MATEMATIKA, 2005, Volume 21, Number 2, pp. 49–65

Improvements on Geometric Means Related to the Tracy-Singh Products of Positive Matrices

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Abstract In this paper, a family of geometric means for positive matrices is studied; we discuss possible definitions of the geometric means of positive matrices, and some counter examples are given. It is still an open problem to find a completely satisfactory definition. Other problems are how to define the geometric, arithmetic, harmonic, $\alpha-$power and operator means of finitely many positive matrices. We generalize these means of two positive matrices to arrive the definitions of the weighted means of k positive matrices. We recover and develop the relationship between the Ando's geometric mean and the Kronecker product to the Tracy-Singh product and other means. Some new attractive inequalities for the Tracy-Singh product, Khatri-Rao product and geometric means of several positive matrices are established. The results lead to the case of Kronecker and Hadamard products of any finite numbers of matrices.

Keywords Tracy-Singh product, Khatri-Rao product, Kronecker product, Hadamard product, Positive definite matrix, Geometric means, weighted geometric means.

1 Introduction and Preliminary Results

Consider matrices $A = [a_{ij}]$ and $C = [c_{ij}]$ of order $m \times n$ and $B = [b_{kl}]$ of order $p \times q$. Let A and B be partitioned as $A = [A_{ij}]$ and $B = [B_{kl}]$ $(1 \leq i \leq t, 1 \leq j \leq c)$, where A_{ij} is an $m_i \times n_j$ matrix and B_{kl} is a $p_k \times q_l$ matrix $(m = \sum_{i=1}^{t} m_i, n = \sum_{j=1}^{c} n_j, p = \sum_{i=1}^{t} p_i, q = \sum_{j=1}^{c} q_j)$. Let $A \otimes B$, $A \circ B$, $A\Theta B$ and $A * B$ be the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, respectively. The definitions of the mentioned four matrix products are given by Liu in [8, 9] as $A \otimes B = (a_{ij}B)_{ij}$, $A \circ C = (a_{ij}c_{ij})_{ij} = C \circ A$, $A * B = (A_{ij} \otimes B_{ij})_{ij}$ and $A\Theta B = (A_{ij}\Theta B)_{ij} = ((A_{ij} \otimes B_{kl})_{kl})_{ij}$ .Additionally, Liu [8] shows that the Khatri-Rao product can be viewed as a generalized Hadamard product and the Tracy-Singh product as a generalized Kronecker product, i.e., for non-partitioned matrices A and B, their $A\Theta B$ is $A \otimes B$ and their $A * B$ is $A \circ B$. The Khatri-Rao and Tracy-Singh products of matrices $A_i (1 \leq i \leq k, k \geq 2)$ will be denoted by

$$\prod_{i=1}^{k} *A_i = A_1 * A_2 * \ldots * A_k \text{ and } \prod_{i=1}^{k} \Theta A_i = A_1 \Theta A_2 \Theta \ldots \Theta A_k,$$

respectively.

For Hermitian matrices A and B, the relation $A > B$ means that $A - B > 0$ is a positive definite and the relation $A \geq B$ means $A - B \geq 0$ is a positive semi-definite. Given a positive definite matrix A, its positive definite square root is denoted by $A^{1/2}$. Notice that for positive definite matrices A and B, the relation $A \geq B$ implies $A^{1/2} \geq B^{1/2}$, $A^2 \geq B^2$ and $B^{-1} \geq A^{-1}$.

Let us introduce some notations. The notation $M_{m,n}(M^+_{m,n})$ is the set of all $m \times n$ (positive definite) matrices over M and when $m = n$, we write $M_m(M^+_m)$ instead of $M_{m,n}(M^+_{m,n})$. The notations A^T, A^*, A^{-1} are the transpose, conjugate transpose and inverse of matrix A, respectively. The Khatri - Rao and Tracy - Singh products are related by the following relation [5,14]:

$$\prod_{i=1}^{k} *A_i = Z_1^T(\prod_{i=1}^{k} \Theta A_i)Z_2. \tag{1-1}$$

Here, $A_i \in M_{m(i),n(i)}(1 \leq i \leq k, k \geq 2)$ are compatibly partitioned matrices, $(m = \prod_{i=1}^{k} m(i),\ n = \prod_{i=1}^{k} n(i)\ ,\ r = \sum_{j=1}^{t}\prod_{i=1}^{k} m_j(i),\ s = \sum_{j=1}^{c}\prod_{i=1}^{k} n_j(i),\ m(i) = \sum_{j=1}^{t} m_j(i),$ $n(i) = \sum_{j=1}^{t} n_j(i))$. Z_1 and Z_2 are real matrices of order $m \times r$ and $n \times s$, respectively such that $Z_1^T Z = I_1$, $Z_2^T Z = I_2$, where I_1 and I_2 are identity matrices of order $r \times r$ and $s \times s$, respectively. In particular, if $m(i) = n(i)$, we then have

$$\prod_{i=1}^{k} *A_i = Z^T(\prod_{i=1}^{k} \Theta A_i)Z. \tag{1-2}$$

We shall make frequent use the following properties of the Tracy-Singh and Khatri-Rao products and their proofs are straightforward by using induction on k. Let A_i and $B_i(1 \leq i \leq k, k \geq 2)$ be compatible partitioned matrices, we have

$$(a) \left(\prod_{i=1}^{k} \Theta A_i\right)\left(\prod_{i=1}^{k} \Theta B_i\right) = \left(\prod_{i=1}^{k} \Theta (A_i B_i)\right). \tag{1-3}$$

$$(b) \left(\prod_{i=1}^{k} \Theta A_i\right)^* = \prod_{i=1}^{k} \Theta A_i^* \quad \text{and} \left(\prod_{i=1}^{k} *A_i\right)^* = \prod_{i=1}^{k} *A_i^*. \tag{1-4}$$

$$(c) \left(\prod_{i=1}^{k} \Theta A_i\right)^r = \prod_{i=1}^{k} \Theta A_i^r \text{ if } A_i \in M^+_{m_i}(1 \leq i \leq k, k \geq 2) \quad \text{and } r \text{ is any real number.} \tag{1-5}$$

$$(d) \left(\prod_{i=1}^{k} (A_i \Theta B_i)\right) = \left(\prod_{i=1}^{k} A_i\right) \Theta \left(\prod_{i=1}^{k} B_i\right). \tag{1-6}$$

In this paper, the results are established in three ways. First, we discuss possible definitions of the geometric means of two matrices and weighted means of k positive matrices, provided some counter examples are given. It is still an open problem to find a completely satisfactory definition. Second, we discover some interesting new inequalities involving Tracy-Singh product, Khatri-Rao product and geometric means of several positive matrices. Finally, the results lead to the case of Kronecker and Hadamard products of any finite numbers of positive matrices.

2 Geometric Means of Two Positive Matrices

If A and B are arbitrary $n \times n$ matrices, then the *arithmetic mean* is defined by

$$A \sim B = \frac{1}{2}(A + B). \tag{2-1}$$

Similarly, when A and B are positive $n \times n$ matrices, their *harmonic mean* can be defined as

$$A!B = \left\{\frac{1}{2}(A^{-1} + B^{-1})\right\}^{-1}. \tag{2-2}$$

However, it is not at all obvious how to define the *geometric mean* of positive matrices. In what follows, if $A \in M_n^+$ and α is any real number, then A^α will denote its unique positive (semi)definite α^{th} power. Ideally, the *geometric mean* $A\#B$ of two positive matrices and should satisfy the following properties:

(i) $A\#B = (AB)^{1/2}$ (when A and B commute) (2-3)

(ii) $A\#B \geq 0$ (Positive property) (2-4)

(iii) $A\#B = B\#A$ (Symmetry property)

(iv) $A!B \leq A\#B \leq A \sim B$ (Arithmetic-Geometric-Harmonic inequality) (2-5)

The obvious candidates for $A\#B$ are not satisfactory. We now explain the problems which arise with some of these candidates in the case that A and B are positive $n \times n$ matrices.

$$\textbf{Candidate (1)} : A\#^1 B = (AB)^{1/2} \tag{2-6}$$

This has the drawback need not a positive semi definite square root unless $AB = BA$. To see this, observe that if $A > 0$, $B > 0$ and $AB \geq 0$, then

$$AB = (AB)^* = B^*A^* = BA$$

So, if $AB \neq BA$, we must conclude that $AB \ngeq 0$ and so it is impossible for any square root of AB to be positive semi definite.

$$\textbf{Candidate (2)} : A\#^2 B = A^{1/2}B^{1/2} \tag{2-7}$$

Again, there are problems with the positivity property. Notice that if $A > 0$, $B > 0$ and $A^{1/2}B^{1/2} = B^{1/2}A^{1/2}$, then $AB = BA$. Hence if $AB \neq BA$, we must conclude that $A^{1/2}B^{1/2} \neq B^{1/2}A^{1/2}$, and so, as above, $A\#^2 B \ngeq A^{1/2}B^{1/2}$.

$$\textbf{Candidate (3)} : A\#^3 B = A^{1/4}B^{1/2}A^{1/4} \tag{2-8}$$

Certainly $A\#^3B$ satisfies property (i) and by design, also satisfies the positivity property (ii). However, $A\#^3B$ need not be symmetric. To see this, consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1.1 & 1 \\ 1 & 1 \end{bmatrix}$$

For these matrices, *mathematica* shows that:

$$A\#^3B = \begin{bmatrix} 1.21163 & 0.719418 \\ 0.719418 & 0.796259 \end{bmatrix}, \quad B\#^3A = \begin{bmatrix} 1.15706 & 0.732975 \\ 0.732975 & 0.850833 \end{bmatrix}$$

and these are clearly different. In 1979, Ando [2] gave an apparently complicated variant of candidate3, which turns out to be very useful and satisfy all properties (i)-(iv). When A and B are positive $n \times n$ matrices, *Ando's geometric mean* is defined by

$$A\#B = A^{1/2}D^{1/2}A^{1/2} \quad \text{with} \quad D = A^{-1/2}BA^{-1/2}. \tag{2-9}$$

It is clear to show that *Ando's geometric mean* enjoy the following properties:

$$(a) A\#A = A. \qquad (b) A^p\#A^q = A^{(p+q)/2}, \quad \text{for all} \quad -\infty < p, q < \infty. \tag{2-10}$$

$$(c)(A\#B)A^{-1}(A\#B) = B \qquad (d)(AB^{-1}A)\#B = A. \tag{2-11}$$

$$(e) A^{-1/2}(A\#B)B^{-1/2} \quad \text{is a unitary matrix.} \tag{2-12}$$

Now we use the fact that if X and Y are positive $n \times n$ matrices, then if and only if $X^2 = Y^2$. This follows from the observation that a positive $n \times n$ matrix has a unique positive square root. Thus $A\#B = B\#A$ is equivalent to

$$\left(B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2}\right)^2 = B^{-1/2}AB^{-1/2}. \tag{2-13}$$

To see that this is indeed true, expand out the square

$$\begin{aligned}
&\left(B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2}\right)^2 \\
&= \left(B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2}\right)\left(B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2}\right) \\
&= B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2}B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2} \\
&= B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}(A^{-1/2}BA^{-1/2})^{-1}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}B^{-1/2} \\
&= B^{-1/2}A^{1/2}A^{1/2}B^{-1/2} = B^{-1/2}AB^{-1/2}. \qquad \square
\end{aligned}$$

Ando's geometric mean also satisfies the arithmetic-geometric-harmonic mean inequality. The geometric-arithmetic means inequality can be established by taking inverses. Let $A > 0$ and $B > 0$ be $n \times n$ matrices and write $D = A^{-1/2}BA^{-1/2}$. It is clear that $D > 0$. Now

$$\begin{aligned}
A\#B \le A \sim B &\Leftrightarrow A^{1/2}\left\{A^{-1/2}BA^{-1/2}\right\}^{1/2}A^{1/2} \le \frac{1}{2}(A+B) \\
&\Leftrightarrow A^{1/2}\left\{A^{-1/2}BA^{-1/2}\right\}^{1/2}A^{1/2} \le \frac{1}{2}A^{1/2}(I + A^{-1/2}BA^{-1/2})A^{-1/2} \\
&\Leftrightarrow \left\{A^{-1/2}BA^{-1/2}\right\}^{1/2} \le \frac{1}{2}(I + A^{-1/2}BA^{-1/2}) \Leftrightarrow D^{1/2} \le \frac{1}{2}(I+D).
\end{aligned} \tag{2-14}$$

This last inequality is the arithmetic-geometric mean inequality for the positive definite matrix. Since $A > 0$, there are a unitary matrix U and a diagonal matrix $S = \text{diag}[s_1, ..., s_n]$ with $s_i > 0$ $(1 \leq i \leq n)$ satisfying $A = U^*SU$. Thus $D^{1/2} \leq \frac{1}{2}(I + D)$ is equivalent to $s_i^{1/2} \leq \frac{1}{2}(1 + s_i)$, for all $(1 \leq i \leq n)$ and this is obviously true.

Ando [2] established further nice properties of his geometric mean of two positive $n \times n$ matrices

$$(v) C^*(A\#B)C = (C^*AC)\#(C^*BC), \quad \text{for all} \quad C \in M_n. \quad (\textit{Distributive property}) \tag{2-15}$$

$$(vi) (A_1 \otimes B_1)\#(A_2 \otimes B_2) = (A_1\#A_2) \otimes (B_1\#B_2). \quad (\textit{Mixed property}) \tag{2-16}$$

$$(vii) (A\#B)^{-1} = A^{-1}\#B^{-1}. \quad (\textit{Inverse property}) \tag{2-17}$$

We examine these in the next section, where we will give some extensions and generalizations. Now, we have

$$(A \sim B)\#(A!B) = A\#B \tag{2-18}$$

To see that this is indeed true:

$$\begin{aligned}(A \sim B)\#(A!B) &= A^{1/2}\left\{(I \sim D)\#(I!D)\right\}A^{1/2} \\ &= A^{1/2}\left\{((1/2)(I + D))^{1/2}\left((1/2)(I + D^{-1})\right)^{-1/2}\right\}A^{1/2} \\ &= A^{1/2}D^{1/2}A^{1/2} = A\#B.\end{aligned}$$

Ando used his definition to study monotone functions of matrices and obtained many nice results [2]. In spite of all this, Ando's definition is not completely satisfactory. It fails to satisfy one very desirable property:

$$(viii) \ (A\#B)^2 \quad \text{is similar to} \quad AB. \quad (\textit{Eigenvalue property}) \tag{2-19}$$

To see why this is a problem, consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then, computing with *Mathematica*, we find

$$A\#B = \begin{bmatrix} 1.84776 & 0.541196 \\ 0.541196 & 0.92388 \end{bmatrix}, \quad \text{and} \quad AB = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$$

Here, the eigenvalues of AB are 0.438447 and 4.56155, while the eigenvalues of $(A\#B)^2$ are 0.45466 and 4.39889 . Hence $(A\#B)^2$ is not similar to AB, since similar matrices must have the same eigenvalues.

In 1997, Fiedler and Ptak [7] succeeded in finding a definition of geometric mean that does satisfy the eigenvalue property. They used

$$A \diamondsuit B = (A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2}, \tag{2-20}$$

for positive matrices A and B. Unfortunately, this definition fails the arithmetic-geometric -harmonic inequality. It satisfy properties from (i) to (iii) and from (v) to (viii). We give a new direct proof of the eigenvalue property, assuming that $A \diamondsuit B = B \diamondsuit A$, for positive matrices A and B. Observe that

$$A \diamondsuit B = (A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2} = (B^{-1}\#A)^{1/2}B(B^{-1}\#A)^{1/2}. \tag{2-21}$$

Thus, by property (iii) of Ando's geometric mean, we have

$$\begin{aligned}(A\Diamond B)^2 &= (A\Diamond B)(B\Diamond A) = (A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2}(B^{-1}\#A)^{1/2}B(B^{-1}\#A)^{1/2}\\ &= (A^{-1}\#B)^{1/2}A\left((B^{-1}\#A)^{1/2}\right)^{-1}(B^{-1}\#A)^{1/2}B(B^{-1}\#A)^{1/2}\\ &= (A^{-1}\#B)^{1/2}AB(B^{-1}\#A)^{1/2} = \left((B^{-1}\#A)^{1/2}\right)^{-1}(AB)(B^{-1}\#A)^{1/2}\\ &= Q^{-1}(AB)Q \qquad (Q = (B^{-1}\#A)^{1/2}).\end{aligned}$$

Hence, $(A\Diamond B)^2$ is not similar to AB. $\square$

In 1998, Ando and Hiai [4] succeeded in generalizing Ando's geometric mean to the $\alpha-$*power mean* that satisfy properties from (i) to (vii). They used

$$A \underset{\alpha}{\#} B = A^{1/2}D^{\alpha}A^{1/2} \quad \text{with} \quad D = A^{-1/2}BA^{-1/2}, \tag{2-22}$$

for any real number α and positive matrices A and B. In particular if $\alpha = \frac{1}{2}$, we get Ando's geometric mean, i.e., $A \underset{1/2}{\#} B = A\#B$.

It is clear to show that $\alpha-$ *power mean* enjoy the following new properties:

$$(a) A \underset{\alpha}{\#} A = A \qquad (b) A^p \underset{\alpha}{\#} A^q = A^{(1-\alpha)p+\alpha q}, \quad \text{for all} \quad -\infty < p, q < \infty. \tag{2-23}$$

In 2000, Micic, Pecaric and Seo [10] succeeded in generalizing the $\alpha-$ power mean to the *operator mean*. They used

$$A\sigma B = A^{1/2}f(D)A^{1/2} \quad \text{with} \quad D = A^{-1/2}BA^{-1/2}, \tag{2-24}$$

for any non-negative operator monotone function $f(t)$ on $[0, \infty)$ and positive matrices A and B. As a matter of fact, the $\alpha-$ power means are determined by the operator monotone function $f(t) = t^{\alpha}$ when $0 < \alpha \le 1$ or by the operator monotone function $f(t) = t^{1/\alpha}$ when $1 \le \alpha < \infty$.

It remains an **open question** whether there is a definition of the geometric mean of two positive $n \times n$ matrices that satisfies all of properties from (i) - (viii).

We extend the mixed property to include the Tracy-Singh product as in next result.

Lemma 2.1 Let A_i and $B_i \in M_n^+$ $(i = 1, 2)$ be compatible partitioned matrices. Then

$$(a)\ (A_1\Theta B_1)\#(A_2\Theta B_2) = (A_1\#A_2)\Theta(B_1\#B_2) \tag{2-25}$$

$$(b)\ (A_1\#B_1)\Theta(A_2\#B_2) = (A_1\Theta A_2)\#(B_1\Theta B_2) \tag{2-26}$$

Proof (a) By using (1-4) and (1-5), we have

$$
\begin{aligned}
&(A_1\Theta B_1)\#(A_2\Theta B_2)\\
&\quad=(A_1\Theta B_1)^{1/2}\left\{(A_1\Theta B_1)^{-1/2}(A_2\Theta B_2)(A_1\Theta B_1)^{-1/2}\right\}^{1/2}(A_1\Theta B_1)^{1/2}\\
&\quad=(A_1^{1/2}\Theta B_1^{1/2})\left\{(A_1^{-1/2}\Theta B_1^{-1/2})(A_2\Theta B_2)(A_1^{-1/2}\Theta B_1^{-1/2})\right\}^{1/2}(A_1^{1/2}\Theta B_1^{1/2})\\
&\quad=(A_1^{1/2}\Theta B_1^{1/2}\left\{A_1^{-1/2}A_2A_1^{-1/2}\Theta B_1^{-1/2}B_2B_1^{-1/2}\right\}(A_1^{1/2}\Theta B_1^{1/2})\\
&\quad=(A_1^{1/2}\Theta B_1^{1/2})\left\{(A_1^{-1/2}A_2A_1^{-1/2})^{1/2}\Theta(B_1^{-1/2}B_2B_1^{-1/2})^{1/2}\right\}(A_1^{1/2}\Theta B_1^{1/2})\\
&\quad=A_1^{1/2}\left\{(A_1^{-1/2}A_2A_1^{-1/2})^{1/2}\right\}A_1^{1/2}\Theta B_1^{1/2}\left\{(B_1^{-1/2}B_2B_1^{-1/2})^{1/2}\right\}B_1^{1/2}\\
&\quad=(A_1\#A_2)\Theta(B_1\#B_2).
\end{aligned}
$$

We can prove (b) in a similar manner. □

We extend also Lemma (2.1) to include the Fiedler and Ptak definition and $\alpha-$ *power mean* definition related to the Tracy -Singh product as in next result.

Lemma 2.2 For any real number α and positive compatible partitioned matrices A_i and $B_i(i=1,2)$, we have

$$(a)(A_1\Theta B_1)\underset{\alpha}{\#}(A_2\Theta B_2)=(A_1\underset{\alpha}{\#}A_2)\Theta(B_1\underset{\alpha}{\#}B_2). \tag{2-27}$$

$$(b)(A_1\Theta B_1)\Diamond(A_2\Theta B_2)=(A_1\Diamond B_1)\Theta(A_2\Diamond B_2). \tag{2-28}$$

$$(c)A_1\underset{\alpha}{\#}B_1)\Theta(A_2\underset{\alpha}{\#}B_2)=(A_1\Theta A_2)\underset{\alpha}{\#}(B_1\Theta B_2). \tag{2-29}$$

$$(d)A_1\Diamond B_1)\Theta(A_2\Diamond B_2)=(A_1\Theta A_2)\Diamond(B_1\Theta B_2). \tag{2-30}$$

Proof (a) To see that this is indeed true, let

$$D_1=A_1^{-1/2}A_2A_1^{-1/2}\text{ and }D_2=B_1^{-1/2}B_2B_1^{-1/2}.$$

Then

$$
\begin{aligned}
&(A_1\Theta B_1)\underset{\alpha}{\#}(A_2\Theta B_2)\\
&\quad=(A_1\Theta B_1)^{1/2}\Big((A_1\Theta B_1)^{-1/2}(A_2\Theta B_2)(A_1\Theta B_1)^{-1/2}\Big)^{\alpha}(A_1\Theta B_1)^{1/2}\\
&\quad=(A_1^{1/2}\Theta B_1^{1/2})\Big((A_1^{-1/2}\Theta B_1^{-1/2})(A_2\Theta B_2)(A_1^{-1/2}\Theta B_1^{-1/2})\Big)^{\alpha}(A_1^{1/2}\Theta B_1^{1/2})\\
&\quad=(A_1^{1/2}\Theta B_1^{1/2})\left\{\Big(A_1^{-1/2}A_2A_1^{-1/2})^{\alpha}\Theta(B_1^{-1/2}B_2B_1^{-1/2}\Big)^{\alpha}\right\}(A_1^{1/2}\Theta B_1^{1/2})\\
&\quad=(A_1^{1/2}\Theta B_1^{1/2})(D_1^{\alpha}\Theta D_2^{\alpha})(A_1^{1/2}\Theta B_1^{1/2})=(A_1^{1/2}D_1^{\alpha}A_1^{1/2})\Theta(B_1^{1/2}D_1^{\alpha}B_1^{1/2})\\
&\quad=(A_1\underset{\alpha}{\#}A_2)\Theta(B_1\underset{\alpha}{\#}B_2).
\end{aligned}
$$

$$\begin{aligned}
(b)&(A_1\Theta B_1)\Diamond(A_2\Theta B_2)\\
&=\left\{(A_1\Theta B_1)^{-1}\#(A_2\Theta B_2)\right\}^{1/2}(A_1\Theta B_1)\left\{(A_1\Theta B_1)^{-1}\#(A_2\Theta B_2)\right\}^{1/2}\\
&=\left\{(A_1^{-1}\Theta B_1^{-1})\#(A_2\Theta B_2)\right\}^{1/2}(A_1\Theta B_1)\left\{(A_1^{-1}\Theta B_1^{-1})\#(A_2\Theta B_2)\right\}^{1/2}\\
&=\left\{(A_1^{-1}\#A_2)\Theta(B_1^{-1}\#B_2)\right\}^{1/2}(A_1\Theta B_1)\left\{(A_1^{-1}\#A_2)\otimes(B_1^{-1}\#B_2)\right\}^{1/2}\\
&=\left\{(A_1^{-1}\#A_2)^{1/2}\Theta(B_1^{-1}\#B_2)^{1/2}\right\}(A_1\Theta B_1)\left\{(A_1^{-1}\#A_2)^{1/2}\Theta(B_1^{-1}\#B_2)^{1/2}\right\}\\
&=\left\{(A_1^{-1}\#A_2)^{1/2}A_1(A_1^{-1}\#A_2)^{1/2}\right\}\Theta\left\{(B_1^{-1}\#B_2)^{1/2}B_1(B_1^{-1}\#B_2)^{1/2}\right\}\\
&=(A_1\Diamond B_1)\Theta(A_2\Diamond B_2)
\end{aligned}$$

We can prove (c) and (d) in a similar manner. □

3 Geometric Means of Several Positive Matrices

Other problems are how to define the geometric, arithmetic, harmonic and operator means of finitely many positive matrices. We generalize these means of two positive matrices to arrive the definitions of the *weighted means* of k positive matrices.

Definition 3.1 Let $w_1, w_2, ..., w_k$ be positive numbers such that $\sum_{i=1}^{k} w_i = 1$ and let $A_i \in M_n^+$ $(1 \le i \le k,\ k \ge 2)$. The *weighted geometric*, *weighted arithmetic* and *weighted harmonic means* of $A_i (1 \le i \le k,\ k \ge 2)$ are defined by
(a) *Weighted geometric mean:*

$$\begin{aligned}
&\prod_{i=1}^{k} \underset{(w)}{\#}\, A_i\\
&= A_k^{1/2}\left\{A_k^{-1/2}A_{k-1}^{1/2}\dots\right.\\
&\left.\left(A_3^{-1/2}A_2^{1/2}\left(A_2^{-1/2}A_1A_2^{-1/2}\right)^{u_1}A_2^{1/2}A_3^{-1/2}\right)^{u_2}\dots A_{k-1}^{1/2}A_k^{-1/2}\right\}^{u_{k-1}}A_k^{1/2},
\end{aligned}\tag{3-1}$$

where $u_s = 1 - \left(w_{s+1}/\sum_{j=1}^{s+1} w_j\right)$ for $s = 1, 2, ..., k-1$.
(b) *Weighted arithmetic mean:*

$$\prod_{i=1}^{k} \underset{(w)}{\sim} A_i = w_1A_1 + ... + w_kA_k = \sum_{i=1}^{k} w_iA_i. \tag{3-2}$$

(c) *Weighted harmonic mean:*

$$\prod_{i=1}^{k} \underset{(w)}{!}\, A_i = (w_1A_1^{-1} + ... + w_1A_k^{-1})^{-1} = \left(\sum_{i=1}^{k} w_iA_i^{-1}\right)^{-1}. \tag{3-3}$$

Definition 3.2 Let $A_i \in M_n^+$ $(1 \le i \le k,\ k \ge 2)$ and f_j $(1 \le j \le k-1)$ be non-negative operator monotone functions on $[0, \infty)$. The *weighted operator mean* of A_i $(1 \le i \le k)$ is defined by

$$\begin{aligned}&\prod_{i=1}^{k} \underset{(w)}{\sigma} A_i \\ &= A_k^{1/2} f_{k-1} \left\{ A_k^{-1/2} A_{k-1}^{1/2} ... f_2 \left(A_3^{-1/2} A_2^{1/2} f_1 \left(A_2^{-1/2} A_1 A_2^{-1/2} \right) A_2^{1/2} A_3^{-1/2} \right) \right. \\ &\qquad \left. ... A_{k-1}^{1/2} A_k^{-1/2} \right\} A_k^{1/2}. \end{aligned} \tag{3-4}$$

In fact, the weighted geometric means are determined by the operator monotone functions

$$f_{k-1}(t) = t^{u_{k-1}},\ f_{k-2}(t) = t^{u_{k-2}}, ..., f_2(t) = t^{u_2}, f_1(t) = t^{u_1}$$

where $u_j = 1 - \left(w_{j+1} / \sum\limits_{r=1}^{j+1} w_r \right)$, $0 < u_j \le 1$, for $j = 1, 2, \cdots, k-1$ and $w_1, w_2, \cdots, w_k$ are positive numbers such that $\sum\limits_{i=1}^{k} w_i = 1$. Note that also the weighted geometric mean becomes Ando's geometric mean, weighted arithmetic mean becomes arithmetic mean and weighted harmonic mean becomes harmonic mean when $k = 2$ and $w_1 = w_2 = \frac{1}{2}$. These general definitions have many good properties analogous. The fundamental relationship is no surprise.

Theorem 3.3 Let $w_1, w_2, ..., w_k$ be positive numbers such that $\sum\limits_{i=1}^{k} w_i = 1$ and let $A_i \in M_n$ $(1 \le i \le k,\ k \ge 2)$ be positive matrices. Then (see [12]).

$$\prod_{i=1}^{k} \underset{(w)}{!} A_i \le \prod_{i=1}^{k} \underset{(w)}{\#} A_i \le \prod_{i=1}^{k} \underset{(w)}{\sim} A_i \tag{3-5}$$

All the inequalities are strict unless $A_1 = A_2 = ... = A_k$.

Surprisingly, it has not previously been observed that the geometric mean for more than two matrices fails the symmetry property. For example, if $k = 3$ and $w_1 = w_2 = w_3 = \frac{1}{3}$, corresponding geometric means of three positive matrices A_1, A_2 and A_3 are

$$A_3 \# A_2 \# A_1 = A_1^{1/2} \left\{ A_1^{-1/2} A_2^{1/2} (A_2^{-1/2} A_3 A_2^{-1/2})^{1/2} A_2^{1/2} A_1^{-1/2} \right\}^{2/3} A_1^{1/2},$$

and

$$A_1 \# A_2 \# A_3 = A_3^{1/2} \left\{ A_3^{-1/2} A_2^{1/2} (A_2^{-1/2} A_1 A_2^{-1/2})^{1/2} A_2^{1/2} A_3^{-1/2} \right\}^{2/3} A_3^{1/2}.$$

But if $A_1 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Then, after computing with *mathematica*, we find

$$A_3 \# A_2 \# A_1 = \begin{bmatrix} 1.64446 & 0.614542 \\ 0.614542 & 1.19496 \end{bmatrix} \text{ and } A_1 \# A_2 \# A_3 = \begin{bmatrix} 1.61321 & 0.605703 \\ 0.605703 & 1.21141 \end{bmatrix}.$$

In other words, $A_3 \# A_2 \# A_1 \neq A_1 \# A_2 \# A_1$.

Even though property (iii) fails, properties (vi) and (vii) are still applicable.

4 Further Developments and Applications

In this section, we prove several theorems related to geometric means and Tracy-Singh products for compatible partitioned matrices.

Theorem 4.1 Let A_i and $B_i \in M_n^+$, $(1 \le i \le k)$ be compatible partitioned matrices. Then

$$\begin{array}{ll} (a) & \prod_{i=1}^{k} \diamondsuit(A_i \Theta B_i) = \left(\prod_{i=1}^{k} \diamondsuit A_i\right) \Theta \left(\prod_{i=1}^{k} \diamondsuit B_i\right). \\ (b) & \prod_{i=1}^{k} \Theta(A_i \diamondsuit B_i) = \left(\prod_{i=1}^{k} \Theta A_i\right) \diamondsuit \left(\prod_{i=1}^{k} \Theta B_i\right). \end{array} . \tag{4-1}$$

Proof (a) The proof is by induction on k; we know by (2-28) that

$$(A_1 \Theta B_1)\diamondsuit(A_2 \Theta B_2) = (A_1 \diamondsuit B_1)\Theta(A_2 \diamondsuit B_2).$$

This gives the claimed when $k = 2$. Suppose that $\prod_{i=1}^{k-1} \diamondsuit(A_i \Theta B_i) = \left(\prod_{i=1}^{k-1} \diamondsuit A_i\right) \Theta \left(\prod_{i=1}^{k-1} \diamondsuit B_i\right)$ holds. Then

$$\begin{aligned} \prod_{i=1}^{k} \diamondsuit(A_i \Theta B_i) &= \left(\prod_{i=1}^{k-1} \diamondsuit(A_i \Theta B_i)\right) \diamondsuit (A_k \Theta B_k) = \left(\left(\prod_{i=1}^{k-1} \diamondsuit A_i\right) \Theta \left(\prod_{i=1}^{k-1} \diamondsuit B_i\right)\right) \diamondsuit (A_k \Theta B_k) \\ &= \left(\left(\prod_{i=1}^{k-1} \diamondsuit A_i\right) \diamondsuit A_k\right) \Theta \left(\left(\prod_{i=1}^{k-1} \diamondsuit B_i\right) \diamondsuit B_k\right) = \left(\prod_{i=1}^{k} \diamondsuit A_i\right) \Theta \left(\prod_{i=1}^{k} \diamondsuit B_i\right). \end{aligned}$$

We can prove (b) in a similar manner. □

In general case, for any real number α and positive matrices A_i and $B_i (i = 1, 2)$. The $\alpha-$ power mixed property related to the Tracy-Singh product can be extended to any finite number of positive matrices as in next results (Theorem (4.2) and Theorem (4.3)).

Theorem 4.2 Let A_i and $B_i \in M_n^+ (1 \le i \le k, k \ge 2)$ be compatible partitioned matrices. Then

$$\begin{array}{ll} (a) & \prod_{i=1}^{k} \underset{\alpha}{\#}(A_i \Theta B_i) = \left(\prod_{i=1}^{k} \underset{\alpha}{\#} A_i\right) \Theta \left(\prod_{i=1}^{k} \underset{\alpha}{\#} B_i\right). \\ (b) & \prod_{i=1}^{k} \Theta(A_i \underset{\alpha}{\#} B_i) = \left(\prod_{i=1}^{k} \Theta A_i\right) \underset{\alpha}{\#} \left(\prod_{i=1}^{k} \Theta B_i\right). \end{array} \tag{4-2}$$

Proof Follows immediately by on induction on.

When $A_i \in M_n^+ (1 \le i \le k, k \ge 2)$ and α_i are real scalars, we have defined

$$A_1 \underset{\alpha_1}{\#} A_2 = A_2^{\frac{1}{2}} (A_2^{-\frac{1}{2}} A_1 A_2^{-\frac{1}{2}})^{\alpha_1} A_2^{\frac{1}{2}} \tag{4-3}$$

Now continue recurrently, setting

$$\prod_{i=1}^{k} \underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#} A_i = A_1 \underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#} A_2 \underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#} \cdots \underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#} A_k$$

$$= A_k^{1/2}\left\{A_k^{-1/2}\left(\prod_{i=1}^{k-1}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-2}}{\#}A_i\right)A_k^{-1/2}\right\}^{\alpha_k}A_k^{1/2}. \tag{4-4}$$

Theorem 4.3 Let A_i and $B_i \in M_n^+(1 \leq i \leq k)$ and be compatible partitioned matrices and let $\alpha_i(1 \leq i \leq k)$ be real scalars. Then for any $m \geq 2$,

$$\prod_{i=1}^{m}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#}(A_i\Theta B_i) = \left(\prod_{i=1}^{m}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#}A_i\right)\Theta\left(\prod_{i=1}^{m}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#}B_i\right). \tag{4-5}$$

Proof We use induction on m. In particular, when $m = 2$ we obtain Theorem (4.2). Suppose that $\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}(A_i\Theta B_i) = \left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}A_i\right)\Theta\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}B_i\right)$ holds. Then

$$\prod_{i=1}^{k+1}\underset{\alpha_1,\alpha_2,\ldots,\alpha_k}{\#}(A_i\Theta B_i) = \left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_k}{\#}(A_i\Theta B_i)\right)\underset{\alpha_1,\alpha_2,\ldots,\alpha_k}{\#}(A_{k+1}\Theta B_{k+1})$$

$$= (A_{k+1}\Theta B_{k+1})^{1/2}\left\{(A_{k+1}\Theta B_{k+1})^{-1/2}\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}(A_i\Theta B_i)\right)(A_{k+1}\Theta B_{k+1})^{-1/2}\right\}^{\alpha_k}$$
$$(A_{k+1}\Theta B_{k+1})^{1/2}$$

$$= \left(A_{k+1}^{1/2}\Theta B_{k+1}^{1/2}\right)\left\{\left(A_{k+1}^{-1/2}\Theta B_{k+1}^{-1/2}\right)\left(\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}\right)\Theta\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}B_i\right)\right)\right.$$
$$\left.\left(A_{k+1}^{-1/2}\Theta B_{k+1}^{-1/2}\right)\right\}^{\alpha_k}\left(A_{k+1}^{1/2}\Theta B_{k+1}^{1/2}\right)$$

$$= \left(A_{k+1}^{1/2}\Theta B_{k+1}^{1/2}\right)\left(\left(A_{k+1}^{-1/2}\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}A_i\right)A_{k+1}^{-1/2}\right)^{\alpha_k}\right.$$
$$\left.\Theta\left(B_{k+1}^{-1/2}\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}B_i\right)B_{k+1}^{-1/2}\right)^{\alpha_k}\right)\left(A_{k+1}^{1/2}\Theta B_{k+1}^{1/2}\right)$$

$$= \left(A_{k+1}^{1/2}\left\{A_{k+1}^{-1/2}\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}A_i\right)A_{k+1}^{-1/2}\right\}^{\alpha_k}A_{k+1}^{1/2}\right)$$
$$\Theta\left(B_{k+1}^{1/2}\left\{B_{k+1}^{-1/2}\left(\prod_{i=1}^{k}\underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#}B_i\right)B_{k+1}^{-1/2}\right\}^{\alpha_k}B_{k+1}^{1/2}\right).$$

$$= \left(\prod_{i=1}^{k+1}\underset{\alpha_1,\alpha_2,\ldots,\alpha_k}{\#}A_i\right)\Theta\left(\prod_{i=1}^{k+1}\underset{\alpha_1,\alpha_2,\ldots,\alpha_k}{\#}B_i\right).$$ □

Theorem (4.3) can be extended by using induction as given in the following theorem.

Theorem 4.4 Let $A_{ij} \in M_n^+(1 \leq i \leq m, 1 \leq j \leq k)$ be compatible partitioned matrices and let $\alpha_i(1 \leq i \leq m)$ be real scalars. Then

$$\left(\prod_{j=1}^{k} \Theta A_{1j}\right) \underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#} \cdots \underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#} \left(\prod_{j=1}^{k} \Theta A_{mj}\right)$$
$$= \prod_{j=1}^{k} \Theta \left(A_{1j} \underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#} \cdots \underset{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}{\#} A_{mj}\right). \tag{4.6}$$

Property (vii) for two positive matrices namely $(A \# B)^{-1} = A^{-1} \# B^{-1}$, can be extended to $\alpha-$power mean of two positive matrices:

$$\left(A \underset{\alpha}{\#} B\right)^{-1} = A^{-1} \underset{\alpha}{\#} B^{-1} \tag{4-7}$$

To see that this is indeed true:

$$A^{-1} \underset{\alpha}{\#} B^{-1} = (A^{-1})^{\frac{1}{2}} \left((A^{-1})^{-\frac{1}{2}} B^{-1} (A^{-1})^{-\frac{1}{2}}\right)^{\alpha} (A^{-1})^{\frac{1}{2}}$$

$$= (A^{\frac{1}{2}})^{-1} \left(\{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\}^{-1}\right)^{\alpha} (A^{\frac{1}{2}})^{-1} = \left\{A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha_1} A^{\frac{1}{2}}\right\}^{-1} = \left(A \underset{\alpha}{\#} B\right)^{-1}. \ \square$$

A quick induction is sufficient to extend this to means of more than two matrices.
Lemma 4.5 Let $A_i \in M_n^+(1 \leq i \leq k, k \geq 2)$ and let $\alpha_i(1 \leq i \leq k)$ be real scalars. Then

$$\left(\prod_{i=1}^{k} \underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#} A_i^{-1}\right) = \left(\prod_{i=1}^{k} \underset{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}{\#} A_i\right)^{-1}. \tag{4-8}$$

We now turn to a generalization of the arithmetic-geometric-harmonic mean inequality related to Tracy-Singh Product.
Theorem 4.6 Let $A_{ij} \in M_n^+(1 \leq i \leq m, 1 \leq j \leq k)$ be compatible partitioned matrices and let $w_i(1 \leq i \leq m)$ be positive scalars such that $\sum_{i=1}^{m} w_i = 1$. Let $u_i = 1-\left(w_{i+1}/\sum_{s=1}^{i+1} w_s\right)$, $i = 1, 2, ..., m-1$. Then

$$\text{(a)} \left(\sum_{i=1}^{m} w_i \prod_{j=1}^{k} \Theta A_{ij}^{-1}\right)^{-1} \leq \prod_{j=1}^{k} \Theta \left(A_{1j} \underset{(w)}{\#} \cdots \underset{(w)}{\#} A_{mj}\right) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \Theta A_{ij}. \tag{4-9}$$

$$\text{(b)} \left(\sum_{i_1,\cdots,i_m=1}^{m} w_{i_1\cdots i_m} \prod_{j=1}^{k} \Theta A_{i_j j}^{-1}\right)^{-1} \leq \left(\left(\prod_{j=1}^{k} \Theta A_{ij}\right) \underset{(w)}{\#} \cdots \left(\underset{(w)}{\#} \prod_{j=1}^{k} \Theta A_{mj}\right)\right)$$
$$\leq \sum_{i_1=1}^{m} \cdots \sum_{i_m=1}^{m} w_{i_1} \cdots w_{i_m} \prod_{j=1}^{k} \Theta A_{i_j j} \tag{4-10}$$

Equalities in (4-9) and (4-10) hold if and only if $\prod_{j=1}^{k} \Theta A_{1j} = \ldots = \prod_{j=1}^{k} \Theta A_{mj}$.

Proof By Theorem (3.3) and Theorem (4.4), we have

$$\prod_{j=1}^{k}\Theta\left(A_{1j}\underset{(w)}{\#}\ldots\underset{(w)}{\#}A_{mj}\right)=\left(\prod_{j=1}^{k}\Theta A_{1j}\right)\underset{(w)}{\#}\ldots\underset{(w)}{\#}\left(\prod_{j=1}^{k}\Theta A_{mj}\right)\leq\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\Theta A_{ij}.$$

Substituting A_{ij}^{-1} for A_{ij} and using Lemma (4.5), we obtain

$$\prod_{j=1}^{k}\Theta\left(A_{1j}\underset{(w)}{\#}\ldots\underset{(w)}{\#}A_{mj}\right)^{-1}=\prod_{j=1}^{k}\Theta\left(A_{1j}^{-1}\underset{(w)}{\#}\ldots\underset{(w)}{\#}A_{mj}^{-1}\right)\leq\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\Theta A_{ij}^{-1}.$$

Now, we have

$$\left(\prod_{j=1}^{k}\Theta\left(A_{1j}\underset{(w)}{\#}\ldots\underset{(w)}{\#}A_{mj}\right)\right)^{-1}\leq\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\Theta A_{ij}^{-1},$$

and hence

$$\left(\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\Theta A_{ij}^{-1}\right)^{-1}\leq\prod_{j=1}^{k}\Theta\left(A_{1j}\underset{(w)}{\#}\ldots\underset{(w)}{\#}A_{mj}\right).$$

Similarly, we have the second inequalities (b). Since

$$\left(\prod_{j=1}^{k}\Theta A_{1j}\right)\underset{(w)}{\#}\ldots\underset{(w)}{\#}\left(\prod_{j=1}^{k}\Theta A_{mj}\right)=\prod_{j=1}^{k}\Theta\left(A_{1j}\underset{(w)}{\#}\ldots\underset{(w)}{\#}A_{mj}\right)$$
$$\leq\prod_{j=1}^{k}\Theta\left(\sum_{i=1}^{m}w_iA_{ij}\right)=\sum_{i_1=1}^{m}\ldots\sum_{i_m=1}^{m}w_{i_1}\ldots w_{i_m}\prod_{j=1}^{k}\Theta A_{i_j j}. \qquad \square$$

We shall use Theorem (4.6) to extend a result of Ando [2, *pp.* 229, *Theorem* 12] in the following theorem.

Theorem 4.7 Let $A_i \in M_n^+(1 \leq i \leq m)$ be commutative matrices. Then (*see* [2])

$$\prod_{j=1}^{m}\circ\left(\prod_{i=1}^{m}A_i\right)^{\frac{1}{m}}\leq\prod_{i=1}^{m}\circ A_i.. \tag{4-11}$$

We need some further ingredients before we address Ando's result. In 1997, Alic and others [1] gave a generalization of Theorem (3.3) in the negative weight case. The various means are defined just as before, even if there are negative weights.

Theorem 4.8 Let $A_i(1 \leq i \leq k)$ be positive matrices and let $w_i(1 \leq i \leq k)$ be real numbers such that $w_1 > 0$, $w_i < 0(2 \leq i \leq k)$ and $\sum_{i=1}^{k} w_i = 1$. Then (*see* [1])

$$\prod_{i=1}^{k}\underset{(w)}{\sim}A_i\leq\prod_{i=1}^{k}\underset{(w)}{\#}A_i. \tag{4-12}$$

If $\sum_{i=1}^{k} w_i A_i^{-1} > 0$, then

$$\prod_{i=1}^{k} \underset{(w)}{\#} A_i \leq \prod_{i=1}^{k} \underset{(w)}{!} A_i. \tag{4-13}$$

Equalities hold if and only if $A_1 = ... = A_k$.

Developing Theorem (4.8), we have the following.

Theorem 4.9 Let $A_{ij} \in M_n^+ (1 \leq i \leq m, 1 \leq j \leq k)$ and let $w_i (1 \leq i \leq m)$ be real numbers such that $w_1 > 0$, $w_i < 0 (2 \leq i \leq m)$, and $\sum_{i=1}^{k} w_i = 1$. Let $u_i = 1 - (w_{i+1} / \sum_{s=1}^{i+1} w_s)$, for $i = 1, 2, \cdots, m-1$. Then

$$(a) \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \Theta A_{ij} \leq \prod_{j=1}^{k} \Theta \left(A_{1j} \underset{(w)}{\#} ... \underset{(w)}{\#} A_{mj} \right) \tag{4-14}$$

$$(b) \sum_{i_1=1}^{m} ... \sum_{i_m=1}^{m} w_{i_1} ... w_{i_m} \prod_{j=1}^{k} \Theta A_{i_j j} \leq \left(\prod_{j=1}^{k} \Theta A_{1j} \right) \underset{(w)}{\#} ... \underset{(w)}{\#} \left(\prod_{j=1}^{k} \Theta A_{mj} \right). \tag{4-15}$$

If $w_1 \prod_{j=1}^{k} \Theta A_{1j} + ... + w_m \prod_{j=1}^{k} \Theta A_{mj} > 0$, then

$$(c) \prod_{j=1}^{k} \Theta \left(A_{1j} \underset{(w)}{\#} ... \underset{(w)}{\#} A_{mj} \right) \leq \left(\sum_{i=1}^{m} w_i \prod_{j=1}^{k} \Theta A_{ij}^{-1} \right)^{-1} \tag{4-16}$$

$$(d) \left(\left(\prod_{j=1}^{k} \Theta A_{ij} \right) \underset{(w)}{\#} ... \underset{(w)}{\#} \left(\prod_{j=1}^{k} \Theta A_{mj} \right) \right) \leq \left(\sum_{i_1, ..., i_m=1}^{m} w_{i_1 ... i_m} \prod_{j=1}^{k} \Theta A_{i_j j}^{-1} \right)^{-1}. \tag{4-17}$$

Equalities hold if and only if $\prod_{j=1}^{k} \Theta A_{1j} = ... = \prod_{j=1}^{k} \Theta A_{mj}$.

Using Theorem (4.6) and the connection between the Khatri-Rao and Tracy-Singh products in (1-1), we have the following two theorems:

Theorem 4.10 Let $A_{ij} \in M_n^+ (1 \leq i \leq m, 1 \leq j \leq k)$ be compatible partitioned matrices and let $w_i (1 \leq i \leq m)$ be positive real numbers such that $\sum_{i=1}^{m} w_i = 1$. Let $u_i = 1 - (w_{i+1} / \sum_{s=1}^{i+1} w_s)$, for $i = 1, 2, \cdots, m-1$. Then

$$(a) \prod_{j=1}^{k} * \left(A_{1j} \underset{(w)}{\#} ... \underset{(w)}{\#} A_{mj} \right) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} * A_{ij}. \tag{4-18}$$

$$(b) \prod_{j=1}^{k} * \left(A_{1j} \underset{(w)}{\#} ... \underset{(w)}{\#} A_{mj} \right) \leq \sum_{i_1=1}^{m} ... \sum_{i_m=1}^{m} w_{i_1} ... w_{i_m} \prod_{j=1}^{k} * A_{i_j j} \tag{4-19}$$

Equalities hold if and only if $\prod_{j=1}^{k} * A_{1j} = ... = \prod_{j=1}^{k} * A_{mj}$.

Proof It follows immediately by using Theorem (4.6) and Eq.(1-1). □

Finally, we arrive at our extension and generalization of Ando's result quoted in Theorem (4.7).

Theorem 4.11 Let $A_i \in M_n^+ (1 \le i \le m)$ be compatible partitioned matrices. Then

$$\left(A_1 \underset{(w)}{\#} ... \underset{(w)}{\#} A_m\right) * \left(A_2 \underset{(w)}{\#} ... \underset{(w)}{\#} A_m \underset{(w)}{\#} A_1\right) * ... * \left(A_m \underset{(w)}{\#} A_1 \underset{(w)}{\#} ... \underset{(w)}{\#} A_{m-1}\right) \le \prod_{i=1}^{m} * A_i. \tag{4-20}$$

Proof In Theorem (4.10), let

$$(A_{11}, ..., A_{m1}) = (A_1, ..., A_m)$$

$$(A_{12}, ..., A_{m2}) = (A_2, ..., A_m, A_1)$$

$$\vdots$$

$$(A_{1m}, A_{2m}, ..., A_{mm}) = (A_m, A_1, ..., A_{m-1}).$$

Then

$$\left(A_1 \underset{(w)}{\#} ... \underset{(w)}{\#} A_m\right) * \left(A_2 \underset{(w)}{\#} ... \underset{(w)}{\#} A_m \underset{(w)}{\#} A_1\right) * ... * \left(A_m \underset{(w)}{\#} A_1 \underset{(w)}{\#} ... \underset{(w)}{\#} A_{m-1}\right)$$
$$= \prod_{j=1}^{k} * \left(A_{1j} \underset{(w)}{\#} ... \underset{(w)}{\#} A_{mj}\right) \le \sum_{i=1}^{m} w_i \prod_{j=1}^{k} * A_{ij} = \sum_{i=1}^{m} w_i \prod_{j=1}^{m} * A_j = \prod_{i=1}^{m} * A_i$$

To see how Ando's result is a special case of Theorem (4.11), note that if $A_i (1 \le i \le m)$ commute and non-partitioned matrices, then

$$A_1 \underset{(w)}{\#} ... \underset{(w)}{\#} A_m = A_2 \underset{(w)}{\#} ... \underset{(w)}{\#} A_m \underset{(w)}{\#} A_1$$

$$\vdots$$

$$= A_m \underset{(w)}{\#} A_1 \underset{(w)}{\#} ... \underset{(w)}{\#} A_{m-1} \tag{4-21}$$

Theorem 4.12 Let $A_i \in M_n^+ (i = 1, 2)$ be compatible partitioned matrices. Then for any real number α and for all $-\infty < p, q < \infty$

$$(A_1 \Theta A_2)^p \underset{\alpha}{\#} (A_1 \Theta A_2)^q = (A_1 \Theta A_2)^{(1-\alpha)p + \alpha q} \tag{4-22}$$

In particular if $\alpha = \frac{1}{2}$, we have

$$(A_1 \Theta A_2)^p \# (A_1 \Theta A_2)^q = (A_1 \Theta A_2)^{(p+q)/2} \tag{4-23}$$

Proof By using (1-5), (2-23) and Lemma (2.1), we have

$$\begin{aligned}(A_1 \Theta A_2)^p \underset{\alpha}{\#} (A_1 \Theta A_2)^q &= (A_1^p \Theta A_2^p) \underset{\alpha}{\#} (A_1^q \Theta A_2^q) = (A_1^p \underset{\alpha}{\#} A_1^q) \Theta (A_2^p \underset{\alpha}{\#} A_2^q) \\ &= A_1^{(1-\alpha)p+\alpha q} \Theta A_2^{(1-\alpha)p+\alpha q} = (A_1 \Theta A_2)^{(1+\alpha)p+\alpha q}. \qquad \square\end{aligned}$$

Theorem 4.13 Let $A_i \in M_n^+ (1 \leq i \leq k)$ be compatible partitioned matrices. Then for any real number α and for all $-\infty < p, q < \infty$

$$\left(\prod_{i=1}^{k} \Theta A_i^p\right) \underset{\alpha}{\#} \left(\prod_{i=1}^{k} \Theta A_i^q\right) = \left(\prod_{i=1}^{k} \Theta A_i^{(1-\alpha)p+\alpha q}\right) \tag{4-24}$$

In particular if $\alpha = \frac{1}{2}$, we have

$$\left(\prod_{i=1}^{k} \Theta A_i^p\right) \# \left(\prod_{i=1}^{k} \Theta A_i^q\right) = \left(\prod_{i=1}^{k} \Theta A_i^{(p+q)/2}\right) \tag{4-25}$$

Proof The proof is a consequence of Theorem (4.12). This gives the claimed when $k = 2$. We can now proceed by induction on k. Assume that the corollary holds for products of $k - 1$ matrices. Then

$$\begin{aligned}\left(\prod_{i=1}^{k} \Theta A_i^p\right) \underset{\alpha}{\#} \left(\prod_{i=1}^{k} \Theta A_i^q\right) &= \left[\left(\prod_{i=1}^{k-1} \Theta A_i^p\right) \underset{\alpha}{\#} \left(\prod_{i=1}^{k-1} \Theta A_i^q\right)\right] \Theta [A_k^p \underset{\alpha}{\#} A_k^q] \\ &= \left(\prod_{i=1}^{k-1} \Theta A_i^{(1-\alpha)p+\alpha q}\right) \Theta A_k^{(1-\alpha)p+\alpha q} = \left(\prod_{i=1}^{k} \Theta A_i^{(1-\alpha)p+\alpha q}\right). \quad \square\end{aligned}$$

Remark: The results obtained in Section 2, Section 3 and Section 4 are quite general. As a special case, consider the matrices in above sections are non-partitioned, we then have equalities and inequalities involving Kronecker and Hadamard products by replacing Θ by $\bigotimes$ and $*$ by $\circ$.

5 Conclusion

In the present work we study a family of geometric means for positive matrices. It is still an open problem to find a completely satisfactory definition that satisfies all properties (i)-(viii). To find a completely satisfactory definition is subject for the future study. In this study we have succeeded to find new connections between geometric means and Tracy-Singh products and establish several equalities and inequalities for geometric means and Tracy-Singh products of several positive matrices.

Acknowledgment

The authors gratefully acknowledge that this research has been partially supported by Universiti Putra Malaysia (UPM) under the Grant IRPA 09-02-04-0259-EA001.

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