MATEMATIKA, 2006, Volume 22, Number 2, pp. 147–153

Positivity-Preserving Piecewise Rational Cubic Interpolation

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Abstract A piecewise rational cubic spline [5] has been used to visualize the positive data in its natural form. The spline representation is interpolatory and applicable to the scalar valued data. The shape parameters in the description of a rational cubic have been constrained in such a way that they preserve the shape of the positive data in the view of positive curve. As far as visual smoothness is concerned, the curve scheme under discussion is C^1.

Keywords Data visualization, interpolation, rational cubic.

1 Introduction

The problem of shape preserving interpolation has been considered by a number of authors [1-8]. This paper examines the problem of shape preserving positive data arising from scientific phenomena or from mathematical functions. Positivity is a very important aspect of shape. There are many physical situations where entities only have meaning when their values are positive. For example, in a probability distribution the representation is always positive or when dealing with samples of populations, the data are always positive. Recently some interest has been shown in this area, by Schmidt and Hess [8] and Butt and Brodlie [2]. Schmidt and Hess [8] have used cubic polynomials and derived necessary and sufficient conditions for preserving the shape of positive data. Butt and Brodlie [2] also used cubic polynomials. The algorithm of Butt and Brodlie [2] works by inserting one or two extra knots, where necessary, to preserve the shape of positive data. Whereas Sarfraz et al in [6] have used C^1 piecewise rational cubic functions to preserve the shape of positive data. This work is also related to positive interpolation. The subject of this paper is to make the constraints on the shape parameters in the description of rational functions so that the resultant curve is positive for a positive data and visually acceptable.

2 Rational Cubic Functions

Let(x_i, f_i), for $i = 1, 2, \cdots, n$, be a given set of data points, where $x_1 < x_2 < \cdots < x_n$, and f_i, d_i are respectively, the function values and the derivative values at the knots x_i. Let $h_i = x_{i+1} - x_i$, and $\Delta_i = \frac{f_{i+1}-f_i}{h_i}$, $i = 1, 2, \cdots, n$.

In each interval $I_i = [x_i, x_{i+1}]$, a rational function $S_i(x)$ may be defined as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} \tag{1}$$

with

$$\begin{aligned} p_i(\theta) &= (1-\theta)^3 v_i f_i + \theta(1-\theta)^2[(2u_i v_i + v_i)f_i + v_i h_i d_i] \\ &\quad + \theta^2(1-\theta)[(2u_i v_i + u_i)f_{i+1} - u_i h_i d_{i+1}] + \theta^3 f_{i+1} \\ q_i(\theta) &= (1-\theta)^2 v_i + 2u_i v_i \theta(1-\theta) + \theta^2 u_i, \end{aligned}$$

where

$$\theta = \frac{x - x_i}{h_i}, \ 0 \leq \theta \leq 1;$$

u_i and v_i are named as shape parameters.

The rational cubic function (1) has the following properties:

$$\begin{aligned} S(x_i) &= f_i, & S(x_{i+1}) &= f_{i+1} \\ S^{(1)}(x_i) &= d_i, & S^{(1)}(x_{i+1}) &= d_{i+1} \end{aligned}$$

where $S^{(1)}$ denotes differentiation with respect to x, and d_i denote derivative values (given or estimated by some method) at the knots x_i. We note that in each interval I_i , when we take $u_i = v_i = 1$, the piecewise rational cubic function reduces to the standard cubic Hermite.

3 Determination of Derivatives

In most applications, derivative parameters d_i are not given and hence must be determined from the data (x_i, f_i). In this article, they are computed from the given data in such a way that the C^1 smoothness of the interpolant (1) is maintained. An obvious choice is stated below.

3.1 Arithmetic Mean Method

This is a three-point difference approximation with

$$d_i = 0, \text{ if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0,$$

otherwise

$$d_i = \frac{h_i \Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i+1}}, \ i = 2, 3, \cdots, n-1.$$

and the end conditions are given as:

$$d_1 = 0, \text{ if } \Delta_1 = 0 \text{ or } \operatorname{sgn}(d_1^*) \neq \operatorname{sgn}(\Delta_1),$$

otherwise

$$d_1 = d_1^* = \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2},$$

and

$$d_n = 0, \text{ if } \Delta_{n-1} = 0 \text{ or } \operatorname{sgn}(d_n^*) \neq \operatorname{sgn}(\Delta_{n-1}),$$

otherwise

$$d_n = d_n^* = \Delta_{n-1} + \frac{(\Delta_{n-1} - \Delta_{n-2})h_{n-1}}{h_{n-1} + h_{n-2}}.$$

4 Positivity-Preserving Spline Interpolation

The rational spline method, described in Section 2, has deficiencies as far as the shape preserving issue is concerned. For example, the rational cubic of Section 2 does not preserve the shape of positive data (see in Figure 1) in Table 1.

Table 1:

x	2	3	7	8	9	13	14
f	10	2	3	7	2	3	10

Clearly, the curve in Figure 1 is not preserving the shape of the data. Similarly an unwanted behavior can be observed in Figure 2 for the data set in Table 1.

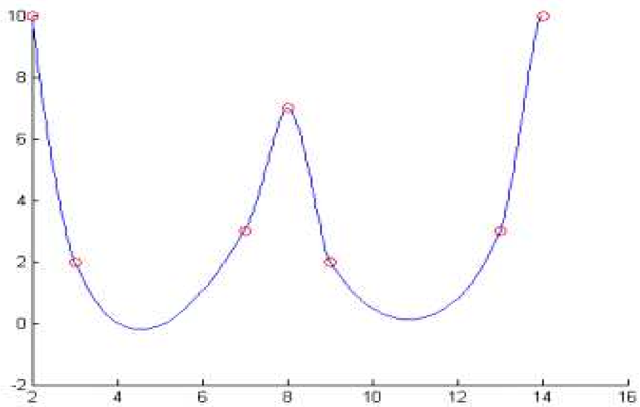

Figure 1: $u_i = 1, \ v_i = 2$

It can be seen that the ordinary spline curve does not guarantee to preserve the shape. Some negative values can be seen in the presentation of the curve. Thus, it is required to assign appropriate values to the shape parameters u_i, v_i so that it generates a curve which preserves the shape of the data. To proceed with this strategy certain mathematical treatment is needed as follows:

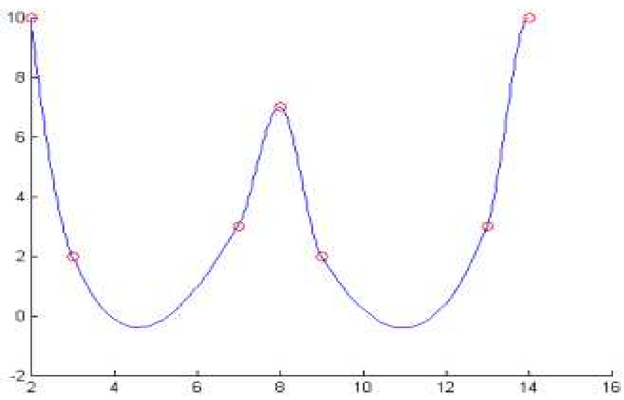

Figure 2: $u_i = 1, \ v_i = 1$

For given data points $(x_1, f_1), (x_2, f_2), \cdots, (x_n, f_n)$ with $x_1 < x_2 < \cdots < x_n$ and $f_1 > 0, f_2 > 0, \cdots, f_n > 0$, construct an interpolant $S(x)$ which is positive on the whole interval $[x_1, x_n]$ that is $S(x) > 0, \ \ x_1 \leq x \leq x_n$.

The key idea to preserving positivity using $S(x)$ is to assign suitable values u_i, v_i in each interval. The condition u_i, $v_i > 0$ guarantees strictly positive denominator of $S(x)$. So, the initial conditions on u_i, v_i are $u_i > 0, \ v_i > 0$. Now the problem reduces to, for what values of u_i, v_i the nominator of $S(x)$ is positive? The nominator of $S(x)$ is a cubic function. Schmidt and Hess [8] have developed the following:

A cubic Hermite $C(x)$ is non-negative in $[x_i, x_{i+1}]$ if and only if

$$(d_i, d_{i+1}) \in R$$

where

$$R = R_1 + R_2$$

with

$$R_1 = \left\{ (a,b) : \ a \geq \frac{-3f_i}{h_i}, \ b \leq \frac{3f_{i+1}}{h_i} \right\}$$

$$R_2 = \left\{ \begin{array}{l} (a,b) : 36 f_i f_{i+1}(a^2 + b^2 + ab - 3\Delta_i(a+b) \\ +3(f_{i+1}a - f_i b)(2h_i ab - 3f_{i+1}a + 3f_i b) \\ +4b_i(f_{i+1}a^3 - f_i b^3) - h_i^2 a^2 b^2 \geq 0 \end{array} \right\}$$

where f_i and d_i denote, respectively, the data values and first derivative values of the cubic polynomial at the knots $x_i, \ i = 1, 2, \cdots, n$.

Now according to Schmidt and Hess [8], $p_i(\theta) > 0$ if and only if

$$u_i > \frac{h_i d_i}{-2f_i} + 1 \text{ and } v_i > \frac{h_i d_{i+1}}{2f_{i+1}} - 1 \tag{2}$$

Hence $S(x) > 0$ if and only if

$$u_i > \text{Max}\left\{0, \frac{h_i d_i}{-2f_i} + 1\right\} \text{ and } v_i > \text{Max}\left\{0, \frac{h_i d_{i+1}}{2f_{i+1}} - 1\right\} \tag{2}$$

The above results can be summarized in the following theorem:

Theorem 1 *The rational cubic polynomial* (1) *preserves positivity if and only if equation* (2) *is satisfied.*

Demonstration

The first example relates to a lot of computers which were bought by a university and were installed in its Computer Centre. Due to continuous usage and availability of latest technology in the market, the computer depreciated in market prices. Due to depreciation, the valuation of the market price was noticed at different stages of time, which is displayed in Table 2.

Table 2:

x	1	2	3	8	10	11	12	14
f	14	8	3	0.8	0.5	0.45	0.4	0.37

At the end of first year, the price was evaluated as \$140000 and it depreciated to \$3700 after 14 years. One can note that all the data values in Table 2 are positive. Visualization of this information is shown in Figures 3 and 4, implementing the schemes in Section 2 and 4, respectively. Application of the Hermite cubic spline method produces the curve in Figure 3. This curve shows negative price which is misleading. We now apply piecewise rational cubic of Section 4 to the same data. Figure 4 is produced by setting the parameter u_i, v_i satisfying the positive conditions derived in Section 4.

Now we visualize the data in Table 1 using the scheme of Section 4. This data is obtained from the known volume of $\mathrm{N}a\mathrm{OH}$ taken in a beaker and its conductivity was determined. HCl solution was added from the burette in steps drop by drop. After each addition volume of HCl(x) was stirred by gentle shaking, the conductance (f) was determined. The data is shown in Table 1. One can note that all the data values are positive. However, Figures 1 and 2 show negative values of conductance which is ridiculous. This flaw is recovered nicely in Figure 5 using the positivity preserving rational cubic scheme of Section 4.

5 Conclusions

C^1 piecewise rational spline has been utilized to obtain C^1 positivity preserving curve method. Constraints are derived on the shape parameters to ensure the positive shape preservation of the data. Choice of the derivative parameters is constrained to be approximation through arithmetic means. The scheme has been implemented for scalar valued curves whereas the search, for the planar curve, is being made by the authors. This curve scheme could also be generalized to the surface case. The authors are keen to discuss it in a subsequent paper.

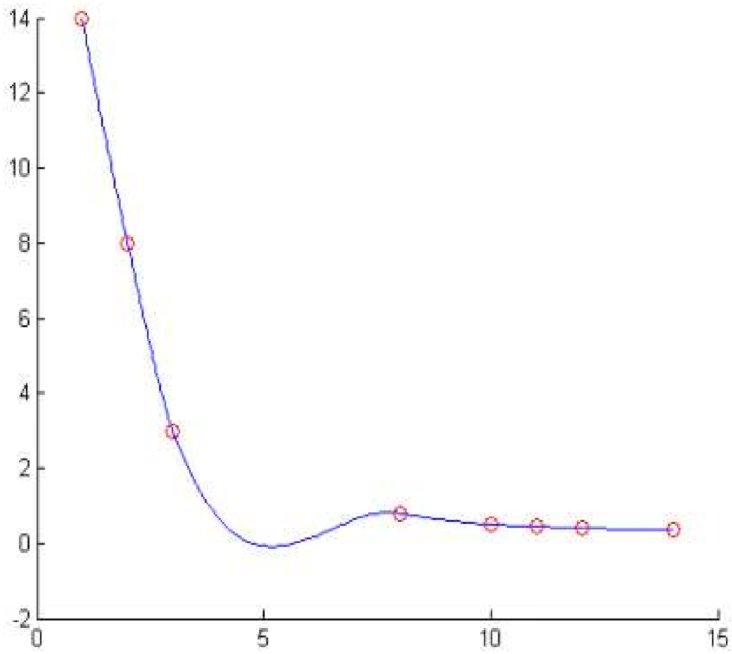

Figure 3: Cubic Hermite Spline curve to the data in Table 2.

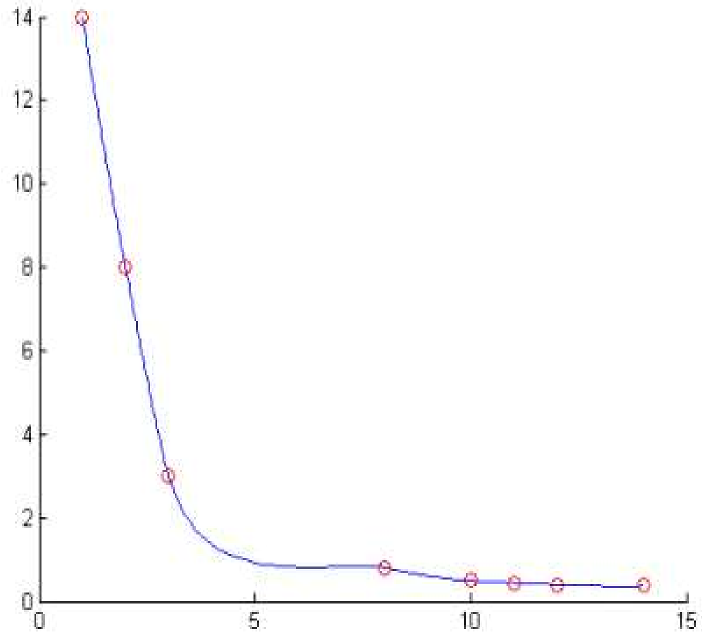

Figure 4: Shape Preserving Rational Cubic Spline curve to the datain Table 2.

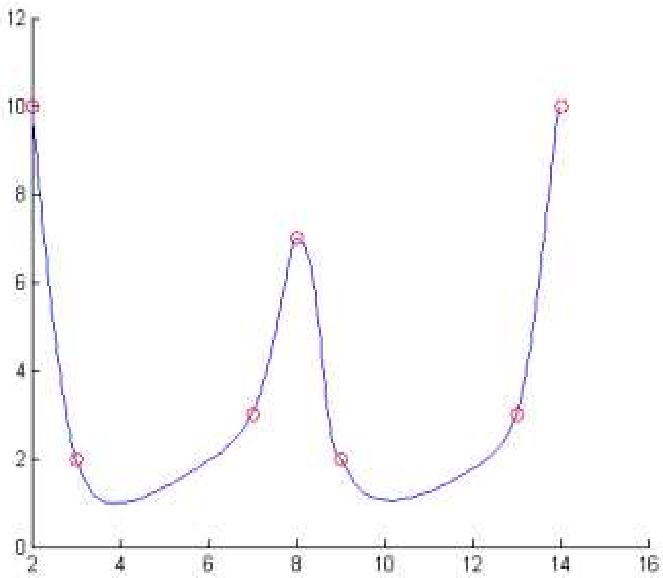

Figure 5: Shape Preserving Rational Cubic Spline curve to the data in Table 1.

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Acknowledgement This work is supported by visiting grant of the Universiti Sains Malaysia to the first author.