \frac{\partial}{\partial\xi}S^{-1}[\xi S\{N\{\zeta(\omega,\xi)\}\}]\right]d\xi,$$

we can write it as

$$\zeta_{n+1}(\omega,\breve{t}) = \zeta_n - S^{-1}[uS\{N\{\zeta_n(\omega,\breve{t})\}\}]. \tag{11}$$

Therefore, the solution from this method is given by

$$\zeta(\omega,\breve{t}) = \lim_{n\to\infty}\zeta_n(\omega,\breve{t}).$$

3.4 Finite Difference Method (FDM)

Discretization of domain: Divide the space domain ω which may be $\{\breve{x}, y\}$ or $\{\breve{x}, y, z\}$, into a grid using spacing h. Divide the time $\breve{t}$ into the time steps using k.
Use the grid points
2D: $\omega = (\breve{x}_i, y_j)$, $\breve{t} = \breve{t}^n$,
3D: $\omega = (\breve{x}_i, y_j, z_k)$, $\breve{t} = \breve{t}^n$.

Let the approximate numerical solution at $(\breve{x}_i, y_j, \breve{t}_n)$ in 2D be denoted by $\zeta^n_{i,j}$, and at $(\breve{x}_i, y_j, z_k, \breve{t}_n)$ in 3D by $\zeta^n_{i,j,k}$. The spatial derivatives are approximated using central difference formulas and time derivative is discretized using a forward difference scheme.
Time derivative:

$$\frac{\partial \zeta}{\partial \breve{t}} = \frac{\zeta^{n+1}_{i,j} - \zeta^n_{i,j}}{\Delta \breve{t}} = \frac{\zeta^{n+1}_{i,j} - \zeta^n_{i,j}}{k},$$
$$\frac{\partial^2 \zeta}{\partial \breve{x}^2} = \frac{\zeta^n_{i+1,j} - 2\zeta^n_{i,j} + \zeta^n_{i-1,j}}{(\Delta \breve{x})^2} = \frac{\zeta^n_{i+1,j} - 2\zeta^n_{i,j} + \zeta^n_{i-1,j}}{h^2}.$$

Then using all these values in given equation, we get the discretized equation.

4 Computational Work

Here in this section, we use Sumudu transform homotopy perturbation method, Sumudu transform Adomian decomposition method, and Sumudu transform variational iteration method in two-dimensional, three-dimensional Burgers' equation and system of Burgers' equation to understand the procedure of proposed schemes.

Example 1. Consider the 2-D Burgers' equation

$$\zeta_{\breve{t}} + \zeta\zeta_{\breve{x}} + \zeta\zeta_y - (\zeta_{\breve{x}\breve{x}} + \zeta_{yy}) = 0, \tag{12}$$

with initial condition $\zeta(\breve{x}, y, 0) = \breve{x} + y, \quad 0 \le \breve{x} \le 1,\ 0 \le y \le 1,\ \breve{t} > 0$.

Solution:

Method 1: Sumudu Transform based Homotopy Perturbation Method (STHPM)
By applying STHPM on Eq. (12)

$$\sum_{n=0}^{\infty} p^n \zeta_n = (\breve{x} + y) + pS^{-1}\Big\{uS\Big[\big(\sum_{n=0}^{\infty} p^n \zeta_n\big)_{\breve{x}\breve{x}} + \big(\sum_{n=0}^{\infty} p^n \zeta_n\big)_{yy} - \sum_{n=0}^{\infty} p^n H^1_n - \sum_{n=0}^{\infty} p^n H^2_n\Big]\Big\},$$

where H_n is He's polynomial. Comparing the coefficients of same powers of p, we obtain

$$\begin{aligned}
&p^0 : \zeta_0 = \breve{x} + y,\\
&p^1 : \zeta_1 = S^{-1}\{uS[(\zeta_0)_{\breve{x}\breve{x}} + (\zeta_0)_{yy} - H^1_0 - H^2_0]\},\\
&p^2 : \zeta_2 = S^{-1}\{uS[(\zeta_1)_{\breve{x}\breve{x}} + (\zeta_1)_{yy} - H^1_1 - H^2_1]\},\\
&p^3 : \zeta_3 = S^{-1}\{uS[(\zeta_2)_{\breve{x}\breve{x}} + (\zeta_2)_{yy} - H^1_2 - H^2_2]\},\\
&\quad \vdots
\end{aligned}$$

Some components of He's polynomials are

$$\begin{aligned}
H_0^1 &= \zeta_0\zeta_{0\breve{x}} = \breve{x}+y,\\
H_1^1 &= \zeta_1\zeta_{0\breve{x}} + \zeta_0\zeta_{1\breve{x}} = -2\breve{t}(\breve{x}+y),\\
H_2^1 &= \zeta_2\zeta_{0\breve{x}} + \zeta_1\zeta_{1\breve{x}} + \zeta_0\zeta_{1\breve{x}} = 12\breve{t}^2(\breve{x}+y),\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
H_0^2 &= \zeta_0\zeta_{0y} = \breve{x}+y,\\
H_1^2 &= \zeta_1\zeta_{0y} + \zeta_0\zeta_{1y} = -2\breve{t}(\breve{x}+y),\\
H_2^2 &= \zeta_2\zeta_{0y} + \zeta_1\zeta_{1y} + \zeta_0\zeta_{1y} = 12\breve{t}^2(\breve{x}+y),\\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
p^0 &: \zeta_0 = \breve{x}+y,\\
p^1 &: \zeta_1 = -2\breve{t}(\breve{x}+y),\\
p^2 &: \zeta_2 = 4\breve{t}^2(\breve{x}+y),\\
p^3 &: \zeta_3 = -8\breve{t}^3(\breve{x}+y),\\
&\vdots
\end{aligned}$$

The series solution is given by

$$\zeta(\breve{x},y,\breve{t}) = \sum_{n=0}^{\infty}\zeta_n = \frac{\breve{x}+y}{1+2\breve{t}}, \quad \text{here } 2\breve{t} \neq -1.$$

which is the required exact solution.

Method 2: Sumudu Transform based Adomian Decomposition Method (STADM)
Applying STADM on Eq. (12),

$$\sum_{n=0}^{\infty}\zeta_n(\breve{x},y,\breve{t}) = (\breve{x}+y) + S^{-1}\Big\{uS\Big[\big(\sum_{n=0}^{\infty}\zeta_n\big)_{\breve{x}\breve{x}} + \big(\sum_{n=0}^{\infty}\zeta_n\big)_{yy} - \sum_{n=0}^{\infty}A_n(\zeta)\Big]\Big\},$$

where A_n is Adomian's polynomial.

$$\begin{aligned}
\zeta_0 &= \breve{x}+y,\\
\zeta_1 &= -2\breve{t}(\breve{x}+y),\\
\zeta_2 &= 4\breve{t}^2(\breve{x}+y),\\
\zeta_3 &= -8\breve{t}^3(\breve{x}+y),
\end{aligned}$$

and so on. The series solution is given by

$$\zeta(\breve{x},y,\breve{t}) = \sum_{n=0}^{\infty}\zeta_n = \frac{\breve{x}+y}{1+2\breve{t}}, \quad \text{here } 2\breve{t} \neq -1.$$

which is the exact solution.

Method 3: Sumudu Transform based Variational Iteration Method (STVIM)
Applying STVIM on Eq. (12),

$$\zeta_{n+1}(\breve{x}, y, \breve{t}) = (\breve{x} + y) + S^{-1}uS\{\zeta_{n\breve{x}\breve{x}} + \zeta_{nyy} - \zeta_n\zeta_{n\breve{x}} - \zeta_n\zeta_{ny}\},$$

we obtain

$$\zeta_0 = (\breve{x} + y),$$
$$\zeta_1 = (\breve{x} + y)\{1 - 2\breve{t}\},$$
$$\zeta_2 = (\breve{x} + y)\{1 - 2\breve{t} + 4\breve{t}^2 - \frac{8}{3}\breve{t}^3\},$$
$$\zeta_3 = (\breve{x} + y)\{1 - 2\breve{t} + 4\breve{t}^2 - 8\breve{t}^3 - \frac{32}{3}\breve{t}^4 - \frac{32}{3}\breve{t}^5 + \frac{64}{9}\breve{t}^6 - \frac{128}{63}\breve{t}^7\},$$

and so on. The exact solution is obtained as

$$\zeta(\breve{x}, y, \breve{t}) = \lim_{n\to\infty} \zeta_n = \frac{(\breve{x} + y)}{1 + 2\breve{t}}, \quad \text{here } 2\breve{t} \neq -1.$$

Method 4: Finite Difference Method (FDM)
By putting the values of forward difference and central difference in equation (12), we get

$$\zeta^{n+1}_{i,j} = \zeta^n_{i,j} + \Delta\breve{t}\Bigg[- \zeta^n_{i,j}\left(\frac{\zeta^n_{i+1,j} - \zeta^n_{i-1,j}}{2\Delta\breve{x}} + \frac{\zeta^n_{i+1,j} - \zeta^n_{i-1,j}}{2\Delta y}\right) + \left(\frac{\zeta^n_{i+1,j} - 2\zeta^n_{i,j} + \zeta^n_{i-1,j}}{(\Delta\breve{x})^2} + \frac{\zeta^n_{i+1,j} - 2\zeta^n_{i,j} + \zeta^n_{i-1,j}}{(\Delta y)^2}\right)\Bigg], \tag{13}$$

Now, the grid size is $\Delta\breve{x} = \Delta y = \frac{1}{4} = 0.25$, and time step is $\Delta\breve{t} = 0.1$. Therefore spatial the grid is $\breve{x} = y = \{0, 0.25, 0.5, 0.75, 1.0\}$. Next, the first time step $\zeta^0_{i,j}$ is calculated from the values of the initial condition.

As the semi-analytical solutions obtained from the three different techniques exhibit similar results at each computational level, Figure 1 until Figure 4 illustrate the physical and dynamical behavior of the solutions for Example 1 over various ranges of $\breve{x}$, y, and $\breve{t}$.

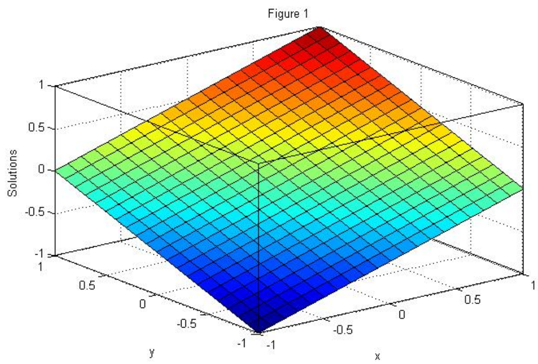

Figure 1: Physical behavior of the solution of Example 1 for $-2 \leq \breve{x}, y \leq 2$.

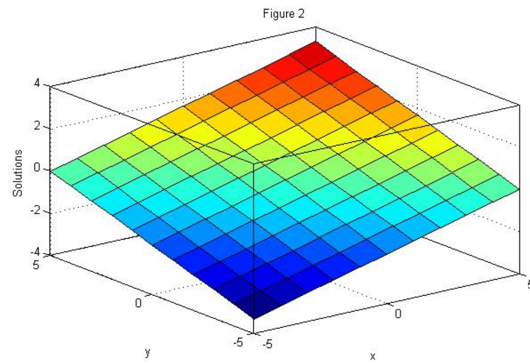

Figure 2: Physical behavior of the solution of Example 1 for $-5 \leq \breve{x}, y \leq 5$.

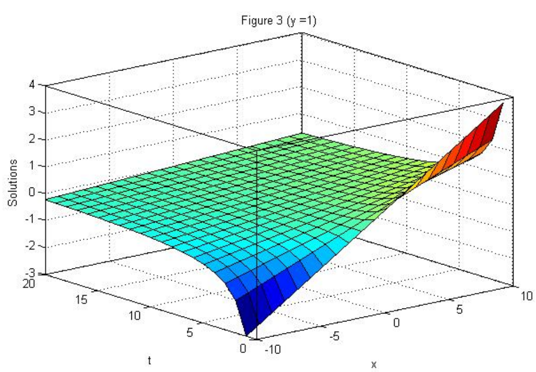

Figure 3: Physical behavior of the solution of Example 1 for $y = 1$.

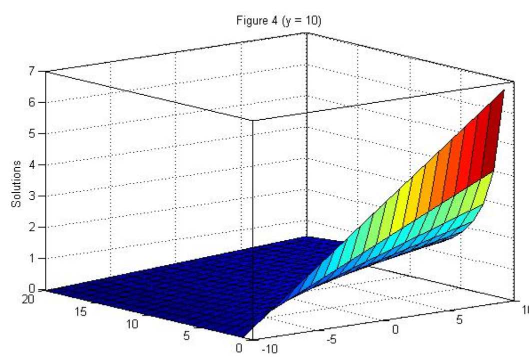

Figure 4: Physical behavior of the solution of Example 1 for $y = 10$.

In addition, Table 1 and Table 2 compare the exact solution with the STHPM and FDM methods at $\breve{x} = y = 0.5$. Both methods give results close to the exact solution. However, STHPM is more accurate. Its absolute and relative errors are very small for all values of $\breve{t}$. On the other hand, FDM shows larger errors, which increase as $\breve{t}$ increases. Therefore, STHPM is more efficient and reliable than FDM.

Table 1: Comparison between the exact solution and the solution obtained via STHPM along $\breve{x} = y = 0.5$.

$\breve{x} = y$	$\breve{t}$	Exact Solution	STHPM (first four terms)	Absolute Error	Relative Error
0.5	0.02	0.961538	0.961538	2.462×10^{-6}	2.560×10^{-6}
0.5	0.03	0.943396	0.943396	1.223×10^{-5}	1.296×10^{-5}
0.5	0.04	0.925925	0.925925	3.793×10^{-5}	4.096×10^{-5}
0.5	0.05	0.909091	0.909090	9.091×10^{-5}	1.000×10^{-4}

Note: The same numerical results (with identical absolute and relative errors) were obtained using STADM and STVIM methods as well. Hence, only the STHPM results are shown here.

Table 2: Comparison between the exact solution and the solution obtained via FDM along $\breve{x} = y = 0.5$.

$\breve{x} = y$	$\breve{t}$	Exact Solution	FDM Solution	Absolute Error	Relative Error
0.5	0.02	0.961538	0.960792	7.465×10^{-4}	7.763×10^{-4}
0.5	0.03	0.943396	0.944100	7.041×10^{-4}	7.464×10^{-4}
0.5	0.04	0.925925	0.930359	4.433×10^{-3}	4.788×10^{-3}
0.5	0.05	0.909091	0.919257	1.017×10^{-2}	1.118×10^{-2}

Example 2. Consider the system of 2-D Burgers' equations

$$\begin{cases} \zeta_{\breve{t}} + \zeta\zeta_{\breve{x}} + \eta\zeta_y - \dfrac{1}{Re}(\zeta_{\breve{x}\breve{x}} + \zeta_{yy}) = 0, \\ \eta_{\breve{t}} + \zeta\eta_{\breve{x}} + \eta\eta_y - \dfrac{1}{Re}(\eta_{\breve{x}\breve{x}} + \eta_{yy}) = 0, \end{cases} \tag{14}$$

with initial conditions

$$\begin{cases} \zeta(\breve{x}, y, 0) = -\sin(\breve{x} + y), \\ \eta(\breve{x}, y, 0) = \sin(\breve{x} + y). \end{cases}$$

Solution:

Method 1: Sumudu Transform Homotopy Perturbation Method (STHPM)
Applying the STHPM on Eq. (14), we have

$$\zeta(\breve{x},y,\breve{t}) = -\sin(\breve{x}+y) + S^{-1}\Big\{uS\Big[\frac{1}{Re}\Big(\sum_{n=0}^{\infty}p^n\zeta_n\Big)_{\breve{x}\breve{x}} + \frac{1}{Re}\Big(\sum_{n=0}^{\infty}p^n\zeta_n\Big)_{yy} - \sum_{n=0}^{\infty}p^nH_n^1 - \sum_{n=0}^{\infty}p^nH_n^2\Big]\Big\},$$

$$\eta(\breve{x},y,\breve{t}) = \sin(\breve{x}+y) + S^{-1}\Big\{uS\Big[\frac{1}{Re}\Big(\sum_{n=0}^{\infty}p^n\eta_n\Big)_{\breve{x}\breve{x}} + \frac{1}{Re}\Big(\sum_{n=0}^{\infty}p^n\eta_n\Big)_{yy} - \sum_{n=0}^{\infty}p^nH_n^3 - \sum_{n=0}^{\infty}p^nH_n^4\Big]\Big\}.$$

Comparing the coefficients of like powers of p

$$\begin{aligned}
p^0: &\quad \zeta_0 = -\sin(\breve{x}+y),\\
p^1: &\quad \zeta_1 = S^{-1}uS\left\{\frac{1}{Re}(\zeta_0)_{\breve{x}\breve{x}} + \frac{1}{Re}(\zeta_0)_{yy} - H_0^1 - H_0^2\right\},\\
p^2: &\quad \zeta_2 = S^{-1}uS\left\{\frac{1}{Re}(\zeta_1)_{\breve{x}\breve{x}} + \frac{1}{Re}(\zeta_1)_{yy} - H_1^1 - H_1^2\right\},\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
p^0: &\quad \eta_0 = \sin(\breve{x}+y),\\
p^1: &\quad \eta_1 = S^{-1}uS\left\{\frac{1}{Re}(\eta_0)_{\breve{x}\breve{x}} + \frac{1}{Re}(\eta_0)_{yy} - H_0^3 - H_0^4\right\},\\
p^2: &\quad \eta_2 = S^{-1}uS\left\{\frac{1}{Re}(\eta_1)_{\breve{x}\breve{x}} + \frac{1}{Re}(\eta_1)_{yy} - H_1^3 - H_1^4\right\},\\
&\vdots
\end{aligned}$$

and so on, therefore

$$\begin{cases} p^0: \quad \zeta_0 = -\sin(\breve{x}+y),\\ p^0: \quad \eta_0 = \sin(\breve{x}+y),\end{cases} \qquad \begin{cases} p^1: \quad \zeta_1 = \dfrac{2\breve{t}}{Re}\sin(\breve{x}+y),\\ p^1: \quad \eta_1 = -\dfrac{2\breve{t}}{Re}\sin(\breve{x}+y).\end{cases}$$

$$\begin{cases} p^2: \quad \zeta_2 = -\dfrac{4}{Re^2}\dfrac{\breve{t}^2}{2!}\sin(\breve{x}+y),\\ p^2: \quad \eta_2 = \dfrac{4}{Re^2}\dfrac{\breve{t}^2}{2!}\sin(\breve{x}+y),\end{cases} \qquad \begin{cases} p^3: \quad \zeta_3 = \dfrac{8}{Re^3}\dfrac{\breve{t}^3}{3!}\sin(\breve{x}+y),\\ p^3: \quad \eta_3 = -\dfrac{8}{Re^3}\dfrac{\breve{t}^3}{3!}\sin(\breve{x}+y),\end{cases}$$

$$\vdots$$

Therefore, the series solution of system of 2D coupled Burgers' equation is given as

$$\zeta(\breve{x}, y, \breve{t}) = \sum_{n=0}^{\infty} \zeta_n(\breve{x}, y, \breve{t}),$$

$$\zeta(\breve{x}, y, \breve{t}) = -\sin(\breve{x}+y) + \frac{2\breve{t}}{Re}\sin(\breve{x}+y) - \frac{4}{Re^2}\frac{\breve{t}^2}{2!}\sin(\breve{x}+y) + \frac{8}{Re^3}\frac{\breve{t}^3}{3!}\sin(\breve{x}+y) + \cdots,$$

$$\zeta(\breve{x}, y, \breve{t}) = -\sin(\breve{x}+y)\, e^{-\frac{2\breve{t}}{Re}},$$

and

$$\eta(\breve{x}, y, \breve{t}) = \sum_{n=0}^{\infty} \eta_n(\breve{x}, y, \breve{t}) = \eta_0 + \eta_1 + \eta_2 + \eta_3 + \cdots,$$

$$\eta(\breve{x}, y, \breve{t}) = \sin(\breve{x}+y) - \frac{2\breve{t}}{Re}\sin(\breve{x}+y) + \frac{4}{Re^2}\frac{\breve{t}^2}{2!}\sin(\breve{x}+y) - \frac{8}{Re^3}\frac{\breve{t}^3}{3!}\sin(\breve{x}+y) + \cdots.$$

This implies

$$\eta(\breve{x}, y, \breve{t}) = \sin(\breve{x}+y)\, e^{-\frac{2\breve{t}}{Re}}.$$

Thus,

$$\begin{cases} \zeta(\breve{x}, y, \breve{t}) = -\sin(\breve{x}+y)\, e^{-\frac{2\breve{t}}{Re}}, \\ \eta(\breve{x}, y, \breve{t}) = \sin(\breve{x}+y)\, e^{-\frac{2\breve{t}}{Re}}, \end{cases}$$

which is the exact solution of 2-D coupled Burgers' equation.

Method 2: Sumudu Transform Adomian Decomposition Method (STADM)
Applying STADM on Eq. (14). Few components of Adomian's polynomial are

$$A_n(\zeta) = \sum_{s=0}^{n} \zeta_s \zeta_{n-s_{\breve{x}}} + \sum_{s=0}^{n} \zeta_s \zeta_{n-s_y},$$

also

$$A_n(\eta) = \sum_{s=0}^{n} \zeta_s \eta_{n-s_{\breve{x}}} + \sum_{s=0}^{n} \eta_s \eta_{n-s_y}.$$

We obtain

$$\begin{cases} \zeta_0 = -\sin(\breve{x}+y), & \eta_0 = \sin(\breve{x}+y), \\ \zeta_1 = \frac{2\breve{t}}{Re}\sin(\breve{x}+y), & \eta_1 = -\frac{2\breve{t}}{Re}\sin(\breve{x}+y), \\ \zeta_2 = -\frac{4}{Re^2}\frac{\breve{t}^2}{2!}\sin(\breve{x}+y), & \eta_2 = \frac{4}{Re^2}\frac{\breve{t}^2}{2!}\sin(\breve{x}+y), \\ \zeta_3 = \frac{8}{Re^3}\frac{\breve{t}^3}{3!}\sin(\breve{x}+y), & \eta_3 = -\frac{8}{Re^3}\frac{\breve{t}^3}{3!}\sin(\breve{x}+y), \\ \vdots \end{cases}$$

The series solution is given by

$$\begin{cases} \zeta(\breve{x},y,\breve{t}) = \sum_{n=0}^{\infty} \zeta_n(\breve{x},y,\breve{t}) = -\sin(\breve{x}+y)e^{-\frac{2\breve{t}}{Re}}, \\ \eta(\breve{x},y,\breve{t}) = \sum_{n=0}^{\infty} \eta_n(\breve{x},y,\breve{t}) = \sin(\breve{x}+y)e^{-\frac{2\breve{t}}{Re}}. \end{cases}$$

which is an exact solution.

Method 3: Sumudu Transform Variational Iteration Method (STVIM)
Applying STVIM on Eq. (14)

$$\begin{cases} \zeta_{n+1}(\breve{x},y,\breve{t}) = -\sin(\breve{x}+y) + S^{-1}u_S \left\{ \zeta_{n,\breve{x}\breve{x}} + \zeta_{n,yy} - \zeta_n\zeta_{n,\breve{x}} - \zeta_n\zeta_{n,y} \right\}, \\ \eta_{n+1}(\breve{x},y,\breve{t}) = \sin(\breve{x}+y) + S^{-1}u_S \left\{ \eta_{n,\breve{x}\breve{x}} + \eta_{n,yy} - \eta_n\eta_{n,\breve{x}} - \eta_n\eta_{n,y} \right\}. \end{cases}$$

After simplification we get

$$\begin{cases} \zeta_0 = -\sin(\breve{x}+y), \\ \eta_0 = \sin(\breve{x}+y), \end{cases} \quad \begin{cases} \zeta_1 = -\sin(\breve{x}+y)\left[1 - \frac{2\breve{t}}{\mathrm{Re}}\right], \\ \eta_1 = \sin(\breve{x}+y)\left[1 - \frac{2\breve{t}}{\mathrm{Re}}\right], \end{cases}$$

$$\begin{cases} \zeta_2 = -\sin(\breve{x}+y)\left[1 - \frac{2\breve{t}}{\mathrm{Re}} + \frac{4\breve{t}^2}{2!\,\mathrm{Re}^2}\right], \\ \eta_2 = \sin(\breve{x}+y)\left[1 - \frac{2\breve{t}}{\mathrm{Re}} + \frac{4\breve{t}^2}{2!\,\mathrm{Re}^2}\right], \end{cases} \quad \begin{cases} \zeta_3 = -\sin(\breve{x}+y)\left[1 - \frac{2\breve{t}}{\mathrm{Re}} + \frac{4\breve{t}^2}{2!\,\mathrm{Re}^2} - \frac{8\breve{t}^3}{3!\,\mathrm{Re}^3}\right], \\ \eta_3 = \sin(\breve{x}+y)\left[1 - \frac{2\breve{t}}{\mathrm{Re}} + \frac{4\breve{t}^2}{2!\,\mathrm{Re}^2} - \frac{8\breve{t}^3}{3!\,\mathrm{Re}^3}\right]. \end{cases}$$

The closed-form solution is

$$\begin{cases} \zeta(\breve{x},y,\breve{t}) = \lim_{n\to\infty} \zeta_n = -\sin(\breve{x}+y)\, e^{-\frac{2\breve{t}}{\mathrm{Re}}}, \\ \eta(\breve{x},y,\breve{t}) = \lim_{n\to\infty} \eta_n = \sin(\breve{x}+y)\, e^{-\frac{2\breve{t}}{\mathrm{Re}}}. \end{cases}$$

which is the exact solution.

Similarly, as in Example 1, the semi-analytical solutions obtained by three different techniques are similar at each level of the procedures (Taking Re = 1). Figure 5 until Figure 8 show the physical and dynamical behavior of the solutions of Example 2 at different ranges of $\breve{x}, y$ and $\breve{t}$.

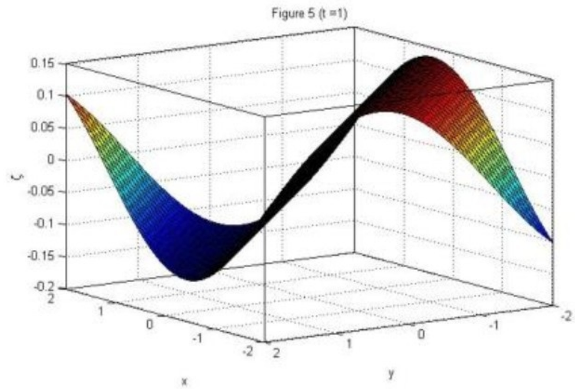

Figure 5: Physical behavior of the solution ζ of Example 2 for $-2 \leq \breve{x}, y \leq 2$.

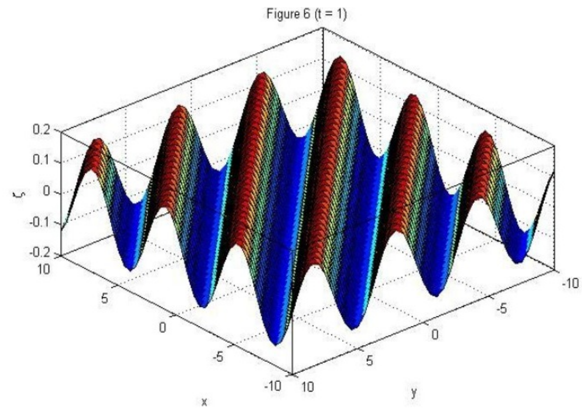

Figure 6: Physical behavior of the solution ζ of Example 2 for $-10 \leq \breve{x}, y \leq 10$.

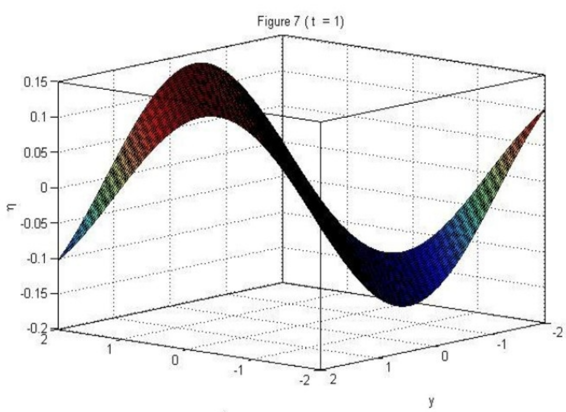

Figure 7: Physical behavior of the solution η of Example 2 for $-2 \leq \breve{x}, y \leq 2$.

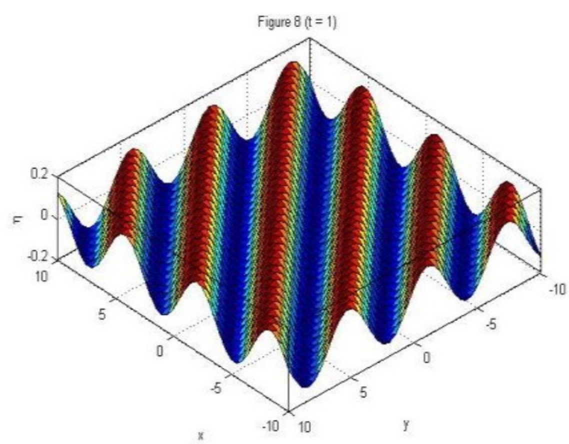

Figure 8: Physical behavior of the solution η of Example 2 for $-10 \leq \breve{x}, y \leq 10$.

Example 3. Consider the 3-D Burgers' equation

$$\zeta_t + \zeta\zeta_{\breve{x}} + \zeta\zeta_y + \zeta\zeta_z - (\zeta_{\breve{x}\breve{x}} + \zeta_{yy} + \zeta_{zz}) = 0, \tag{15}$$

with the initial condition $\zeta(\breve{x}, y, z, 0) = \breve{x} + y + z$.

Solution:

Method 1: Sumudu Transform Homotopy Perturbation Method (STHPM)
Applying the STHPM on Eq. (15) and comparing the coefficients of the same powers of p,

$$\begin{aligned}
p^0 &= \zeta_0 = \breve{x} + y + z,\\
p^1 &= \zeta_1 = -S^{-1}uS\{(\zeta_0)_{\breve{x}\breve{x}} + (\zeta_0)_{yy} + (\zeta_0)_{zz} - H_0^{(1)} - H_0^{(2)} - H_0^{(3)}\},\\
p^2 &= \zeta_2 = -S^{-1}uS\{(\zeta_1)_{\breve{x}\breve{x}} + (\zeta_1)_{yy} + (\zeta_1)_{zz} - H_1^{(1)} - H_1^{(2)} - H_1^{(3)}\},\\
p^3 &= \zeta_3 = -S^{-1}uS\{(\zeta_2)_{\breve{x}\breve{x}} + (\zeta_2)_{yy} + (\zeta_2)_{zz} - H_2^{(1)} - H_2^{(2)} - H_2^{(3)}\},\\
&\vdots
\end{aligned}$$

and so on. Here, $H_n^{(1)}, H_n^{(2)}, H_n^{(3)}$ are He's polynomials. This implies

$$\begin{aligned}
p^0 &= \zeta_0 = \breve{x} + y + z,\\
p^1 &= \zeta_1 = -3\breve{t}(\breve{x} + y + z),\\
p^2 &= \zeta_2 = 9\breve{t}^2(\breve{x} + y + z),\\
p^3 &= \zeta_3 = -27\breve{t}^3(\breve{x} + y + z),\\
&\vdots
\end{aligned}$$

Hence, the series solution $\zeta(\breve{x}, y, z, \breve{t})$ is given by

$$\zeta(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \zeta_n(\breve{x}, y, z, \breve{t}) = \frac{\breve{x} + y + z}{1 + 3\breve{t}}, \quad \text{here } 3\breve{t} \neq -1,$$

which is the exact solution of the 3-D Burgers' equation.

Method 2: Sumudu Transform Adomian Decomposition Method (STADM)
Applying STADM on Eq. (15),

$$A_n(\zeta) = \sum_{s=0}^{n} \zeta_s \zeta_{n-s\breve{x}} + \sum_{s=0}^{n} \zeta_s \zeta_{n-sy} + \sum_{s=0}^{n} \zeta_s \zeta_{n-sz},$$

where $A_n(\zeta)$ represents the Adomian's polynomial. After solving, we obtain

$$\begin{aligned}
\zeta_0 &= \breve{x} + y + z,\\
\zeta_1 &= -3\breve{t}(\breve{x} + y + z),\\
\zeta_2 &= 9\breve{t}^2(\breve{x} + y + z),\\
\zeta_3 &= -27\breve{t}^3(\breve{x} + y + z),\\
&\vdots
\end{aligned}$$

The series solution is given by

$$\zeta(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \zeta_n(\breve{x}, y, z, \breve{t}) = \frac{\breve{x} + y + z}{1 + 3\breve{t}}, \quad \text{here } 3\breve{t} \neq -1.$$

This is the exact solution of the 3-D Burgers' equation.

Method 3: Sumudu Transform based Variational Iteration Method (STVIM)
Applying STVIM on Eq. (15),

$$\begin{aligned}
\zeta_0 &= \breve{x} + y + z, \\
\zeta_1 &= (\breve{x} + y + z)\{1 - 3\breve{t}\}, \\
\zeta_2 &= (\breve{x} + y + z)\{1 - 3\breve{t} + 9\breve{t}^2 - 9\breve{t}^3\}, \\
\zeta_3 &= (\breve{x} + y + z)\{1 - 3\breve{t} + 9\breve{t}^2 - 27\breve{t}^3 + 54\breve{t}^4 - 81\breve{t}^5 + 81\breve{t}^6 - \frac{243}{7}\breve{t}^7\}.
\end{aligned}$$

The closed-form solution is

$$\zeta(\breve{x}, y, z, \breve{t}) = \lim_{n\to\infty} \zeta_n = \frac{\breve{x} + y + z}{1 + 3\breve{t}}, \quad \text{here } 3\breve{t} \neq -1,$$

which is the exact solution.

Similarly, as in Example 1 and Example 2, the semi–analytical solutions obtained by the three different techniques are similar at each level of the procedures. Figure 9 until Figure 12 show the physical and dynamical behavior of the solutions of Example 3 at different ranges of $\breve{x}$, y, and $\breve{t}$.

Example 4. Consider the system of 3-D Burgers' equations

$$\begin{cases}
\zeta_t + \zeta\zeta_{\breve{x}} + \eta\zeta_y + \mu\zeta_z - (\zeta_{\breve{x}\breve{x}} + \zeta_{yy} + \zeta_{zz}) = 0, \\
\eta_t + \zeta\eta_{\breve{x}} + \eta\eta_y + \mu\eta_z - (\eta_{\breve{x}\breve{x}} + \eta_{yy} + \eta_{zz}) = 0, \\
\mu_t + \zeta\mu_{\breve{x}} + \eta\mu_y + \mu\mu_z - (\mu_{\breve{x}\breve{x}} + \mu_{yy} + \mu_{zz}) = 0,
\end{cases} \tag{16}$$

where $\breve{x}, y, z \in \Delta$, and $\breve{t} > 0$, with initial conditions are

$$\begin{cases}
\zeta(\breve{x}, y, z, 0) = \breve{x} + y, \\
\eta(\breve{x}, y, z, 0) = \breve{x} - y, \\
\mu(\breve{x}, y, z, 0) = 1 - z.
\end{cases}$$

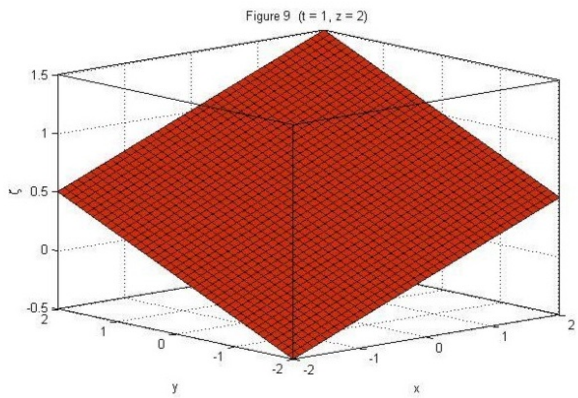

Figure 9: Physical behavior of the solution ζ of Example 3 for $-2 \leq \breve{x}, y \leq 2$, $z = 2$.

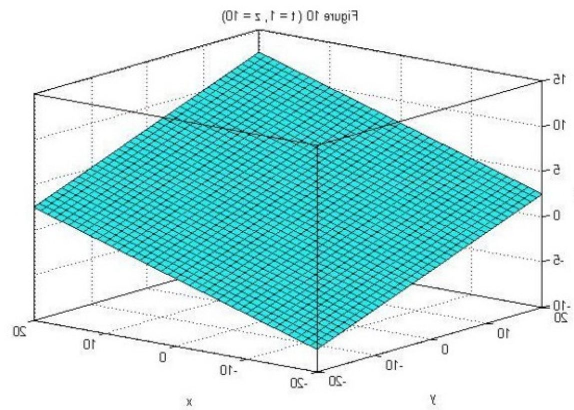

Figure 10: Physical behavior of the solution ζ of Example 3 for $-10 \leq \breve{x}, y \leq 10$, $z = 10$.

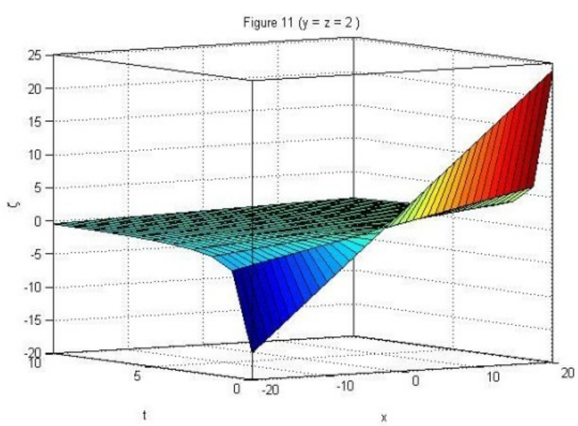

Figure 11: Physical behavior of the solution η of Example 3 for $y = z = 2$.

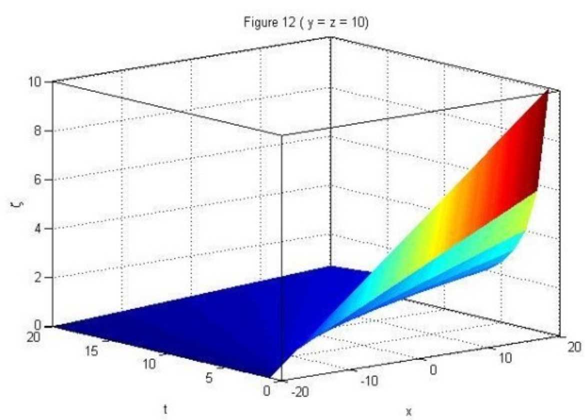

Figure 12: Physical behavior of the solution η of Example 3 for $y = z = 10$.

Solution:

Method 1: Sumudu Transform based Homotopy Perturbation Method (STHPM)
Applying STHPM on Eq. (16) and comparing the coefficients of the same powers of p

$$\begin{aligned}
p^0 &= \zeta_0 = \breve{x} + y,\\
p^1 &= \zeta_1 = -2\breve{x}\breve{t},\\
p^2 &= \zeta_2 = 2(\breve{x} + y)\breve{t}^2,\\
p^3 &= \zeta_3 = -4\breve{x}\breve{t}^3,\\
&\vdots
\end{aligned}$$

Similarly,

$$\begin{aligned}
p^0 &= \eta_0 = \breve{x} - y,\\
p^1 &= \eta_1 = -2y\breve{t},\\
p^2 &= \eta_2 = 2(\breve{x} - y)\breve{t}^2,\\
p^3 &= \eta_3 = -4y\breve{t}^3,\\
&\vdots
\end{aligned}$$

Also

$$\begin{aligned}
p^0 &= \mu_0 = 1 - z,\\
p^1 &= \mu_1 = (1 - z)\breve{t},\\
p^2 &= \mu_2 = (1 - z)\breve{t}^2,\\
p^3 &= \mu_3 = (1 - z)\breve{t}^3,\\
&\vdots
\end{aligned}$$

Here, $H_n^{(1)}, H_n^{(2)}, H_n^{(3)}, H_n^{(4)}, H_n^{(5)}, H_n^{(6)}, H_n^{(7)}, H_n^{(8)}, H_n^{(9)}$ are He's polynomials. Therefore, the solutions of the system of $(3+1)$-D coupled Burgers' equations are given as

$$\zeta(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \zeta_n(\breve{x}, y, z, \breve{t}) = \frac{\breve{x} + y - 2\breve{x}\breve{t}}{1 - 2\breve{t}^2}, \quad \text{here } 2\breve{t}^2 \neq 1,$$

$$\eta(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \eta_n(\breve{x}, y, z, \breve{t}) = \frac{\breve{x} - y - 2y\breve{t}}{1 - 2\breve{t}^2}, \quad \text{here } 2\breve{t}^2 \neq 1,$$

$$\mu(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \mu_n(\breve{x}, y, z, \breve{t}) = \frac{1 - z}{1 - \breve{t}}, \quad \text{here } \breve{t} \neq 1.$$

These are the exact solutions of the 3-D coupled Burgers' equations.

Method 2: Sumudu Transform Adomian Decomposition Method (STADM)
Applying STADM on Eq. (16), where $A_n(\zeta)$, $A_n(\eta)$, and $A_n(\mu)$ are Adomian's polynomials. Few of the components of Adomian polynomial are

$$A_n(\zeta) = \sum_{s=0}^{n} \left(\zeta_s \zeta_{n-s\,x}\right) + \sum_{s=0}^{n} \left(\zeta_s \zeta_{n-s\,y}\right) + \sum_{s=0}^{n} \left(\mu_s \zeta_{n-s\,z}\right),$$
$$A_n(\eta) = \sum_{s=0}^{n} \left(\zeta_s \eta_{(n-s)\,x}\right) + \sum_{s=0}^{n} \left(\eta_s \eta_{n-s\,y}\right) + \sum_{s=0}^{n} \left(\mu_s \eta_{n-s\,z}\right),$$
$$A_n(\mu) = \sum_{s=0}^{n} \left(\zeta_s \mu_{(n-s)\,x}\right) + \sum_{s=0}^{n} \left(\eta_s \mu_{n-s\,y}\right) + \sum_{s=0}^{n} \left(\mu_s \mu_{n-s\,z}\right).$$

The series solution is given by

$$\begin{cases} \zeta(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \zeta_n(\breve{x}, y, z, \breve{t}) = \dfrac{(\breve{x} + y - 2\breve{x}\breve{t})}{(1 - 2\breve{t}^2)}, & \text{here } 2\breve{t}^2 \neq 1, \\ \eta(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \eta_n(\breve{x}, y, z, \breve{t}) = \dfrac{(\breve{x} - y - 2y\breve{t})}{(1 - 2\breve{t}^2)}, & \text{here } 2\breve{t}^2 \neq 1, \\ \mu(\breve{x}, y, z, \breve{t}) = \sum_{n=0}^{\infty} \mu_n(\breve{x}, y, z, \breve{t}) = \dfrac{(1 - z)}{(1 - \breve{t})}, & \text{here } \breve{t} \neq 1, \end{cases}$$

which are also the exact solutions.

Method 3: Sumudu Transform based Variational Iteration Method (STVIM)
Applying STVIM on Eq. (16), and after simplification, we obtain

$$\begin{cases} \zeta_0 = \breve{x} + y, \\ \eta_0 = \breve{x} - y, \\ \mu_0 = 1 - z, \end{cases}$$

$$\begin{cases} \zeta_1 = (\breve{x} + y) - 2\breve{x}\breve{t}, \\ \eta_1 = (\breve{x} - y) - 2y\breve{t}, \\ \mu_1 = (1 - z)(1 + \breve{t}), \end{cases}$$

$$\begin{cases} \zeta_2 = (\breve{x} + y) - 2\breve{x}\breve{t} + 2\breve{x}\breve{t}^2 + 2y\breve{t}^2 - \dfrac{4\breve{x}\breve{t}^3}{3}, \\ \eta_2 = (\breve{x} - y) - 2y\breve{t} + 2\breve{x}\breve{t}^2 - 2y\breve{t}^2 - \dfrac{4y\breve{t}^3}{3}, \\ \mu_2 = (1 - z)(1 + \breve{t} + \breve{t}^2 + \dfrac{\breve{t}^3}{3}), \end{cases}$$

$$\begin{cases} \zeta_3 = (\breve{x} + y) - 2\breve{x}\breve{t} + 2\breve{x}\breve{t}^2 + 2y\breve{t}^2 - 4\breve{x}\breve{t}^3 + \dfrac{8}{3}\breve{x}\breve{t}^4 + \dfrac{8}{3}y\breve{t}^4 - \dfrac{8}{3}\breve{x}\breve{t}^5 + \dfrac{8}{9}\breve{x}\breve{t}^6 + \dfrac{8}{9}y\breve{t}^6 - \dfrac{16}{63}\breve{x}\breve{t}^7, \\ \eta_3 = (\breve{x} - y) - 2y\breve{t} + 2\breve{x}\breve{t}^2 - 2y\breve{t}^2 - 4y\breve{t}^3 + \dfrac{8}{3}\breve{x}\breve{t}^4 + \dfrac{8}{3}y\breve{t}^4 - \dfrac{8}{3}\breve{x}\breve{t}^5 + \dfrac{8}{9}\breve{x}\breve{t}^6 + \dfrac{8}{9}y\breve{t}^6 - \dfrac{16}{63}\breve{x}\breve{t}^7, \\ \mu_3 = (1 - z)\left(1 + \breve{t} + \breve{t}^2 + \breve{t}^3 + \dfrac{2}{3}\breve{t}^4 + \dfrac{1}{3}\breve{t}^5 + \dfrac{1}{9}\breve{t}^6 + \dfrac{1}{63}\breve{t}^7\right), \end{cases}$$

and so on. The exact solution like the previous two methods can be obtained in the form of a convergent series

$$\begin{cases} \zeta = \lim_{n\to\infty} \zeta_n = \dfrac{(\breve{x} + y - 2\breve{x}\breve{t})}{(1 - 2\breve{t}^2)}, & \text{here } 2\breve{t}^2 \neq 1, \\ \eta = \lim_{n\to\infty} \eta_n = \dfrac{(\breve{x} - y - 2y\breve{t})}{(1 - 2\breve{t}^2)}, & \text{here } 2\breve{t}^2 \neq 1, \\ \mu = \lim_{n\to\infty} \mu_n = \dfrac{(1 - z)}{(1 - \breve{t})}, & \text{here } \breve{t} \neq 1. \end{cases}$$

5 Discussion and Conclusion

The primary objective of this work is to perform a comparative analysis between the STADM, STHPM and STVIM. These methods are effective, powerful and providing closed form solutions. Importantly, the comparative analysis highlighted a significant alignment across these methodologies, indicating their comparable effectiveness in solving intricate problems. These hybrid schemes reduced the computation size as compared to the classical methods. This study verified that the results obtained from these three methods are identical, which ensure the efficacy and reliability of these methods. The comparative analysis of these three hybrid methods indicates that, in comparison to STADM and STHPM, STVIM streamlines calculations and offers a faster solution.

Additionally, the results obtained using STHPM are significantly more accurate than those from traditional FDM, as clearly observed in 2D case. The FDM comparison was conducted only for the 2D Burgers' equation in Example 1 to keep the analysis concise and manageable, as extending FDM to 3D or systems increasing complexity without adding substantial insight in the context of hybrid method validation. Figure 1 to Figure 12 are also included to show the physical behavior of the solution for different values of $\breve{x}$, y, z, and t. They help us understand how the solution changes with these variables. The graphs show the trends and patterns of the solution. They also confirm that the results are stable and meaningful. These figures support the results discussed in the paper.

Furthermore, the Burgers' equation addressed in this study has direct relevance to modeling turbulent flow motion. The effectiveness of the proposed methods in solving both 2D and 3D forms demonstrate their strong potential in simulating complex fluid dynamics problems. The results obtained here are accurate, reliable and computationally efficient. These findings contribute meaningfully toward advancing the numerical understanding of turbulent flow behaviors, particularly in multi-dimensional settings.

Thus, the hybrid techniques explored in this study, not only offer analytical strength but also practical relevance in simulating real-world complex systems.

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