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Solution of Abel's Integral Equation by Modified Laplace Transform

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Abstract This paper provides a comprehensive and detailed analysis of the modified Laplace transform, exploring its applications in various mathematical and engineering domains. Initially, the modified Laplace transform is applied to various special functions including Mittag-Leffler function, k-Struve function, Bessel-Maitland function, and Generalized function. Following this, the study examines the modified Laplace transform of nth order derivative of a function. The paper also demonstrates how the modified Laplace transform can be utilized to solve Abel's integral equation. Lastly, the research explores some practical applications of the modified Laplace transform in addressing real-world problems like chemical sciences and impulsive forces.

Keywords Modified Laplace transforms; Mittag-Leffler function; Bessel-Maitland function; k-Struve function; Generalized function.

Mathematics Subject Classification 42A38, 42B10.

1 Introduction

The integral transform known as the Laplace transform, named in honor of Pierre-Simon Laplace, converts a function originally defined in terms of a real variable, typically denoted as t and representing the time domain, into a function expressed in terms of a complex variable s. This transformation occurs within the complex frequency domain, often referred to as the s-domain or s-plane, and was first commenced in 1782 while venturing into the domain of probability theory. It has now developed as a critical tool for dealing with ordinary or partial differential equations with linear constant coefficients, particularly when addressing appropriate initial and boundary value problems, it transforms ordinary differential equations into algebraic equations and convolution into multiplication [1-5].

The Laplace transform plays a crucial role in the study of linear time-invariant systems, encompassing areas such as harmonic oscillators, electrical circuits, optical devices, and mechanical systems. Its primary function is to convert signals or functions from the time domain into the frequency domain. While it shares similarities with the Fourier transform [6], the key distinction lies in their respective approaches: the Fourier transform decomposes a signal into a series of vibration modes, while the Laplace transform dissects a function into its constituent moments. This versatile mathematical tool finds extensive applications across a wide spectrum of disciplines including physics, electrical engineering, control engineering, optics, and signal processing [7]. In this paper, we provide a detailed explanation of the modified Laplace transform introduced by Mohd Saif et al. [8]. A comprehensive discussion on the properties and applications of the modified Laplace transform can be found in their work. Unlike the classical Laplace transform, the modified version introduces a parameter a which provides additional flexibility in the region of convergence. This feature allows us to handle Abel-type singular kernels more systematically. We demonstrate that, although analytically equivalent to the classical Laplace, the Modified Laplace transform framework highlights new structural properties and offers alternative perspectives in solving Abel's equation. For further detailed information refer to [9-14].

2 Preliminaries

This section introduces the fundamental definitions, notation, and mathematical results required for solving Abel's integral equation using the Modified Laplace Transform, thereby providing the necessary foundation for the derivation and analysis of the solution method presented in the subsequent sections.

Definition 2.1 Laplace Transform ($\mathscr{L}.\mathscr{T}.$)
The $\mathscr{L}.\mathscr{T}.$ of a function $f(t)$ is given as

$$\mathscr{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt \tag{1}$$

where the function $f(t)$ is piecewise continuous and exponential order.
The inverse $\mathscr{L}.\mathscr{T}$ is as follows

$$f(t) = \mathscr{L}^{-1}\left(F(s)\right) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\int_0^\infty e^{-st} f(t)dt \tag{2}$$

where $Re(s) > 0$ and $b > 0$.

Definition 2.2 Modified Laplace Transform ($\mathscr{M}.\mathscr{L}.\mathscr{T}.$)
The $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ [8] for a function $f(t)$ which is piece-wise continuous and of exponential order α i.e. $|f(t)| \leq Me^{\alpha t}$ is established through the integral

$$\mathscr{L}_a f(t) = F(s; a) = \int_0^\infty a^{-st} f(t)dt,\ Re(s) > 0, a \in (0, \infty) \setminus 1 \tag{3}$$

provided that the integral prevails. For $a = e$, the $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ in equation (3) becomes simple $\mathscr{L}.\mathscr{T}.$ given by

$$\mathscr{L}_e\left(f\left(t\right)\right) = F\left(s; e\right) = \mathscr{L}\left(f\left(t\right)\right) = \int_0^\infty e^{-st} f\left(t\right)dt. \tag{4}$$

Definition 2.3 Gamma Function ($\mathscr{G}.\mathscr{F}.$)
The $\mathscr{G}.\mathscr{F}.$ [15] is defined as

$$\Gamma(\varkappa) = \int_0^\infty e^{-t} t^{\varkappa-1} dt, \ Re(\varkappa) > 0 \tag{5}$$

and the k-$\mathscr{G}.\mathscr{F}.$ [16] is defined as

$$\Gamma_k(\varkappa) = \int_0^\infty e^{-t^k/k} t^{\varkappa-1} dt, \ k \in \mathbb{R}^+, Re(\varkappa) > 0. \tag{6}$$

Definition 2.4 Pochhammer Symbol ($\mathscr{P}.\mathscr{S}$)
The $\mathscr{P}.\mathscr{S}$ [16] is defined as

$$(\delta)_n = \begin{cases} \delta(\delta+1)(\delta+2)\cdots(\delta+(n-1)), & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases} \tag{7}$$

where $n \in \mathbb{N}$, δ is neither zero nor a negative integer given by

$$(\delta)_n = \frac{\Gamma(\delta+n)}{\Gamma(\delta)}. \tag{8}$$

Definition 2.5 Mittag-Leffler Function ($\mathscr{M}.\mathscr{L}.\mathscr{F}.$)
The $\mathscr{M}.\mathscr{L}.\mathscr{F}.$ [17] is defined as

$$E_{\vartheta,\rho,\mu}^{\lambda,r,w}(\varkappa) = \sum_{n=0}^{\infty} \frac{(\lambda)_{nr}}{\Gamma(\vartheta\ n+\rho)(\mu)_{nw}} \frac{\varkappa^n}{n!} \tag{9}$$

where $\varkappa, \vartheta, \rho, \lambda, \mu \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\vartheta) > 0$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\mu) > 0$, $r, w > 0$, and $r \leq \mathrm{Re}(\vartheta) + w$. Moreover, $(\lambda)_{nr}$ denotes the generalized Pochhammer symbol, defined by

$$(\lambda)_{nr} = \frac{\Gamma(\lambda+nr)}{\Gamma(\lambda)}.$$

Definition 2.6 k-Struve Function
The generalized k-Struve function [18] was defined as

$$S_{\eta,\gamma}^{k}(\varkappa) = \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{\Gamma_k\left(nk+\eta+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{\varkappa}{2}\right)^{2n+\frac{\eta}{k}+1}, \ \left(k \in \mathbb{R}^+; \gamma \in \mathbb{R}; \eta > -1\right). \tag{10}$$

When $k = 1$ and $\gamma = 1$ equation (9) simplifies to produce the familiar Struve function [19] of order η, which is expressed as:

$$H_\eta(\varkappa) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(nk+\eta+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)n!} \left(\frac{\varkappa}{2}\right)^{2n+\eta+1}. \tag{11}$$

Definition 2.7 Bessel Maitland Function ($\mathscr{B}.\mathscr{M}.\mathscr{F}.$)
Let $\rho, \delta, \tau, \varrho \in \mathbb{C}$ with $Re(\rho) > 0,\ Re(\delta) > -1,\ Re(\tau) > 0,\ Re(\varrho) > 0$. Also let $q, t \in \mathbb{R}^+$ with $q < Re(\rho) + t$. The generalized $\mathscr{B}.\mathscr{M}.\mathscr{F}.$ [20] is defined as

$$J^{\rho,q,t}_{\delta,\tau,\varrho}(\varkappa) = \sum_{n=0}^{\infty} \frac{(\tau)_{nq}(-\varkappa)^n}{\Gamma(\rho\, n + \delta + 1)(\varrho)_{nt}}\ . \tag{12}$$

Definition 2.8 Generalized Function ($\mathscr{G}_{\ell}$)
Lorenzo and Hartley [21] defined the following special function named as $\mathscr{G}_{\ell}$

$$G_{\gamma,\vartheta,\alpha}[\mathfrak{a}, \varkappa] = \varkappa^{\alpha\gamma-\vartheta-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\mathfrak{a}\varkappa^{\gamma})^n}{\Gamma(\gamma n + \gamma\alpha - \vartheta)\, n!},\ Re(\alpha\gamma - \vartheta) > 0. \tag{13}$$

On taking $\alpha = 1$ and $\varkappa = \varkappa - c$ in equation (12), it reduces to special R-function [22] which is defined by Lorenzo and Hartley [21] as

$$R_{\gamma,\vartheta}[\mathfrak{a}, c, \varkappa] = \varkappa^{\gamma-\vartheta-1} \sum_{n=0}^{\infty} \frac{[\mathfrak{a}(\varkappa - c)^{\gamma}]^n}{\Gamma(\gamma n + \gamma - \vartheta)\, n!},\ \gamma \geq 0, \gamma \geq \vartheta. \tag{14}$$

3 Results and Discussion

3.1 Modified Laplace Transforms on Some Special Functions

Theorem 3.1 Let $\vartheta, \rho, \lambda, \mu \in \mathbb{C},\ Re(\lambda) > 0,\ Re(\vartheta) > 0,\ Re(\rho) > 0,\ Re(\mu) > 0,\ r, w > 0,\ \ r \leq Re(\vartheta) + w$, then

$$\mathscr{L}_{\mathscr{a}}(E^{\lambda,r,w}_{\vartheta,\rho,\mu}(\mathscr{t})) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\mathscr{s}\log \mathscr{a}} E^{\lambda,r,w}_{\vartheta,\rho,\mu}\left(\frac{1}{\mathscr{s}\log \mathscr{a}}\right). \tag{15}$$

Proof. Taking L.H.S. of (15) and applying definition of Modified laplace transfrom (3) on Mittag-Leffler function we have

$$\begin{aligned}
\mathscr{L}_{\mathscr{a}}\Big(E^{\lambda,r,w}_{\vartheta,\rho,\mu}(\mathscr{t})\Big) &= \int_0^{\infty} \mathscr{a}^{-\mathscr{s}\mathscr{t}} E^{\lambda,r,w}_{\vartheta,\rho,\mu}(\mathscr{t})\, d\mathscr{t} && (16)\\
&= \sum_{n=0}^{\infty} \frac{(\lambda)_{nr}}{\Gamma(\vartheta n + \rho)(\mu)_{nw} n!} \int_0^{\infty} \mathscr{a}^{-\mathscr{s}\mathscr{t}} \mathscr{t}^n\, d\mathscr{t} && (17)\\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\mathscr{s}\log \mathscr{a}} \frac{(\lambda)_{nr}}{\Gamma(\vartheta n + \rho)(\mu)_{nw}} \frac{1}{(\mathscr{s}\log \mathscr{a})^n} && (18)\\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\mathscr{s}\log \mathscr{a}} E^{\lambda,r,w}_{\vartheta,\rho,\mu}\left(\frac{1}{\mathscr{s}\log \mathscr{a}}\right) && (19)
\end{aligned}$$

Theorem 3.2 Let $k \in \mathbb{R}^+; \gamma \in \mathbb{R}; \eta > -1$ then

$$\mathscr{L}_{\mathscr{a}}\left(S^k_{\eta,\gamma}(\mathscr{t})\right) = \frac{\Gamma\left(2n + \frac{\eta}{k} + 1\right)}{\mathscr{s}\log \mathscr{a}} S^k_{\eta,\gamma}\left(\frac{1}{\mathscr{s}\log \mathscr{a}}\right). \tag{20}$$

Proof. Taking L.H.S. of (20) and applying definition of Modified laplace transfrom (3) on k-Struve function we have

$$\mathscr{L}_{a}\big(S^{k}_{\eta,\gamma}(t)\big) = \int_0^\infty a^{-st} S^{k}_{\eta,\gamma}(t)\,dt \tag{21}$$

$$= \int_0^\infty a^{-st} \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{\Gamma_k\big(nk+\eta+\frac{3k}{2}\big)\,\Gamma\big(n+\frac{3}{2}\big)} \left(\frac{t}{2}\right)^{2n+\frac{\eta}{k}+1} dt \tag{22}$$

$$= \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{\Gamma_k\big(nk+\eta+\frac{3k}{2}\big)\,\Gamma\big(n+\frac{3}{2}\big)} \frac{1}{2^{2n+\frac{\eta}{k}+1}} \int_0^\infty a^{-st} t^{2n+\frac{\eta}{k}+1}\,dt \tag{23}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\big(2n+\frac{\eta}{k}+2\big)\,(-\gamma)^n}{s\log a\,\Gamma_k\big(nk+\eta+\frac{3k}{2}\big)\,\Gamma\big(n+\frac{3}{2}\big)} \frac{1}{2^{2n+\frac{\eta}{k}+1}} \frac{1}{(s\log a)^{2n+\frac{\eta}{k}+2}}. \tag{24}$$

$$= \frac{\Gamma\left(2n+\frac{\eta}{k}+1\right)}{s\log a} S^{k}_{\eta,\gamma}\left(\frac{1}{s\log a}\right) \tag{25}$$

Theorem 3.3 Let $\rho,\delta,\tau,\varrho \in \mathbb{C}$ with $Re\,(\rho) > 0,\ Re\,(\delta) > -1, Re\,(\tau) > 0,\ Re\,(\varrho) > 0$. Also let $q,t \in \mathbb{R}^+$ with $q < Re\,(\rho) + t$, then

$$\mathscr{L}_{a}\left(J^{\rho,q,t}_{\delta,\tau,\varrho}(t)\right) = \frac{\Gamma\,(n+1)}{s\log a} J^{\rho,q,t}_{\delta,\tau,\varrho}\left(\frac{1}{s\log a}\right). \tag{26}$$

Proof. Taking L.H.S. of (26) and applying definition of Modified laplace transfrom (3) on Bessel-Maitand function we have

$$\mathscr{L}_{a}\big(J^{\rho,q,t}_{\delta,\tau,\varrho}(t)\big) = \int_0^\infty a^{-st} J^{\rho,q,t}_{\delta,\tau,\varrho}(t)\,dt \tag{27}$$

$$= \sum_{n=0}^{\infty} \frac{(\tau)_{nq}(-1)^n}{\Gamma(\rho n+\delta+1)(\varrho)_{nt}} \int_0^\infty a^{-st} t^n\,dt \tag{28}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)(\tau)_{nq}(-1)^n}{s\log a\,\Gamma(\rho n+\delta+1)(\varrho)_{nt}} \frac{1}{(s\log a)^n} \tag{29}$$

$$= \frac{\Gamma(n+1)}{s\log a} J^{\rho,q,t}_{\delta,\tau,\varrho}\Big(\frac{1}{s\log a}\Big) \tag{30}$$

Theorem 3.4 Let $\gamma,\vartheta,\alpha \in \mathbb{C}$ and $Re(s) > 0$ be such that

$$\mathscr{L}_{a}\left(G_{\gamma,\vartheta,\alpha}\left[\mathfrak{a},t\right];s\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,\mathfrak{a}^n}{n!\,(s\log a)^{n\gamma+\alpha\gamma-\vartheta}}. \tag{31}$$

Proof. Taking L.H.S. of (31) and applying definition of Modified laplace transfrom (3) on Generalized function we have

$$\mathscr{L}_{e}(G_{\gamma,\vartheta,\alpha}[\mathfrak{a}, t]; s) = \int_0^{\infty} e^{-st} G_{\gamma,\vartheta,\alpha}[\mathfrak{a}, t]\, dt \tag{32}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n \mathfrak{a}^n}{\Gamma(\gamma n + \gamma\alpha - \vartheta)\, n!} \int_0^{\infty} e^{-st} t^{n\gamma+\alpha\gamma-\vartheta-1}\, dt \tag{33}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n \mathfrak{a}^n}{n!\, (s \log e)^{n\gamma+\alpha\gamma-\vartheta}} \tag{34}$$

3.2 Modified Laplace Transform of Derivatives

The $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ of the n^{th}-order derivative of a function is given by

$$\begin{aligned} \mathscr{L}_{e}\left[f^{(n)}(t)\right] &= (s \log e)^n F_{e}(s) - (s \log e)^{n-1} f(0) \\ &\quad - (s \log e)^{n-2} f'(0) - (s \log e)^{n-3} f''(0) \\ &\quad - \cdots - (s \log e) f^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned} \tag{35}$$

Proof. Suppose that $f(t)$ is continuous on $[0, \infty)$ and $f'(t)$ is piecewise continuous on $(0, \infty)$. Then

$$\mathscr{L}_{e}(f'(t)) = \int_0^{\infty} \underbrace{e^{-st}}_{u} \underbrace{f'(t)\, dt}_{dv} \tag{36}$$

$$= e^{-st} f(t)\Big|_0^{\infty} - \int_0^{\infty} \left(-s \log e\, e^{-st}\right) f(t)\, dt \tag{37}$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s \log e \int_0^{\infty} e^{-st} f(t)\, dt \tag{38}$$

$$= -f(0) + s \log e\, \mathscr{L}_{e}(f(t))\,. \tag{39}$$

If f is of exponential order s_0, then

$$\lim_{t\to\infty} e^{-st} f(t) = 0, \qquad \text{whenever } s > s_0. \tag{40}$$

Hence,

$$\mathscr{L}_{e}(f'(t)) = s \log e\, \mathscr{L}_{e}(f(t)) - f(0), \tag{41}$$

or equivalently,

$$\mathscr{L}_{e}(f'(t)) = s \log e\, F_{e}(s) - f(0). \tag{42}$$

Applying the same result to $f'(t)$ gives

$$\mathscr{L}_{e}(f''(t)) = s \log e\, \mathscr{L}_{e}(f'(t)) - f'(0) \tag{43}$$

$$= s \log e\, [s \log e\, F_{e}(s) - f(0)] - f'(0) \tag{44}$$

$$= (s \log e)^2 F_{e}(s) - (s \log e) f(0) - f'(0). \tag{45}$$

Similarly,

$$\mathscr{L}_a(f'''(t)) = s\log a\, \mathscr{L}_a(f''(t)) - f''(0) \tag{46}$$
$$= (s\log a)^3 F_a(s) - (s\log a)^2 f(0) - (s\log a) f'(0) - f''(0). \tag{47}$$

Proceeding by mathematical induction, we obtain

$$\begin{aligned}\mathscr{L}_a\big(f^{(n)}(t)\big) = (s\log a)^n F_a(s) - (s\log a)^{n-1} f(0)\\ - (s\log a)^{n-2} f'(0) - (s\log a)^{n-3} f''(0) - \cdots\\ - (s\log a) f^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned} \tag{48}$$

3.3 Solution of Abel's Integral Equation by Modified Laplace Transform

In this section, we introduce the use of the Modified Laplace Transform $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ to solve Abel's Integral Equation. Niels Henrik Abel explored the motion of a particle along a smooth curve within a vertical plane, formulating this motion mathematically as Abel's Integral Equation, which is given by

$$f(t) = \int_0^t \frac{1}{\sqrt{t-u}} b(u)\, du. \tag{49}$$

Here the kernel of integral equation, $K(t, u) = \frac{1}{\sqrt{t-u}}$ becomes $\infty atu = t$, the function $f(t)$ is known function and the function $b(t)$ is unknown function.

Now, taking $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ of both sides of (49), we have

$$\mathscr{L}_a\{f(t)\} = \mathscr{L}_a\left\{\int_0^t \frac{b(u)}{\sqrt{t-u}}\, du\right\}, \tag{50}$$
$$= \mathscr{L}_a\left\{t^{-1/2} * b(t)\right\}. \tag{51}$$

Applying convolution theorem of $\mathscr{M}.\mathscr{L}.\mathscr{T}$ in (51), we have

$$\mathscr{L}_a\{f(t)\} = \mathscr{L}_a\left\{t^{-1/2}\right\} \mathscr{L}_a\{b(t)\}. \tag{52}$$

If we calculate $\mathscr{L}_a\left\{t^{-1/2}\right\}$ we have

$$\mathscr{L}_a\left\{t^{-1/2}\right\} = \int_0^\infty a^{-st} t^{-1/2} dt = \int_0^\infty e^{-(s\log a)t} t^{-1/2} dt, \tag{53}$$

where $s\log a$ is constant, resulting in $\mathscr{L}_a\left\{t^{-1/2}\right\} = \sqrt{\frac{\pi}{s\log a}}$. Hence,

$$\mathscr{L}_a\{f(t)\} = \sqrt{\frac{\pi}{s\log a}}\, \mathscr{L}_a\{b(t)\}. \tag{54}$$

Therefore,

$$\mathscr{L}_{a}\{b(t)\} = \frac{(s\log a)^{1/2}}{\sqrt{\pi}}\,\mathscr{L}_{a}\{f(t)\}\,, \tag{55}$$
$$= \frac{s\log a}{\pi}\left[\frac{\sqrt{\pi}}{(s\log a)^{1/2}}\mathscr{L}_{a}\{f(t)\}\right], \tag{56}$$
$$= \frac{s\log a}{\pi}\left[\mathscr{L}_{a}\left\{t^{-1/2}\right\}\mathscr{L}_{a}\{f(t)\}\right], \tag{57}$$
$$= \frac{s\log a}{\pi}\mathscr{L}_{a}\left\{t^{-1/2} * f(t)\right\}, \tag{58}$$
$$= \frac{s\log a}{\pi}\mathscr{L}_{a}\left\{\int_0^t \frac{f(u)}{\sqrt{t-u}}\,du\right\}, \tag{59}$$
$$= \frac{s\log a}{\pi}\mathscr{L}_{a}\{F(t)\}\,, \tag{60}$$

where $F(t) = \displaystyle\int_0^t \frac{1}{\sqrt{t-u}}f(u)du$. Now applying the property, $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ of derivative of the function on $F(t)$, we have

$$\mathscr{L}_{a}(F'(t)) = s\log a\,\mathscr{L}_{a}(F(t)) - F(0), \tag{61}$$
$$= s\log a\,\mathscr{L}_{a}(F(t))\,. \tag{62}$$

So,

$$\mathscr{L}_{a}(F(t)) = \frac{1}{s\log a}\,\mathscr{L}_{a}(F'(t))\,. \tag{63}$$

Now from (60) and (63), we have

$$\mathscr{L}_{a}\left\{b(t)\right\} = \frac{1}{\pi}\mathscr{L}_{a}\left(F'\left(t\right)\right). \tag{64}$$

Applying inverse $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ on both sides of (64), we get

$$b(t) = \frac{1}{\pi}F'\left(t\right) = \frac{1}{\pi}\frac{d}{dt}F(t)\ . \tag{65}$$

Using the function $F(t)$ in (65), we have

$$b(t) = \frac{1}{\pi}\left[\frac{d}{dt}\int_0^t \frac{1}{\sqrt{t-u}}f\left(u\right)du\right] \tag{66}$$

which is the required solution of (49).

3.4 Application of Modified Laplace Transforms

$\mathscr{M}.\mathscr{L}.\mathscr{T}.$ are widely used in physics and engineering problems and even in chemistry etc. In this section, we employ the $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ to address several problems within the domain of physics and chemistry.

Example 1 (Impulsive Force)
The expression for Newton's second law concerning the impulsive force acting on a particle with mass m moving along a linear path can be stated as follows

$$m\frac{d^2 x}{dt^2} = \mathbb{F}(t) \tag{67}$$

where $\mathbb{F}(t) = p\delta(t)$ is an impulsive force acting for an extremely brief duration with constant p. Our aim now is to obtain $x(t)$, given $x(0) = 0,\ x'(0) = 0$. Let

$$\mathscr{L}_a [x(t)] = \mathscr{X}(s). \tag{68}$$

Applying $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ to the equation of motion we get

$$m(s\log a)^2 \mathscr{X}(s) - ms\log a\, x(0) - m\ x'(0) = p,\ with\ \mathscr{L}_a[\delta(t)] = 1. \tag{69}$$

Applying the initial conditions, we have

$$m(s\log a)^2 \mathscr{X}(s) = p, \tag{70}$$

$$\mathscr{X}(s) = \frac{p}{m(s\log a)^2}. \tag{71}$$

Since

$$\mathscr{L}_a{}^{-1}\left(\frac{1}{(s\log a)^2}\right) = t, \tag{72}$$

the inverse $\mathscr{M}.\mathscr{L}.\mathscr{T}.$ yields

$$x(t) = \frac{p}{m}t, \tag{73}$$

and

$$\frac{dx(t)}{dt} = \frac{p}{m} \tag{74}$$

is a constant. Thus the effect of the impulse is to transfer, instantaneously, p units of linear momentum to the particle.

Now, consider a particle of mass $m = 2$ kilogram (kg) that is initially at rest. An impulsive force of magnitude $p = 10$ Ns, modeled by the Dirac delta function $\delta(t)$, acts on the particle at time $t = 0$. Substituting the given numerical values $p = 10$ Ns and $m = 2$ kg in (72), we get

$$x(t) = \frac{10}{2}t = 5t. \tag{75}$$

Thus, the particle moves with a constant velocity of $\dfrac{dx(t)}{dt} = 5$ m/s after the application of the impulse, as shown in Figure 1. This implies an instantaneous transfer of momentum p to the particle, resulting in uniform linear motion.

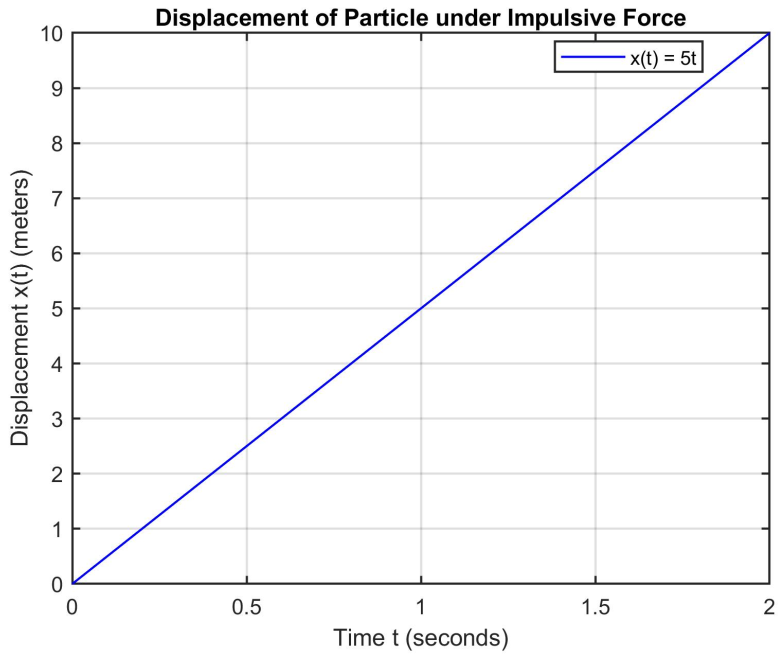

Figure 1: Displacement of a particle under impulsive force

Example 2 (Mixing Problem)
In a classic mixing scenario, there is a fixed-capacity tank filled with a well-mixed substance (e.g., salt). A solution of a specific concentration flows into the tank at a constant rate while the thoroughly mixed mixture exits at a potentially different constant rate. The tank initially contains 5000 litres (L) of water with 20 kilograms (kg) of dissolved salt. Brine with a salt concentration of 0.03 kg/L enters the tank at 25 L/min, and the solution drains out of the tank at the same rate. Suppose that we want to find how much salt remains in the tank after 30 minutes.

Let $y(t)$ denote the quantity (kg) of salt at time t minutes (where $t = 0$ represents the start time). Initially, the tank contains 20 kg of salt, denoted as $y(0) = 20$. Our objective is to determine the remaining quantity of salt at $t = 0$, represented as $y(30)$.

Moreover, if $y(t)$ denotes the quantity of substance in the tank at time t, then $y'(t)$ represents the difference between the addition rate and the removal rate. Thus, it is worth noting that $\dfrac{dy}{dt}$ represents the rate of change in the amount of salt,

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})\,. \tag{76}$$

The rate of salt entering the tank, denoted as rate in, and the rate of salt leaving the tank, denoted as rate out, can be defined. The calculation for the rate in is given by
(0.03 kg/L) (25 L/min) = 0.75 kg/min. Besides, since the tank contains 5000 L of liquid, the concentration at time t can be expressed as $y(t) = 5000$ (measured in kilograms per litre). Furthermore, considering that the brine flows out at a rate of 25 L/min, we have the following

scenario.

$$\text{rate out} = \left(\frac{y(t)}{5000}\right)\left(\frac{\text{kg}}{\text{L}}\right)(25)\left(\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}. \tag{77}$$

Then from equation (76)

$$\frac{dy}{dt} = \frac{3}{4} - \frac{y(t)}{200}, \tag{78}$$

which implies

$$\frac{dy}{dt} + \frac{y(t)}{200} = \frac{3}{4}, y(0) = 20. \tag{79}$$

Equation (79) represents an ordinary differential equation accompanied by an initial value condition. Hence, it can be extended or generalized as

$$\frac{dy}{dt} + ky(t) = r. \tag{80}$$

Applying $\mathscr{M}.\mathscr{L}.\mathscr{T}$. on both sides and simplifies give

$$\mathscr{L}_a\left(\frac{dy}{dt}\right) + k\mathscr{L}_a(y(t)) = r\mathscr{L}_a(1), \tag{81}$$

$$s\log a\mathscr{L}_a(y(t)) - y(0) + k\mathscr{L}_a(y(t)) = \frac{r}{s\log a}, \tag{82}$$

$$(s\log a + k)\mathscr{L}_a(y(t)) - 20 = \frac{r}{s\log a}, \tag{83}$$

$$\mathscr{L}_a(y(t)) = \frac{r + 20s\log a}{s\log a(s\log a + k)}. \tag{84}$$

Using a partial fraction, we have

$$\mathscr{L}_a(y(t)) = \frac{\left(\frac{r}{k}\right)}{s\log a} - \frac{\left(\frac{r-20k}{k}\right)}{(s\log a + k)}. \tag{85}$$

Next, applying inverse $\mathscr{M}.\mathscr{L}.\mathscr{T}$. on both sides will yield to

$$y(t) = \left(\frac{r}{k}\right)\mathscr{L}_a^{-1}\left(\frac{1}{s\log a}\right) - \left(\frac{r-20k}{k}\right)\mathscr{L}_a^{-1}\frac{1}{(s\log a + k)}, \tag{86}$$

$$y(t) = \left(\frac{r}{k}\right)(1) - \left(\frac{r-20k}{k}\right)e^{-kt}. \tag{87}$$

In order to solve equation (79), we put $k = \frac{1}{200}$ and $r = \frac{3}{4}$ in equation (80) as

$$y(t) = 150 - 130e^{-\frac{t}{200}}. \tag{88}$$

Finally, considering at $t = 30$ minutes, the amount of salt remaining in the tank is computed as follows.

$$\begin{aligned} y(30) &= 150 - 130e^{-30/200}, \\ &\approx 150 - 130e^{-0.15}, \\ &\approx 150 - 130 \times 0.8607, \\ &\approx 150 - 111.89, \\ &\approx 38.11 \text{ kg}. \end{aligned} \tag{89}$$

Hence, after 30 minutes, approximately 38.11 kg of salt remains in the tank (see Figure 2 for the graph of $y(t)$ against t).

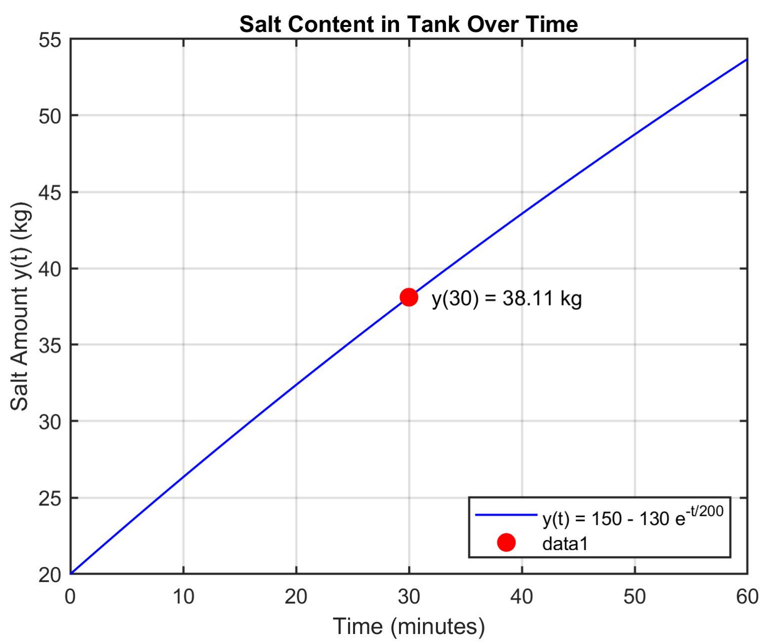

Figure 2: Salt concentration in the mixing tank after 30 minutes

4 Conclusion

This paper has presented a comprehensive study of the Modified Laplace transform, demonstrating its applicability to a range of special functions including the Mittag-Leffler function, k-Struve function, Bessel-Maitland function, and generalized functions. The analysis extended to the Modified Laplace transform of the nth-order derivative, revealing its utility in handling more generalized forms of differential equations and explored its practical relevance in modeling problems from impulsive forces and chemical sciences, emphasizing its potential as a powerful tool for both theoretical advancement and real-world applications and supported the theory with numerical validation. The paper further solves the Abel's integral equation with the help of Modified Laplace transform and for future work one can extend the approach to other singular integral equations and explore stable numerical inversion methods.

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