MATEMATIKA, 2006, Volume 22, Number 2, pp. 161–170

Chromatically Unique Bipartite Graphs With Certain 3-independent Partition Numbers

[1]**Roslan Hasni** & [2]**Y.H. Peng**
[1]Pusat Pengajian Sains Matematik, Universiti Sains Malaysia, 11800 Penang, Malaysia
[2]Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
e-mail: [1]hroslan@cs.usm.my, [2]yhpeng@fsas.upm.edu.my

Abstract For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of $2-$connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. In this paper, we prove that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $1 \leq s \leq q-1$, if the number of 3-independent partitions of G is $2^{p-1} + 2^{q-1} + s + 3$, then G is chromatically unique. This result extends the similar theorem by Dong et al. (Discrete Math. vol. 224 (2000) 107–124).

Keywords Chromatic polynomial, chromatically equivalence, chromatically unique.

1 Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply $\chi-$*disjoint*.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. $2-$connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.

For a bipartite graph $G = (A,B;E)$ with bipartition A and B and edge set E, let $G' = (A',B';E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E,\, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

In [1], Dong et al. proved the following result.

Theorem 1.1 *For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, $\mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed.*

Throughout this paper, we fix the following conditions for p, q and s:

$$p \geq q \geq 3 \qquad \text{and} \qquad 1 \leq s \leq q-1.$$

For a graph G and a positive integer k, a partition $\{ A_1, A_2, \ldots, A_k \}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of $k-$*independent partitions* in G.

For any bipartite graph $G = (A, B; E)$, define

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2).$$

In [1], the authors found the following sharp bounds for $\alpha'(G,3)$.

Theorem 1.2 *For $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq q-1$,*

$$s \leq \alpha'(G,3) \leq 2^s - 1,$$

where $\alpha'(G,3) = s$ iff $\Delta(G') = 1$ and $\alpha'(G,3) = 2^s - 1$ iff $\Delta(G') = s$.

For $t = 0, 1, 2, \ldots$, let $\mathcal{B}(p,q,s,t)$ denote the set of graphs $G \in \mathcal{K}^{-s}(p,q)$ with $\alpha'(G,3) = s+t$. Thus, $\mathcal{K}^{-s}(p,q)$ is partitioned into the following subsets:

$$\mathcal{B}(p,q,s,0), \quad \mathcal{B}(p,q,s,1), \quad \ldots, \mathcal{B}(p,q,s,2^s-s-1).$$

Assume that $\mathcal{B}(p,q,s,t) = \emptyset$ for $t > 2^s - s - 1$.

Lemma 1.1 (Dong et al. [2]) *For $p \geq q \geq 3$ and $0 \leq s \leq q-1$, if $0 \leq t \leq 2^{q-1} - q - 1$, then*

$$\mathcal{B}(p,q,s,t) \subseteq \mathcal{K}_2^{-s}(p,q).$$

Dong et al. [1] have shown that if G is a 2-connected graph in $\mathcal{B}(p,q,s,0) \cup \mathcal{B}(p,q,s,2^s - s - 1)$, then G is $\chi-$unique. In [2], Dong et al. proved that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is $\chi-$unique for $1 \leq t \leq 4$. In this paper, we extend this result by examining the chromatic uniqueness of 2-connected graph in $\mathcal{B}(p,q,s,5)$.

2 Preliminary results and notation

For any graph G of order n, we have (see [3]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 2.1 *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

By Theorem 1.1,the following two results were obtained in [2].

Theorem 2.1 *The set $\mathcal{B}(p,q,s,t) \cap \mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed for all $t \geq 0$.*

Corollary 2.1 *If $0 \leq t \leq 2^{q-1} - q - 1$, then $\mathcal{B}(p,q,s,t)$ is $\chi-$closed.*

Let $\beta_i(G)$, or simply β_i, denote the number of vertices in G with degree i, $n_i(G)$ denote the number of i-cycles in G and P_n denote the path with n vertices. Then Dong et al. [2] established the next two results.

Lemma 2.2 *For $G = (A, B; E) \in \mathcal{K}^{-s}(p, q)$,*

(i) if $\Delta(G') \le 2$, then $\alpha'(G, 3) = s + \beta_2(G') + n_4(G')$;

(ii) if $\Delta(G') = 3$, then $\alpha'(G, 3) \ge s + \beta_2(G') + 4\beta_3(G') + n_4(G')$, where equality holds iff $|N_{G'}(u) \cap N_{G'}(v)| \le 2$ for all u, $v \in A'$ or u, $v \in B'$;

(iii) $\alpha'(G, 3) \ge 2^{\Delta(G')} + s - 1 - \Delta(G')$.

For two disjoint graphs H_1 and H_2, let $H_1 \cup H_2$ denote the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Let $kH = \underbrace{H \cup \cdots \cup H}_{k}$ for $k \ge 1$ and let kH be null if $k = 0$.

Lemma 2.3 *Let $G \in \mathcal{K}^{-s}(p, q)$. If $\alpha'(G, 3) = s + t \le s + 4$, then either*

(i) each component of G' is a path and $\beta_2(G') = t$, or

(ii) $G' \cong K_{1,3} \cup (s - 3)K_2$.

Now for convenient we define the graphs Y_n and Z_1 as in Figure 1.

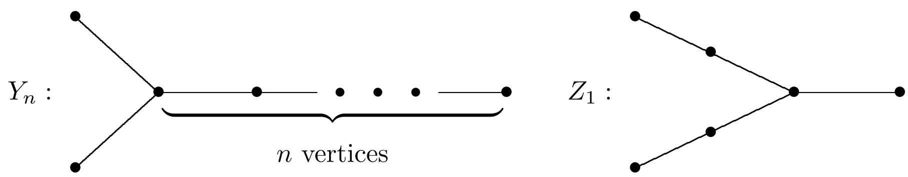

Figure 1: The graphs Y_n and Z_1

The following result is an extension of Lemma 2.3.

Lemma 2.4 *Let $G \in \mathcal{K}^{-s}(p, q)$. If $\alpha'(G, 3) = s + 5$, then either*

(i) each component of G' is a path and $\beta_2(G') = 5$, or

(ii) $G' \cong K_{1,3} \cup P_3 \cup (s - 5)K_2$, or

(iii) $G' \cong C_4 \cup (s - 4)K_2$, or

(iv) $G' \cong Y_3 \cup (s - 4)K_2$.

Proof. Since $\alpha'(G, 3) = s+5$, $\Delta(G') \le 3$ by Lemma 2.2(iii). If $\Delta(G') = 3$,by Lemma 2.2(ii), we have $\beta_2(G') = 1$, $n_4(G') = 0$ and $\beta_3(G') = 1$. Thus $G' \cong K_{1,3} \cup P_3 \cup (s - 5)K_2$ or $G' \cong Y_3 \cup (s-4)K_2$. If $\Delta(G') = 2$, we have $\beta_2(G') + n_4(G') = 5$ by Lemma 2.2(i), and thus either G' contain no cycles or only have one cycle. Hence, when $\Delta(G') = 2$, either each component of G' is a path, and $\beta_2(G') = 5$, or $G' \cong C_4 \cup (s - 4)K_2$, by Lemma 2.2(i). $\square$

By Lemma 2.4, we establish the following result.

Theorem 2.2 *Let $G \in \mathcal{K}^{-s}(p,q)$ and $\alpha'(G,3) = s+5$, then*

$$G' \in \{\ P_7 \cup (s-6)K_2,\ P_6 \cup P_3 \cup (s-7)K_2,\ P_5 \cup P_4 \cup (s-7)K_2,\\ P_5 \cup 2P_3 \cup (s-8)K_2,\ P_4 \cup 3P_3 \cup (s-9)K_2,\ 2P_4 \cup P_3 \cup (s-8)K_2,\\ 5P_3 \cup (s-10)K_2,\ K_{1,3} \cup P_3 \cup (s-5)K_2,\ C_4 \cup (s-4)K_2,\ Y_3 \cup (s-4)K_2\ \}$$

where $H \cup (s-i)K_2$ does not exist if $s < i$.

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{\ Q \mid Q \text{ is an independent sets in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset\ \}.$$

For a bipartite graph $G = (A, B; E)$, the number of 4−independent partitions $\{ A_1, A_2, A_3, A_4 \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned}(2^{|A|-1}-1)(2^{|B|-1}-1) &+ \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3)\\ &= (2^{|A|-1}-2)(2^{|B|-1}-2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.\end{aligned}$$

Define

$$\alpha'(G,4) = \alpha(G,4) - \{\ (2^{|A|-1}-2)(2^{|B|-1}-2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2\ \}.$$

Observe that for $G,\ H \in \mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4) = \alpha(H,4) \quad \text{iff} \quad \alpha'(G,4) = \alpha'(H,4).$$

The following five lemmas (see [2]) will be used to prove our main results.

Lemma 2.5 *For $G = (A, B; E) \in \mathcal{K}^{-s}(p,q)$ with $|A| = p$ and $|B| = q$,*

$$\begin{aligned}\alpha'(G,4) \quad = \quad & \sum_{Q \in \Omega(G)} (2^{p-1-|Q\cap A|} + 2^{q-1-|Q\cap B|} - 2) + \\ & \Big|\{\ \{\ Q_1, Q_2\ \} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset\ \}\Big|.\end{aligned}$$

Lemma 2.6 *For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,*

$$\alpha(G,k) = \alpha(G+uv,k) + \alpha(G - \{u,v\}, k-1) + \alpha(G - \{u,v,w\}, k-1).$$

For a bipartite graph $G = (A, B; E)$, let $\beta_i(G, A)$ (resp., $\beta_i(G, B)$) be the number of vertices in A (resp., B) with degree i.

Lemma 2.7 *For $G \in \mathcal{B}(p,q,s,t)$, if each component of G' is a path, then*

$$\sum_{Q \in \Omega(G)} (2^{p-1-|Q\cap A|} + 2^{q-1-|Q\cap B|} - 2)$$
$$= s(2^{p-2} + 2^{q-2} - 2) + t(2^{p-3} + 2^{q-2} - 2) + (2^{p-3} + 2^{q-3})\beta_2(G', A').$$

Let $p_i(G)$ denote the number of paths P_i in G.

Lemma 2.8 *For $G \in \mathcal{B}(p,q,s,t)$, if each component of G' is a path, then*

$$\left|\{\ \{ Q_1, Q_2 \} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset\ \}\right| = \binom{s+t}{2} - 3t - 3p_4(G') - p_5(G').$$

For $G \in \mathcal{B}(p,q,s,t)$, define

$$\alpha''(G,4) = \alpha'(G,4) - \Big[s(2^{p-2} + 2^{q-2} - 2) + t(2^{p-3} + 2^{q-2} - 2) + (s+t)(s+t-1)/2 - 3t \Big]. \quad (1)$$

Observe that for G, $H \in \mathcal{B}(p,q,s,t)$,

$$\alpha''(G,4) = \alpha''(H,4) \quad \text{iff} \quad \alpha(G,4) = \alpha(H,4).$$

Lemma 2.9 *For $G \in \mathcal{B}(p,q,s,t)$, if each component of G' is a path, then*

$$\alpha''(G,4) = (2^{p-3} + 2^{q-3})\beta_2(G', A') - 3p_4(G') - p_5(G').$$

3 Main result

Dong et al. [1] have shown that any graph G in $\mathcal{B}(p,q,s,0) \cup \mathcal{B}(p,q,s,2^s - s - 1)$, if G is 2-connected, is $\chi-$unique. In [2], Dong et al. proved that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is $\chi-$unique for $1 \le t \le 4$. In this section, we shall prove that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is $\chi-$unique for $t = 5$.

Our main result is Theorem 3.1.

Theorem 3.1 *Let p, q and s be integers with $p \ge q \ge 3$ and $0 \le s \le q-1$. For every $G \in \mathcal{B}(p,q,s,5)$, if G is 2-connected, then G is $\chi-$unique.*

Proof. By Theorem 2.1, $\mathcal{B}(p,q,s,t) \cap \mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed for all $t \ge 0$. Hence, to show that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is $\chi-$unique, it suffices to show that for every two graphs G and H in $\mathcal{B}(p,q,s,t)$, if $G \not\cong H$, then either $\alpha(G,4) \ne \alpha(H,4)$ or $\alpha(G,5) \ne \alpha(H,5)$. Recall that for $\alpha''(G,4) = \alpha''(H,4)$ iff $\alpha(G,4) = \alpha(H,4)$ and $\alpha'(G,4) = \alpha'(H,4)$ iff $\alpha(G,4) = \alpha(H,4)$.

The set $\mathcal{B}(p,q,s,5)$ contain 31 graphs by Theorem 2.2, named as $G_{5,1}$, $G_{5,2}$, $G_{5,3}$, $G_{5,4}, \ldots$, $G_{5,31}$ (see table in [5]). These graphs are shown in this table with the values $\alpha''(G_{5,1},4)$, $\alpha''(G_{5,2},4), \ldots, \alpha''(G_{5,31},4)$. For each graph $G_{5,i}$, if every component of $G'_{5,i}$ is a path, then $\alpha''(G_{5,i},4)$ can be obtained by Lemma 2.9; otherwise, we must find $\alpha'(G_{5,i},4)$ by Lemma 2.5, and then find $\alpha''(G_{5,i},4)$ by Equation (1).

Partition $\mathcal{B}(p,q,s,5)$ into the following ten subsets:

$$\begin{aligned}
\mathcal{T}_1 &= \{\, G_{5,1}, G_{5,2} \,\}\\
\mathcal{T}_2 &= \{\, G_{5,3}, G_{5,4} \,\}\\
\mathcal{T}_3 &= \{\, G_{5,5}, G_{5,6} \,\}\\
\mathcal{T}_4 &= \{\, G_{5,7}, G_{5,8}, G_{5,9}, G_{5,10}, G_{5,11}, G_{5,12} \,\}\\
\mathcal{T}_5 &= \{\, G_{5,13}, G_{5,14} \,\}\\
\mathcal{T}_6 &= \{\, G_{5,15}, G_{5,16}, G_{5,17}, G_{5,18} \,\}\\
\mathcal{T}_7 &= \{\, G_{5,19} \,\}\\
\mathcal{T}_8 &= \{\, G_{5,20}, G_{5,21} \,\}\\
\mathcal{T}_9 &= \{\, G_{5,22}, G_{5,23}, G_{5,24}, G_{5,25} \,\}\\
\mathcal{T}_{10} &= \{\, G_{5,26}, G_{5,27}, G_{5,28}, G_{5,29}, G_{5,30}, G_{5,31} \,\}
\end{aligned}$$

For non-empty sets $W_1, W_2, \ldots, W_k$ of graphs, let $\eta(W_1, W_2, \ldots, W_k) = 0$ if $\alpha(G_1,4) \neq \alpha(G_2,4)$ for every two graphs $G_1 \in W_i$ and $G_2 \in W_j$, where $i \neq j$, and let $\eta(W_1, W_2, \ldots, W_k) = 1$ otherwise.

The values of $\alpha''(G_{5,19},4)$, $\alpha''(G_{5,20},4)$, $\ldots$, $\alpha''(G_{5,25},4)$ cannot be computed using Lemma 2.9, but it can be obtained by Lemma 2.5 and Equation (1). For example, we calculate

$$\begin{aligned}
\text{(i)}\ \alpha''(G_{5,19},4) &= \Bigg[\, s(2^{p-2}+2^{q-2}-2) + 2(2^{p-3}+2^{q-2}-2) + \\
&\quad 2(2^{q-3}+2^{p-2}-2) + (2^{p-3}+2^{q-3}-2) + \\
&\quad \left\{ 9(s-4) + \binom{s-4}{2} + 2 \right\} \Bigg] - \Bigg[\, s(2^{p-2}+2^{q-2}-2) + \\
&\quad 5(2^{p-3}+2^{q-2}-2) + \binom{s+5}{2} - 15 \,\Bigg] \\
&= 2^{p-2} - 3\cdot 2^{q-3} - 19.
\end{aligned}$$

Similarly, we obtain the following (as shown in the table [5]):

(ii) $\alpha''(G_{5,20},4) = 2^{p-4} - 2^{q-3} - 18.$

(iii) $\alpha''(G_{5,21},4) = 2^{p-1} - 9\cdot 2^{q-4} - 18.$

(iv) $\alpha''(G_{5,22},4) = -2^{p-4} - 9.$

(v) $\alpha''(G_{5,23},4) = 2^{p-4} - 2^{q-3} - 9.$

(vi) $\alpha''(G_{5,24},4) = 2^{p-1} - 9\cdot 2^{q-4} - 9.$

(vii) $\alpha''(G_{5,25},4) = 5\cdot 2^{p-3} - 11\cdot 2^{q-4} - 9.$

Claim 1. $\eta(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7, \mathcal{T}_8, \mathcal{T}_9, \mathcal{T}_{10}) = 0$:

Proof of Claim 1. Note that if 2^k (k is an integer ≥ 1) is not a factor of x, then 2^h is also not a factor of x for any integer $h \geq k$. Similarly, if 2^k (k is an integer ≥ 1) is a factor of x, then 2^h is also a factor of x for any integer $1 \leq h \leq k$.

(a) For $s \leq 4$, only $\mathcal{T}_7$ and $\mathcal{T}_8$ are non-empty. Observe that 2^2 is a factor of $\alpha''(G,4)+19$ for $G \in \mathcal{T}_7$ but 2^2 is not a factor of $\alpha''(G,4)+19$ for $G \in \mathcal{T}_8$. Hence $\eta(\mathcal{T}_7, \mathcal{T}_8) = 0$.

(b) For $s \geq 5$, $\alpha''(G,4)$ is odd if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7 \cup \mathcal{T}_9$ and even if $G \in \mathcal{T}_3 \cup \mathcal{T}_5 \cup \mathcal{T}_8 \cup \mathcal{T}_{10}$. Hence $\eta(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7 \cup \mathcal{T}_9, \mathcal{T}_3 \cup \mathcal{T}_5 \cup \mathcal{T}_8 \cup \mathcal{T}_{10}) = 0$.

(c) For $s \geq 5$, 2^2 is a factor of $\alpha''(G,4)+9$ for every $G \in \mathcal{T}_9$, but 4 is not a factor of $\alpha''(G,4)+9$ for every $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7$. Hence $\eta(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7, \mathcal{T}_9) = 0$.

(d) For $s \geq 5$, 2^4 is a factor of $\alpha''(G,4)+15$ for $G \in \mathcal{T}_1$, 2^3 is not a factor of $\alpha''(G,4)+15$ for every $G \in \mathcal{T}_2 \cup \mathcal{T}_6 \cup \mathcal{T}_7$, and 2^3 is a factor of $\alpha''(G,4)+15$ but 2^4 is not a factor for $G \in \mathcal{T}_4$. Hence $\eta(\mathcal{T}_1, \mathcal{T}_4, \mathcal{T}_2 \cup \mathcal{T}_6 \cup \mathcal{T}_7) = 0$.

(e) For $s \geq 5$, 2^4 is a factor of $\alpha''(G,4)+11$ for $G \in \mathcal{T}_2$, but 2^4 is not a factor of $\alpha''(G,4)+11$ for every $G \in \mathcal{T}_6 \cup \mathcal{T}_7$. Hence $\eta(\mathcal{T}_2, \mathcal{T}_6 \cup \mathcal{T}_7) = 0$.

(f) For $s \geq 5$, 2^5 is a factor of $\alpha''(G,4)+3$ for $G \in \mathcal{T}_6$, but 2^5 is not a factor of $\alpha''(G,4)+3$ for every $G \in \mathcal{T}_7$. Hence $\eta(\mathcal{T}_6, \mathcal{T}_7) = 0$.

(g) For $s \geq 5$, 2^4 is a factor of $\alpha''(G,4)+10$ for $G \in \mathcal{T}_3$, 2^2 is not a factor of $\alpha''(G,4)+10$ for $G \in \mathcal{T}_{10}$, but 2^2 is a factor of $\alpha''(G,4)+10$ and 2^3 is not a factor for $G \in \mathcal{T}_5$, and 2^3 is a factor $\alpha''(G,4)+10$ but 2^4 is not for $G \in \mathcal{T}_8$. Hence $\eta(\mathcal{T}_3, \mathcal{T}_5, \mathcal{T}_8, \mathcal{T}_{10}) = 0$.

By (a)–(g), Claim 1 holds.

The remaining work is to compare every two graphs in each $\mathcal{T}_i$, except $\mathcal{T}_7$ which contain only one graph. We shall establish several inequalities of the form $\alpha''(G_{5,i},4) < \alpha''(G_{5,j},4)$ for some i, j. Since the methods used to obtain these inequalities are standard, long and rather repetitive, we shall not discuss all of them here. In the following we shall only show two examples of detailed comparisons. In the first example, we compare all graphs in $\mathcal{T}_6$ when $p = q$ and in the second example, we compare all graphs in $\mathcal{T}_4$ when $p > q$.

(1) $\mathcal{T}_6$

When $p = q$, $G_{5,15} \cong G_{5,18}$, $G_{5,16} \cong G_{5,17}$ and from the table (see [5]), we can easily see that $\alpha''(G_{5,15},4) = \alpha''(G_{5,16},4)$. Thus, we need to calculate $\alpha(G_{5,15},5) - \alpha(G_{5,16},5)$. By using Lemma 2.6,we have

$$\begin{aligned}
&\alpha(G_{5,15},5) - \alpha(G_{5,16},5) \\
&= \Big[\alpha(G_{5,15}+a_5b_5,5) + \alpha(G_{5,15}-\{a_5,b_5\},4) + \alpha(G_{5,15}-\{a_5,b_5,c_5\},4) \Big] - \\
&\qquad \Big[\alpha(G_{5,16}+a_6b_6,5) + \alpha(G_{5,16}-\{a_6,b_6\},4) + \alpha(G_{5,16}-\{a_6,b_6,c_6\},4) \Big] \\
&= \alpha(G_{5,15}-\{a_5,b_5,c_5\},4) - \alpha(G_{5,16}-\{a_6,b_6,c_6\},4) \\
&\quad \text{since} \quad G_{5,15}+a_5b_5 \cong G_{5,16}+a_6b_6, \quad \text{and} \quad G_{5,15}-\{a_5,b_5\} \cong G_{5,16}-\{a_6,b_6\} \\
&= \alpha''(G_{5,15}-\{a_5,b_5,c_5\},4) - \alpha''(G_{5,16}-\{a_6,b_6,c_6\},4).
\end{aligned}$$

Since
$$G_{5,15} - \{a_5, b_5, c_5\} \in \mathcal{B}(p-2, q-1, s-2, 4), \quad \text{and}$$
$$G_{5,16} - \{a_6, b_6, c_6\} \in \mathcal{B}(p-1, q-2, s-2, 4),$$

by Lemma 2.9, we have

$$\begin{aligned}
&\alpha(G_{5,15}, 5) - \alpha(G_{5,16}, 5)\\
&\qquad = \alpha''(G_{5,15} - \{a_5, b_5, c_5\}, 4) - \alpha''(G_{5,16} - \{a_6, b_6, c_6\}, 4)\\
&\qquad = \Big[\ (2^{p-5} - 2^{q-4}) \cdot 1 - 3 \cdot 1\ \Big] - \Big[\ (2^{p-4} - 2^{q-5}) \cdot 1 - 3 \cdot 1\ \Big]\\
&\qquad = 2^{p-5} - 2^{q-4} - 2^{p-4} + 2^{q-5}\\
&\qquad = -2^{q-4} < 0 \quad (\text{ since } p = q \text{ and } q \geq 10)
\end{aligned}$$

(2) $\mathcal{T}_4$

When $p > q$, from the table (see [5]), we can easily see that

(a) $\alpha''(G_{5,7}, 4) - \alpha''(G_{5,8}, 4) < 0$,

(b) $\alpha''(G_{5,8}, 4) - \alpha''(G_{5,9}, 4) < 0$,

(c) $\alpha''(G_{5,9}, 4) - \alpha''(G_{5,12}, 4) < 0$,

(d) $\alpha''(G_{5,8}, 4) - \alpha''(G_{5,10}, 4) = 0$,

(e) $\alpha''(G_{5,9}, 4) - \alpha''(G_{5,11}, 4) = 0$.

Since $\alpha''(G_{5,8}, 4) = \alpha''(G_{5,10}, 4)$ and $\alpha''(G_{5,9}, 4) = \alpha''(G_{5,11}, 4)$, we need to compare $\alpha(G_{5,8}, 5)$ with $\alpha(G_{5,10}, 5)$ and $\alpha(G_{5,9}, 5)$ with $\alpha(G_{5,11}, 5)$. Hence, we have the following claim:

<u>Claim 2.</u> $\alpha(G_{5,8}, 5) - \alpha(G_{5,10}, 5) = 3(2^{p-5} - 2^{q-5})$.

<u>Proof of Claim 2</u>.

By Lemma 2.6,

$$\begin{aligned}
&\alpha(G_{5,8}, 5)\\
&\quad = \alpha(G_{5,8} + a_1b_1, 5) + \alpha(G_{5,8} - \{a_1, b_1\}, 4) + \alpha(G_{5,8} - \{a_1, b_1, c_1\}, 4)\\
&\quad = \Big[\ \alpha(G_{5,8} + a_1b_1 + b_1c_1, 5) + \alpha(G_{5,8} - \{b_1, c_1\}, 4) + \alpha(G_{5,8} - \{b_1, c_1, d_1\}, 4)\ \Big] +\\
&\qquad\quad \alpha(G_{5,8} - \{a_1, b_1\}, 4) + \alpha(G_{5,8} - \{a_1, b_1, c_1\}, 4), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
&\alpha(G_{5,10}, 5)\\
&\quad = \alpha(G_{5,10} + a_2b_2, 5) + \alpha(G_{5,10} - \{a_2, b_2\}, 4) + \alpha(G_{5,10} - \{a_2, b_2, c_2\}, 4)\\
&\quad = \Big[\ \alpha(G_{5,10} + a_2b_2 + b_2c_2, 5) + \alpha(G_{5,10} - \{b_2, c_2\}, 4) + \alpha(G_{5,10} - \{b_2, c_2, d_2\}, 4)\ \Big] +\\
&\qquad\quad \alpha(G_{5,10} - \{a_2, b_2\}, 4) + \alpha(G_{5,10} - \{a_2, b_2, c_2\}, 4).
\end{aligned}$$

Observe that

$$
\begin{aligned}
&G_{5,8} + a_1b_1 + b_1c_1 \cong G_{5,10} + a_2b_2 + b_2c_2,\\
&G_{5,8} - \{a_1, b_1\} \cong G_{5,10} - \{a_2, b_2\},\\
&\alpha(G_{5,8} - \{b_1, c_1\}, 4) - \alpha(G_{5,10} - \{b_2, c_2\}, 4) = 2^{p-4} - 2^{q-4}. \qquad (2)
\end{aligned}
$$

Since

$$
\begin{aligned}
&G_{5,8} - \{a_1, b_1, c_1\} \in \mathcal{B}(p-2, q-1, s-3, 2), \quad \text{and}\\
&G_{5,10} - \{b_2, c_2, d_2\} \in \mathcal{B}(p-2, q-1, s-4, 2),
\end{aligned}
$$

by Lemmas 2.5, 2.7, and 2.8, we have

$$
\begin{aligned}
&\alpha(G_{5,8} - \{a_1, b_1, c_1\}, 4) - \alpha(G_{5,10} - \{b_2, c_2, d_2\}, 4)\\
&\quad = \alpha'(G_{5,8} - \{a_1, b_1, c_1\}, 4) - \alpha'(G_{5,10} - \{b_2, c_2, d_2\}, 4)\\
&\quad = \Bigg[(s-3)(2^{p-4} + 2^{q-3} - 2) + 2(2^{p-5} + 2^{q-3} - 2) + (2^{p-5} - 2^{q-4}) +\\
&\qquad\quad \left\{ \binom{s-1}{2} - 6 \right\} \Bigg] - \Bigg[(s-4)(2^{p-4} + 2^{q-3} - 2) +\\
&\qquad\quad 2(2^{p-5} + 2^{q-3} - 4) + \left\{ \binom{s-2}{2} - 6 \right\} \Bigg]\\
&\quad = (2^{p-4} - 2^{q-3} - 2) + (2^{p-5} - 2^{q-4}) + (s-4) + 2\\
&\quad = 2^{p-4} + 2^{p-5} + 2^{q-4} + s - 4. \qquad (3)
\end{aligned}
$$

Similarly, since

$$
\begin{aligned}
&G_{5,8} - \{b_1, c_1, d_1\} \in \mathcal{B}(p-1, q-2, s-4, 2), \quad \text{and}\\
&G_{5,10} - \{a_2, b_2, c_2\} \in \mathcal{B}(p-1, q-2, s-3, 2),
\end{aligned}
$$

by Lemmas 2.5, 2.7, and 2.8, we have

$$
\begin{aligned}
&\alpha(G_{5,8} - \{b_1, c_1, d_1\}, 4) - \alpha(G_{5,10} - \{a_2, b_2, c_2\}, 4)\\
&\quad = \alpha'(G_{5,8} - \{b_1, c_1, d_1\}, 4) - \alpha'(G_{5,10} - \{a_2, b_2, c_2\}, 4)\\
&\quad = \Bigg[(s-4)(2^{p-3} + 2^{q-4} - 2) + 2(2^{p-4} + 2^{q-4} - 2) + (2^{p-4} - 2^{q-5}) -\\
&\qquad\quad \left\{ \binom{s-2}{2} - 6 \right\} \Bigg] - \Bigg[(s-3)(2^{p-3} + 2^{q-4} - 2) +\\
&\qquad\quad 2(2^{p-4} + 2^{q-4} - 2) + \left\{ \binom{s-1}{2} - 6 \right\} \Bigg]\\
&\quad = -2^{p-4} - 2^{q-4} - 2^{q-5} - s + 4. \qquad (4)
\end{aligned}
$$

By (2) – (4), we have

$$\begin{aligned}
&\alpha(G_{5,8},5)-\alpha(G_{5,10},5)\\
&=\Big[\alpha(G_{5,8}+a_1b_1+b_1c_1,5)+\alpha(G_{5,8}-\{b_1,c_1\},4)+\alpha(G_{5,8}-\{b_1,c_1,d_1\},4)+\\
&\quad\alpha(G_{5,8}-\{a_1,b_1\},4)+\alpha(G_{5,8}-\{a_1,b_1,c_1\},4)\Big]-\\
&\quad\Big[\alpha(G_{5,10}+a_2b_2+b_2c_2,5)+\alpha(G_{5,10}-\{b_2,c_2\},4)+\alpha(G_{5,10}1\{b_2,c_2,d_2\},4)+\\
&\quad\alpha(G_{5,10}-\{a_2,b_2\},4)+\alpha(G_{5,10}-\{a_2,b_2,c_2\},4)\Big]\\
&=\Big[\alpha(G_{5,8}-\{b_1,c_1\},4)-\alpha(G_{5,10}-\{b_2,c_2\},4)\Big]+\\
&\quad\Big[\alpha(G_{5,8}-\{a_1,b_1,c_1\},4)-\alpha(G_{5,10}-\{b_2,c_2,d_2\},4)\Big]+\\
&\quad\Big[\alpha(G_{5,8}-\{b_1,c_1,d_1\},4)-\alpha(G_{5,10}-\{a_2,b_2,c_2\},4)\Big]\\
&=(2^{p-4}-2^{q-4})+(2^{p-4}+2^{p-5}+2^{q-4}+s-4)+\\
&\quad(-2^{p-4}-2^{q-4}-2^{q-5}-s+4)\\
&=3(2^{p-5}-2^{q-5})
\end{aligned}$$

Hence, Claim 2 is proved.

Claim 3. $\alpha(G_{5,9},5)-\alpha(G_{5,11},5)=3(2^{p-5}-2^{q-5})$.

Proof of Claim 3. Similar to the proof of Claim 2.
Similarly, we can show that for $G_{5,i}$ and $G_{5,j}$ in $\mathcal{B}(p,q,s,t)$, if $G_{5,i}\ncong G_{5,j}$, then either $\alpha''(G_{5,i},4)\neq\alpha''(G_{5,j},4)$ or $\alpha(G_{5,i},5)\neq\alpha(G_{5,j},5)$ (see [4]).

This completes the proof of Theorem 3.1. □

References

[1] F.M. Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, *Sharp bounds for the numbers of 3-partitions and the chromaticity of bipartite graphs,* J. Graph Theory, 37(2001) 48–77.

[2] F.M. Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, *Chromatically unique graphs with low 3-independent partition numbers,* Discrete Math. 224(2000), 107–124.

[3] R.C.Read, W.T. Tutte, Chromatic polynomials, in: L.W. Beineke, R.J. Wilson(Eds.), *Selected Topics in Graph Theory III*, Academic Press, New York, 1988, pp. 15–42.

[4] http://www.fsas.upm.edu.my/ yhpeng/publish/report3.pdf.

[5] http://www.fsas.upm.edu.my/ yhpeng/publish/table5.pdf.