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Determining the Optimum Process Mean for the Linear Transformation System

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Abstract The classical Taguchi quality model only considers the control of quality. Huang[9] proposed a model that took both quality and cost into account. Huang's model presents a trade-off problem between quality and cost, in which higher quality is not the final aim but the profit. In Huang's model, the symmetrical quadratic quality loss function was employed to stand for the loss of profit. In this paper, the process mean for the linear transformation is determined based on three types of loss functions: the symmetrical quadratic, asymmetrical quadratic, and asymmetrical linear loss functions. In addition, the application of Huang's model is extended to the process with constant coefficient of variation and the auto-correlated process. Some numerical examples are given for illustration.

Keywords Quality loss function, linear transformation system, mean, standard deviation.

1 Introduction

In 1986, Taguchi redefined product quality and presented quality loss function for evaluating the quality losses of products and/or services. The general expression of quality loss function for the characteristic Y is

$$L(y) = \begin{cases} k_1(y-T)^w, & \text{for } y \le T, \\ k_2(y-T)^w, & \text{for } y > T, \end{cases} \tag{1}$$

where T is the target of Y, k_1 and k_2 are the coefficients of quality loss to the left-hand and right-hand sides of the target respectively, and w is the order of the function (generally, $w = 1$ or 2). If $k_1 = k_2 = k$, equation (1) is called a symmetrical loss function; while if $k_1 \neq k_2$, equation (1) is called an asymmetrical loss function. If $w = 1$, equation (1) is called a linear loss function, and if $w = 2$, equation (1) is called a quadratic loss function. Recently, applications of some specific forms of quality loss function have been found in the literature, such as Wu & Tang [1], Li [2,3], Li & Cherng [4], Maghsoodloo & Li [5], and Duffuaa & Siddiqui [6].

The classical Taguchi quality model ([7]) only considers the control of quality. Huang [8] proposed a model that took both quality and cost into account. Huang's model presents a trade-off problem between quality and cost, in which higher quality is not the final aim but the profit. This model includes two components: the loss of profit and the cost to set the process mean and control the process variance. The loss of profit is assumed to be proportional to the loss of quality, and thus can be described by the quality loss function. Huang [8] applied the symmetrical quadratic loss function to measure the loss of profit. The objective of Huang's model is to determine the optimum process mean and variance such that the total cost, including both the loss of profit and the cost to set the process mean and control the process variance, is minimized.

In the general linear transformation system, given the input characteristic X, the output characteristic Y can be expressed as a linear function of X, namely

$$Y = bX + c, \tag{2}$$

where b and c are constants. In this paper, we will develop the procedure of determining the process mean for the linear transformation system using Huang's model. In the next section, Huang's model will be briefly reviewed. Then, Huang's model will be modified based on the symmetrical quadratic, asymmetrical quadratic and asymmetrical linear loss functions. These modified models will be extended to both the process with constant coefficient of variation and the auto-correlated process. Some numerical examples will be presented for illustration.

2 Review of Huang's Model

There are five assumptions in Huang's model ([5]). These assumptions are as follows:

(i) The input quality characteristic, X, is a normally distributed random variable with process mean μ and process variance σ^2.

(ii) The output quality characteristic, Y, is nominal-the-best and has the target value T.

(iii) The relationships between X and Y are either linear or quadratic. That is, $Y = bX + c$ or $Y = aX^2 + bX + c$ where a, b, and c are all constants.

(iv) The cost of setting μ is proportional to $|\mu|$ or μ^2, while the cost for controlling σ^2 is proportional to $1/\sigma$ or $1/\sigma^2$. Thus, the sum of the cost for process adjustment, denoted by $C_X(\mu,\ \sigma)$, may have one of the following four forms:

$C_X(\mu, \sigma) = \beta_1 |\mu| + \frac{\beta_2}{\sigma}$,
$C_X(\mu, \sigma) = \beta_1 |\mu| + \frac{\beta_2}{\sigma^2}$,
$C_X(\mu, \sigma) = \beta_1 \mu^2 + \frac{\beta_2}{\sigma}$, and
$C_X(\mu, \sigma) = \beta_1 \mu^2 + \frac{\beta_2}{\sigma^2}$,

where β_1 and β_2 are positive constants.

(v) The loss of quality results in the loss of profit, which is assumed to be proportional to the loss of quality.

According to Huang [8], the total cost of the trade-off problem, denoted by $TC(\mu, \sigma)$, is the sum of the loss of profit and the cost of process adjustment, and is the function of μ and σ . Mathematically,

$$TC(\mu, \sigma) = C_X(\mu, \sigma) + \overline{\alpha} k E[(Y - T)^2] \tag{3}$$

where $\overline{\alpha}$ is the coefficient of profit loss which is proportional to the quality loss. Huang's model is desirable to find the optimum process parameters μ^* and σ^* such that equation (3) is minimized.

3 Process Mean Determination for the Linear Transformation System

In this section, we shall modify Huang's model based on the symmetrical quadratic, asymmetrical quadratic and asymmetrical linear loss functions. Assume that the output characteristic Y is a linear function of the input characteristic X, i.e., $Y = bX + c$, where b and c are constants, and X is normally distributed with unknown μ and known variance σ^2. The output characteristic, Y, is assumed to be nominal-the-best. The process with constant standard deviation will be used as a base model, and then will be extended to both the process with constant coefficient of variation and the auto-correlated process.

3.1 The Process with Constant Standard Deviation

3.1.1 Modified Huang's model based on the symmetrical quadratic loss function

From Huang [8, p. 267], the expected loss per item of symmetrical quadratic quality loss function, $E[L(y)]$, and the Huang's cost model with the symmetrical quadratic quality loss function, $TC(\mu)$, for the linear transformation system are as follows:

$$\begin{aligned} E[L(y)] &= E[(Y-T)^2] = \int_{-\infty}^{\infty} k(bx+c-T)^2 f(x)dx \\ &= k[b^2\sigma^2 + (b\mu + c - T)^2] \end{aligned} \tag{4}$$

$$TC(\mu) = C_X(\mu,\sigma) + \alpha[b^2\sigma^2 + (b\mu + c - T)^2] \tag{5}$$

where $\alpha = \overline{\alpha}_k$.

3.1.2 Modified Huang's model based on asymmetric quadratic loss function

From Chen and Chou [9, p. 77], the expected loss per item based on asymmetrical quadratic quality loss function, $E[L(y)]$, and the modified Huang's cost model based on the asymmetrical quadratic loss function , $TC(\mu)$, for the linear transformation system are as follows:

$$\begin{aligned} E[L(y)] &= E[(Y-T)^2] = \int_{-\infty}^{\frac{T-c}{b}} k_1(T-bx-c)^2 f(x)dx + \int_{\frac{T-c}{b}}^{\infty} k_2(bx+c-T)^2 f(x)dx \\ &= [b^2\sigma^2 + (b\mu+c-T)^2](k_1-k_2)\Phi\left(\frac{\frac{T-c}{b}-\mu}{\sigma}\right) \\ &\quad +[b^2\sigma(T-\mu) + 2b\sigma(b\mu+c-T)](k_2-k_1)\phi\left(\frac{\frac{T-c}{b}-\mu}{\sigma}\right) \\ &\quad +[b^2\sigma^2 + (b\mu+c-T)^2]k_2 \end{aligned} \tag{6}$$

$$\begin{aligned} TC(\mu) &= C_x(\mu,\sigma) + \overline{\alpha}\{[b^2\sigma^2 + (b\mu+c-T)^2](k_1-k_2)\Phi\left(\frac{\frac{T-c}{b}-\mu}{\sigma}\right) \\ &\quad +[b^2\sigma(T-\mu) + 2b\sigma(b\mu+c-T)](k_2-k_1)\phi\left(\frac{\frac{T-c}{b}-\mu}{\sigma}\right) \\ &\quad +[b^2\sigma^2 + (b\mu+c-T)^2]k_2\} \end{aligned} \tag{7}$$

where $\Phi(z)$ is the cumulative distribution function for the standard normal random variable with density function $\phi(z)$ and $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ for $-\infty < z < \infty$.

3.1.3 Modified Huang's model based on asymmetric linear loss function

From Chen and Chou [9, p. 78], the expected loss per item based on asymmetrical linear quality loss function, $E[L(y)]$, and the modified Huang's cost model based on the asymmetrical linear loss function, $TC(\mu)$, for the linear transformation system are as follows:

$$\begin{aligned} E[L(y)] &= E[(y-T)] = \int_{-\infty}^{\frac{T-c}{b}} k_1(T-bx-c)f(x)dx + \int_{\frac{T-c}{b}}^{\infty} k_2(bx+c-T)f(x)dx \\ &= k_2(b\mu+c-T) + (k_1+k_2)\left[(T-b\mu-c)\Phi\left(\frac{\frac{T-c}{b}-\mu}{\sigma}\right) + b\sigma\phi\left(\frac{\frac{T-c}{b}-\mu}{\sigma}\right)\right] \end{aligned} \tag{8}$$

$$\begin{aligned} TC(\mu) &= C_X(\mu,\sigma) + \overline{\alpha}\{k_2(b\mu+c-T) \\ &\quad + (k_1+k_2)[(T-b\mu-c)\Phi(\frac{\frac{T-c}{b}-\mu}{\sigma}) + b\sigma\phi(\frac{\frac{T-c}{b}-\mu}{\sigma})]\} \end{aligned} \tag{9}$$

Given a set of model parameters, we can use the direct search method to obtain the optimal values of μ in equations (5), (7) and (9).

3.2 The Process with Constant Coefficient of Variation

The process with constant coefficient of variation has been considered by Li [2] and Maghsoodloo and Li [5] for unbalanced tolerance design. Assume that the standard deviation of the quality characteristic is proportional to the process mean. Then, the coefficient of variation can be defined as

$$r = \sigma/\mu \tag{10}$$

Substituting equation (10) into equations (5), (7), and (9) respectively, the optimal value of process mean which minimizes the TC can be obtained by the direct search method.

3.3 The Auto-Correlated Process

Traditionally, we assume that the observations from the process are independent normal random variables with constant mean and variance. However, the observations from many processes exhibit autocorrelation that may be the result of dynamics that are inherent to the process. Box and Luceno [10, p. 106] point out that a simple way to represent dependence of successive values of a disturbance is by means of autoregressive models. Thus, a stationary time series model - the first-order autoregressive model, denoted by AR(1), which may be used to describe the relationship among the random variables, can be expressed as

$$X_t = (1-\varphi)\mu + \varphi X_{t-1} + \delta_t, \quad \text{for } t = 1, 2, ... \tag{11}$$

where φ is the autoregressive coefficient for $|\varphi| < 1$ and δ_t is the random shock at time t, (δ_t's are assumed to be independent normal random variables with mean zero and variance σ^2). The process standard deviation of AR(1) model is

$$\sigma_t = \sigma/\sqrt{1-\varphi^2} \tag{12}$$

Substituting equation (12) into equations (5), (7), and (9) respectively, the optimal value of process mean which minimizes the $TC(\mu)$ may be obtained by the direct search method.

4 Numerical Examples

In this section, some numerical examples will be given for illustration.

4.1 The Process with Constant Standard Deviation

Example 1 Consider the numerical example given in Huang [8, p. 274]. Let the target value $T = 100$ and assume that $C_X(\mu, \sigma) = 2\mu^2 + \frac{1}{\sigma^2}$, $Y = 5X + 50$, and $\alpha = \overline{\alpha}k = 5$. The optimal solution of Huang [8] is $(\mu^*, \sigma^*) = (9.82, 0.3)$ with $TC(\mu^*, \sigma^*) = 219.21$.

Assume that the process standard deviation is constant. Let $\sigma = 0.5$. By solving equation (5), we obtain the optimal process mean for symmetrical quadratic quality loss function, that is $\mu^* = 9.842$ with $TC(\mu^*) = 232.1004$.

We use the above data, except for $\overline{\alpha} = 0.1, k_1 = 50, \text{and} k_2 = 100$. By solving equation (7), we obtain the optimal process mean for asymmetrical quadratic quality loss function is $\mu^* = 9.746$ with $TC(\mu^*) = 239.737$. By solving equation (9), we obtain the optimal process mean for asymmetrical linear quality loss function, that is $\mu^* = 6.248$ with $TC(\mu^*) = 175.875$.

4.2 The Process with Constant Coefficient of Variation

Example 2 Consider the data in *Example 1*, except for $r = \sigma/\mu = 0.2$ (i.e., $\sigma = 0.2\mu$). By solving equation (5), we obtain the optimal process mean for symmetrical quadratic quality loss function given by $\mu^* = 9.470$ with $TC(\mu^*) = 663.1576$. By solving equation (7), we obtain the optimal process mean for asymmetrical quadratic quality loss function, that is $\mu^* = 9.044$ with $TC(\mu^*) = 768.3335$. By solving equation (9), we obtain the optimal process mean for asymmetrical linear quality loss function, that is $\mu^* = 6.2567$ with $TC(\mu^*) = 172.5492$.

4.3 The Auto-Correlated Process

Example 3 Consider the data in *Example 1*, except for the auto-correlated process with $\varphi = 0.2$ and consequently $\sigma_t = 0.5103$. By solving equation (5), we obtain the optimal process mean for symmetrical quadratic quality loss function, which is $\mu^* = 9.842$ with $TC(\mu^*) = 233.2425$. By solving equation (7), we obtain the optimal process mean for asymmetrical quadratic quality loss function, that is $\mu^* = 9.473$ with $TC(\mu^*) = 241.2785$. By solving equation (9), we obtain the optimal process mean for asymmetrical linear quality loss function, that is $\mu^* = 6.252$ with $TC(\mu^*) = 175.715$. Table 1 lists the summary of optimal solutions for the above numerical examples. From Table 1, we have the following conclusion: the modified Huang's model based on the asymmetric linear loss function has the smallest expected loss per item than those of other modified models.

Table 1: The summary of optimal solutions.

Huang's model		
Symmetric quadratic loss function	: $(\mu^*, \sigma^*) = (9.82, 0.3)$,	$TC(\mu^*, \sigma^*) = 219.21$
Modified Huang's model with constant standard deviation		
Symmetric quadratic loss function	: $\mu^* = 9.842$,	$TC(\mu^*) = 232.1004$
Asymmetric quadratic loss function	: $\mu^* = 9.746$,	$TC(\mu^*) = 239.737$
Asymmetric linear loss function	: $\mu^* = 6.248$,	$TC(\mu^*) = 175.875$
Modified Huang's model with constant coefficient of variance $(r = \sigma/\mu = 0.2)$		
Symmetric quadratic loss function	: $\mu^* = 9.470$,	$TC(\mu^*) = 663.1576$
Asymmetric quadratic loss function	: $\mu^* = 9.044$,	$TC(\mu^*) = 768.3335$
Asymmetric linear loss function	: $\mu^* = 6.2567$,	$TC(\mu^*) = 172.5492$
Modified Huang's model with auto-correlated process $(\varphi = 0.2)$		
Symmetric quadratic loss function	: $\mu^* = 9.842$,	$TC(\mu^*) = 233.2425$
Asymmetric quadratic loss function	: $\mu^* = 9.473$,	$TC(\mu^*) = 241.2785$
Asymmetric linear loss function	: $\mu^* = 6.252$,	$TC(\mu^*) = 175.715$

5 Conclusions

In this paper, we employ Huang's cost model to determine the optimum process mean for the linear transformation system. Three types of quality loss functions are considered in the model: symmetrical quadratic, asymmetrical quadratic, and asymmetrical linear loss functions. Hereof we also determine the optimum process mean for the following three types of processes: the process with constant standard deviation, the process with constant coefficient of variation, and the auto-correlated process. Further research may be extended to the models of multiple input characteristics and the application to the quadratic transformation system.

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