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Determining the Optimum Process Parameters by Asymmetric Quality Loss Function

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Abstract Huang presented a trade-off problem, taking both product quality and process adjustment cost into account, to determine the optimum parameters (i.e., the process mean and process variance) of the input characteristic in the transformation model. In Huang's transformation model, the input characteristic, x, is assumed to be normally distributed and the output characteristic, y, is nominal-the-best with a target value. The relationship between x and y can be either linear or quadratic. When formulating the cost function in the transformation model, Huang used the symmetric quadratic loss function to measure the loss of profit. In this paper, we extend Huang's quadratic transformation model to a more general case by respectively using asymmetric quadratic and asymmetric linear loss functions in the cost function. The modified cost functions using asymmetric quadratic and asymmetric linear loss functions are developed. A numerical example is provided for illustration.

Keywords Asymmetric Quality Loss Function, Trade-Off Problem, Process Mean, Process Variance, Target Value.

1 Introduction

Since Taguchi (1986) presented quality loss function for evaluating the quality losses of products and/or services, quality loss function has been widely applied in the areas of statistical process control, such as design of control charts, tolerance design, parameter design, design of sampling plans, design of specification limits, and so forth. The general expression of quality loss function for the characteristic y is

$$L(y) = \begin{cases} k_1(y-T)^w, \text{for } \ y \le T, \\ k_2(y-T)^w, \text{for } \ y > T, \end{cases} \tag{1}$$

where T is the target of y, k_1 and k_2 are the coefficients of quality loss to the left-hand and right-hand sides of the target respectively, and w is the order of the function (generally, $w = 1$ or 2). If $k_1 = k_2 = k$, equation (1) is called a symmetric loss function; while if $k_1 \neq k_2$, equation (1) is called an asymmetric loss function. If $w = 1$, equation (1) is called a linear loss function, and if $w = 2$, equation (1) is called a quadratic loss function. Recently, applications of some specific forms of quality loss function have been found in the literature, such as Wu and Tang (1998), Li (2000), Li and Cherng (2000), Maghsoodloo and Li (2000), etc.

The classical Taguchi quality model (1986) only considers the control and improvement of quality. Huang (2001) proposed a transformation model that took both quality and cost into account. In Huang's transformation model (2001), the input characteristic, x, is assumed to be normally distributed with mean μ and variance σ^2. The output characteristic, y, is nominal-the-best and has a target value T. The relationship between x and y may be either linear or quadratic. Huang's transformation model presents a trade-off problem between quality and cost, in which higher quality is not the final aim but the profit. This model includes two components: the loss of profit and the cost to set the process mean and to control the process variance. The loss of profit is assumed to be proportional to the loss of quality, and thus can be described by the quality loss function. Huang (2001) applied the symmetric quadratic loss function to measure the loss of profit. The objective of Huang's transformation model is to determine the optimum process mean and variance for the input characteristic such that the total cost, including the loss of profit and the cost to set the process mean and to control the process variance, is minimized.

Since the symmetric quadratic loss function is inappropriate in some situations, we may extend Huang's quadratic transformation model to a more general case by using asymmetric quadratic and asymmetric linear loss functions in the cost function respectively.

2 Review of Huang's Model

There are five assumptions in Huang's transformation model (2001). They are as follows:

(i) The input characteristic, x, is a normally distributed random variable with mean μ and variance σ^2.

(ii) The output characteristic, y, is nominal-the-best and has the target value T.

(iii) The relationships between x and y may be linear or quadratic. That is, $y = bx + c$ or $y = ax^2 + bx + c$ where a, b, and c are all constants.

(iv) The cost of setting μ is proportional to $|\mu|$ or μ^2, while the cost for controlling σ^2 is proportional to $1/\sigma$ or $1/\sigma^2$. Thus, the cost of process adjustment, denoted by $C_x(\mu,\ \sigma)$, may have one of the following four forms:

$$C_x(\mu,\sigma) = \beta_1\,|\mu| + \frac{\beta_2}{\sigma},$$

$$C_x(\mu,\sigma) = \beta_1\,|\mu| + \frac{\beta_2}{\sigma^2},$$

$$C_x(\mu,\sigma) = \beta_1\mu^2 + \frac{\beta_2}{\sigma}, \text{ and}$$

$$C_x(\mu,\sigma) = \beta_1\mu^2 + \frac{\beta_2}{\sigma^2},$$

where β_1 and β_2 are positive constants.

(v) The loss of quality results in the loss of profit, which is assumed to be proportional to the loss of quality.

According to Huang (2001), the total cost of the trade-off problem, denoted by $TC(\mu, \sigma)$, is the sum of the loss of profit and the cost of process adjustment, which is the function of μ and σ. That is,

$$TC(\mu, \sigma) = C_x(\mu, \sigma) + \overline{\alpha} E[(y - T)^2] \tag{2}$$

where $\overline{\alpha}$ is the coefficient of profit loss which is proportional to the quality loss. Huang's model is desirable to find the optimum process parameters $\mu^*$and σ^* such that equation (2) may be minimized.

3 Modified Huand's Cost Function with Asymmetric Quality Loss Function

Huang's quadratic transformation model (2001) is considered here, i.e., $y = ax^2 + bx + c$. The linear transformation model is a special case of the quadratic model as $a = 0$. From the viewpoint of product quality, it would be ideal if the process variance σ^2 can be reduced to zero and each value of the output characteristic y is right on its target T. It can be easily shown that in this ideal case, the value of the input characteristic is

$$x = \frac{-b + \sqrt{b^2 - 4a(c - T)}}{2a}, \quad \text{for} \;\; b^2 > 4a(c - T).$$

We may modify Huang's cost function (2001) by using asymmetric quadratic and asymmetric linear quality loss functions respectively as follows:

(1) *Asymmetric Quadratic Quality Loss Function*

The expected quality loss per unit for the output characteristic is

$$E[L(y)] = E[(y - T)^2] = \int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} k_1(T - ax^2 - bx - c)^2 f(x)dx + \int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} k_2(ax^2 + bx + c - T)^2 f(x)dx, \tag{3}$$

where $f(x)$ is the density function of the normal distribution with mean μ and variance σ^2.

$$\begin{aligned} \Theta\; ax^2 + bx + c - T &= a(x-\mu)^2 + (2a\mu + b)(x-\mu)^2 + a\mu^2 + b\mu + c - T \\ \therefore (ax^2 + bx + c - T)^2 &= a^2(x-\mu)^4 + 2a(2a\mu + b)(x-\mu)^3 + \\ &\quad [(2a\mu + b)^2 + 2a(a\mu^2 + b\mu + c - T)](x-\mu)^2 + \\ &\quad 2(2a\mu + b)(a\mu^2 + b\mu + c - T)(x-\mu) + \\ &\quad (a\mu^2 + b\mu + c - T)^2 \end{aligned}$$

Denote $z = \dfrac{x-\mu}{\sigma}$ and $z_0 = \dfrac{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a} - \mu}{\sigma}$. We have the following six equations:

1. $$\int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} (x-\mu)^4 f(x)dx = \int_{z_0}^{\infty} \sigma^4 z^4 \phi(z)dz = \sigma^4\{z_0^3\phi(z_0) + 3z_0\phi(z_0) + 3[1-\Phi(z_0)]\}$$

2. $$\int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} (x-\mu)^2 f(x)dx = \int_{z_0}^{\infty} \sigma^2 z^2 \phi(z)dz = \sigma^2\{z_0\phi(z_0) + [1-\Phi(z_0)]\}$$

3. $$\int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} (x-\mu) f(x)dx = \int_{z_0}^{\infty} \sigma z \phi(z)dz = \sigma\phi(z_0)$$

4. $$\int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} (x-\mu)^4 f(x)dx = \int_{-\infty}^{z_0} \sigma^4 z^4 \phi(z)dz = \sigma^4[-z_0^3\phi(z_0) - 3z_0\phi(z_0) + 3\Phi(z_0)]$$

5. $$\int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} (x-\mu)^2 f(x)dx = \int_{-\infty}^{z_0} \sigma^2 z^2 \phi(z)dz = \sigma^2[-z_0\phi(z_0) + \Phi(z_0)]$$

6. $$\int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} (x-\mu) f(x)dx = \int_{-\infty}^{z_0} \sigma z \phi(z)dz = -\sigma\phi(z_0)$$

where $\Phi(z)$ and $\phi(z)$ are respectively the cumulative distribution function and the density function of the standard normal random variable.

Hence, equation (3) may be expressed as

$$\begin{aligned} E[L(y)] = k_1\{&a^2\sigma^4[-z_0^3\phi(z_0) - 3z_0\phi(z_0) + 3\Phi(z_0)] - [2a(2a\mu+b)]\sigma^3(z_0^2+2)\phi(z_0) + \\ &[(2a\mu+b)^2 + 2a(a\mu^2+b\mu+c-T)]\sigma^2[-z_0\phi(z_0) + \Phi(z_0)] - 2\sigma\phi(z_0)\cdot \\ &(a\mu^2+b\mu+c-T)(2a\mu+b) + (a\mu^2+b\mu+c-T)^2\Phi(z_0)\} + k_2\{a^2\sigma^4\cdot \\ &[z_0^3\phi(z_0) + 3z_0\phi(z_0) + 3(1-\Phi(z_0))] + [2a(2a\mu+b)]\sigma^3(z_0^2+2)\phi(z_0) + \\ &[(2a\mu+b)^2 + 2a(a\mu^2+b\mu+c-T)]\sigma^2[z_0\phi(z_0) + 1 - \Phi(z_0)] + 2\sigma\phi(z_0)\cdot \\ &(2a\mu+b)(a\mu^2+b\mu+c-T) + [1-\Phi(z_0)](a\mu^2+b\mu+c-T)^2\}, \end{aligned} \tag{4}$$

The modified Huang's cost function with the asymmetric quadratic quality loss function is

$$
\begin{aligned}
TC(\mu,\sigma) &= C_x(\mu,\sigma) + \overline{\alpha}E[L(y)] \\
&= C_x(\mu,\sigma) + \overline{\alpha}k_1\{a^2\sigma^4[-z_0^3\phi(z_0) - 3z_0\phi(z_0) + 3\Phi(z_0)] - [2a(2a\mu+b)]\cdot \\
&\quad \sigma^3(z_0^2+2)\phi(z_0) + [(2a\mu+b)^2 + 2a(a\mu^2+b\mu+c-T)]\sigma^2[-z_0\phi(z_0)+\Phi(z_0)] \\
&\quad - 2\sigma\phi(z_0)(a\mu^2+b\mu+c-T)(2a\mu+b) + (a\mu^2+b\mu+c-T)^2\Phi(z_0)\} + \\
&\quad \overline{\alpha}k_2\{a^2\sigma^4[z_0^3\phi(z_0) + 3z_0\phi(z_0) + 3(1-\Phi(z_0))] + [2a(2a\mu+b)]\sigma^3(z_0^2+2)\phi(z_0) \\
&\quad + [(2a\mu+b)^2 + 2a(a\mu^2+b\mu+c-T)]\sigma^2[z_0\phi(z_0)+1-\Phi(z_0)] + 2\sigma\phi(z_0)\cdot \\
&\quad (2a\mu+b)(a\mu^2+b\mu+c-T) + [1-\Phi(z_0)](a\mu^2+b\mu+c-T)^2\}. \qquad (5)
\end{aligned}
$$

The optimum process parameters μ^* and σ^* for equation (5) can be obtained by using the multidimensional search techniques in Al-Sultan and Rahim (1997, pp.23-27), such as Newton's method or Hooke and Jeeve's pattern search method.

(2) *Asymmetric Linear Quality Loss Function*

The expected quality loss per unit for the output characteristic is

$$
\begin{aligned}
E[L(y)] = E[(y-T)] \quad &= \int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} k_1(T - ax^2 - bx - c)f(x)dx + \\
&\quad \int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} k_2(ax^2+bx+c-T)f(x)dx \qquad (6)
\end{aligned}
$$

Denote $z = \dfrac{x-\mu}{\sigma}$ and $z_0 = \dfrac{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a} - \mu}{\sigma}$. We have the following six equations:

1. $\displaystyle\int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} (c-T)f(x)dx = \int_{z_0}^{\infty}(c-T)\phi(z)dz = (c-T)[1-\Phi(z_0)]$

2. $\displaystyle\int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} bxf(x)dx = b\int_{z_0}^{\infty}(\mu+z\sigma)\phi(z)dz = b\{\mu[1-\Phi(z_0)] + \sigma\phi(z_0)\}$

3. $\displaystyle\int_{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}}^{\infty} ax^2f(x)dx = a\int_{z_0}^{\infty}(\mu+z\sigma)^2\phi(z)dz$

$$= a\{\mu^2[1-\Phi(z_0)] + 2\mu\sigma\phi(z_0) + \sigma^2[z_0\phi(z_0)+1-\Phi(z_0)]\}$$

4. $$\int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} (T-c)f(x)dx = \int_{-\infty}^{z_0} (T-c)\phi(z)dz = (T-c)\Phi(z_0)$$

5. $$\int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} -bxf(x)dx = -b\int_{-\infty}^{z_0} (\mu+z\sigma)\phi(z)dz = -b[\mu\Phi(z_0)-\sigma\phi(z_0)]$$

6. $$\int_{-\infty}^{\frac{-b+\sqrt{b^2-4a(c-T)}}{2a}} -ax^2f(x)dx = -a\int_{-\infty}^{z_0} (\mu+z\sigma)^2\phi(z)dz$$
$$= -a\{\mu^2\Phi(z_0)-2\mu\sigma\phi(z_0)+\sigma^2[-z_0\phi(z_0)+\Phi(z_0)]\}$$

where $\Phi(z)$ and $\phi(z)$ are respectively the cumulative distribution function and the density function of the standard normal random variable.

Hence, equation (6) can be expressed as

$$\begin{aligned} E[L(y)] = k_1\{&(T-c)\Phi(z_0) - b[\mu\Phi(z_0)-\sigma\phi(z_0)] - a\{\mu^2\Phi(z_0)-2\mu\sigma\phi(z_0)+\sigma^2\cdot \\ &[\Phi(z_0)-z_0\phi(z_0)]\}\} + k_2\{(c-T)[1-\Phi(z_0)] + b[\mu(1-\Phi(z_0))+\sigma\phi(z_0)] \\ &+ a\{\mu^2(1-\Phi(z_0))+2\mu\sigma\phi(z_0)+\sigma^2[1-\Phi(z_0)+z_0\phi(z_0)]\}\}. \end{aligned} \tag{7}$$

The modified Huang's cost model with the asymmetric linear quality loss is

$$\begin{aligned} TC(\mu,\sigma) &= C_x(\mu,\sigma) + \overline{\alpha}E[L(y)] \\ &= C_x(\mu,\sigma) + \overline{\alpha}k_1\{(T-c)\Phi(z_0) - b[\mu\Phi(z_0)-\sigma\phi(z_0)] - a\{\mu^2\Phi(z_0)-2\mu\sigma\phi(z_0) \\ &\quad + \sigma^2[\Phi(z_0)-z_0\phi(z_0)]\}\} + \overline{\alpha}k_2\{(c-T)[1-\Phi(z_0)] + b[\mu(1-\Phi(z_0))+\sigma\phi(z_0)] \\ &\quad + a\{\mu^2(1-\Phi(z_0))+2\mu\sigma\phi(z_0)+\sigma^2[1-\Phi(z_0)+z_0\phi(z_0)]\}\}. \end{aligned} \tag{8}$$

The optimum process parameters μ^* and σ^* for equation (8) can be obtained by using the multidimensional search techniques in Al-Sultan and Rahim (1997, pp.23-27), such as Newton's method or Hooke and Jeeve's pattern search method.

4 Numerical Example

Consider the example given in Huang (2001, p. 275): $T = 100, C_x(\mu,\sigma) = 0.1\mu^2 + \dfrac{0.1}{\sigma^2}$, and $y = 0.1x^2 + 2x + 50$. Let $\overline{\alpha} = 0.1$, $k_1 = 100$ and $k_2 = 150$. By applying Newton's method to solve equation (5), the optimum parameters (μ^*, σ^*) for using asymmetric quadratic loss function in the cost function are (14.47, 0.13) with $TC(\mu^*, \sigma^*) = 31.991$. By applying Newton's method to solve equation (8), the optimum parameters (μ^*, σ^*) for using asymmetric linear loss function in the cost function are (14.44, 0.16) with $TC(\mu^*, \sigma^*) = 32.348$.

5 Conclusions

In this study, Huang's quadratic transformation model is extended by respectively using asymmetric quadratic and asymmetric linear loss functions in the cost function. The modified cost functions are developed and a numerical example is given for illustration. This study improves the applications of quality loss function to determine the optimum process parameters for the input characteristic in the quadratic transformation models. The extension to the smaller-the-better and larger-the-better characteristics, or to the model with multiple input or output characteristics may be left for further study.

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