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Effects of Control on the Onset of Marangoni-Bénard Convection with Uniform Internal Heat Generation

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Abstract The effect of control on the onset of Marangoni-Bénard convection in a horizontal layer of fluid with internal heat generation heated from below and cooled from above is investigated. The resulting eigenvalue problem is solved exactly. The effects of control are studied by examining the critical Marangoni numbers and wave numbers. It is found that the onset of Marangoni-Bénard convection with internal heat generation can be delayed through the use of control.

Keywords Marangoni-Bénard convection, control, internal heat generation.

1 Introduction

Effect of buoyancy or surface tension can become a major mechanism of driving a possible convective instability for a horizontal fluid layer heated from below and cooled from above. The instability of convection driven by buoyancy is referred to as the Rayleigh-Bénard convection and the instability convection driven by surface tension is referred to as the Marangoni-Bénard convection while the instability due to the combined effects of the thermal buoyancy and surface tension is called the Bénard-Marangoni convection. Theoretical analysis of Marangoni-Bénard convection was started with the linear analysis by Pearson [7] who assumed an infinite fluid layer, nondeformable case and zero gravity in the case of no-slip boundary conditions at the bottom. He showed that thermocapillary forces can cause convection when the Marangoni number exceeds a critical value in the absence of buoyancy forces. Pearson [7] obtained the critical Marangoni number, $M_c = 79.607$ and the critical wave number $a_c = 1.99$.

In the above Marangoni-Bénard instability analysis, the convective instability is induced by the temperature gradient which is decreasing linearly with fluid layer height. Sparrow et al. [9] and Roberts [8] analyze the thermal instability in a horizontal fluid layer with the nonlinear temperature distribution which is created by an internal heat generation. The effect of a quadratic basic state temperature profile caused by internal heat generation was first addressed by Char and Chiang [4] for Bénard-Marangoni convection. Later, Wilson [12] investigated the effect of the internal heat generation on the onset of Marangoni-Bénard convection when the lower boundary is conducting and when it is insulating to temperature perturbations. He found that the effect of increasing the internal heat generation is always to destabilize the layer.

The present work attempts to delay the onset of convection by applying the control. The objective of the control is to delay the onset of convection while maintaining a state of no motion in the fluid layer. Tang and Bau [10],[11] and Howle [5] have shown that the critical Rayleigh number for the onset of Rayleigh-Bénard convection can be delayed. Or et al. [6] studied theoretically the use of control strategies to stabilize long wavelength instabilities in the Marangoni-Bénard convection. Bau [2] has shown independently how such a control can delay the onset of Marangoni-Bénard convection on a linear basis with no-slip boundary conditions at the bottom. Recently, Arifin et al. [1] have shown that a control can also delay the onset of Marangoni-Bénard convection with free-slip boundary conditions at the bottom. Therefore, in this paper, we use a linear controller to delay the onset of Marangoni-Bénard convection in a fluid layer with internal heat generation. First, we derive the analytical expressions for the critical Marangoni-Bénard convection, and next we demonstrate that the no-motion state in the Marangoni-Bénard convection with internal heat generation can controlled for a particular choice of parameter values.

2 The Mathematical Formulation

Consider a horizontal fluid layer of depth d with a free upper surface heated from below and subject to a uniform vertical temperature gradient. The fluid layer is bounded below by a horizontal solid boundary at a constant temperature T_1 and above by a free surface at constant temperature T_2 which is in contact with a passive gas at constant pressure P_0 and constant temperature T_∞ (see Figure 1).

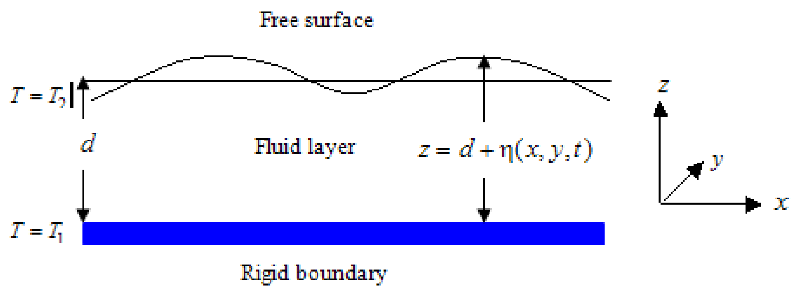

Figure 1: Problem set-up

We use Cartesian coordinates with two horizontal x-axis and y-axis located at the lower solid boundary and a positive z-axis is directed towards the free surface. The surface tension, τ is assumed to be a linear function of the temperature

$$\tau = \tau_0 - \gamma (T - T_0) , \tag{1}$$

where τ_0 is the value of τ at temperature T_0 and the constant γ is positive for most fluids. The density of the fluid is given by

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{2}$$

where α is the positive coefficient of the thermal liquid expansion and ρ_0 is the value of ρ at the reference temperature T_0.

The fluid is assumed to be an incompressible Newtonian Liquid satisfying the continuity equation together with the Navier-Stokes and the heat equations. These equations are, respectively

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial \mathrm{t}} + \mathbf{u} \cdot \nabla\right) \mathbf{u} = -\frac{\nabla p}{\rho} + v \nabla u, \tag{4}$$

$$\left(\frac{\partial}{\partial \mathrm{t}} + \mathbf{u} \cdot \nabla\right) T = \kappa \nabla^2 T + q \tag{5}$$

where $\mathbf{u}$, T, p, ρ, v, κ and q denote the velocity , temperature, pressure, density, kinematic viscosity, thermal diffusivity and uniformly distributed volumetric internal heat generation in the fluid layer, respectively. When motion occurs, the upper free surface of the layer will be deformable with its position at $z = d + f\,(x, y, t)$. At the free surface, we have the usual kinematic condition together with the conditions of continuity for the normal and tangential stresses. The temperature obeys the Newton's law of cooling, $k \partial T / \partial \mathbf{n} = h\,(T - T_\infty)$, where k and h are the thermal conductivity of the fluid and the heat transfer coefficient between the free surface and the air, respectively, and $\mathbf{n}$ is the outward unit normal to the free surface. The boundary conditions at the bottom wall, $z = 0$, are no-slip and conducting to the temperature perturbations.

To simplify the analysis, it is convenient to write the governing equations and the boundary conditions in a dimensionless form. In the dimensionless formulation, scales for length, velocity, time and temperature gradient are taken to be d, κ/d, d^2/κ and ΔT, respectively. Furthermore, six dimensionless groups appearing in the problem are the Marangoni number $M = \gamma \Delta T d / (\rho_0 \kappa v)$, the Biot number, $B_{\mathrm{i}} = hd/k$, the Bond number, $B_{\mathrm{o}} = \rho_0 g d^2 / \tau_0$, the Prandtl number, $P_r = \nu/\kappa$, the Crispation number, $C_r = \rho_0 \nu \kappa / (\tau_0 d)$ and the internal heating, $Q = q d^2 / (2\kappa\, \Delta T)$.

Our control strategy basically applies a principle similar to that used by Bau [2], which is as follows:
Assumed that the sensors and actuators are continuously distributed and that each sensor directs an actuator installed directly beneath it at the same $\{x, y\}$ location. The sensor detects the deviation of the free surface temperature from its conductive value. The actuator modifies the heated surface temperature according to the following rule (Bau [2]):

$$T(x, y, 0, t) = \frac{1 + B_i}{B_i} - K\left(T\,(x, y, 1, t) - \frac{1}{B_i}\right) \tag{6}$$

where K is the scalar controller gain. Equation (6) can be rewritten more conveniently as

$$T'(x, y, 0, t) = -K\,(T'(x, y, 1, t)) \tag{7}$$

where T' is the deviation of the fluid's temperature from its conductive value. The control strategy in equation (7), in which K is a scalar will be used to demonstrate that our system can be controlled.

3 Linearized Problem

We study the linear stability of the basic state by seeking perturbed solutions for any quantity $\Phi\,(x, y, z, t)$ in terms of normal modes in the form

$$\Phi(x, y, z, t) = \Phi_0(x, y, z) + \phi(z) \exp[i(\alpha_x x + \alpha_y y) + st], \tag{8}$$

where Φ_0 is the value of Φ in the basic state, $a = (\alpha_x^2 + \alpha_y^2)^{1/2}$ is the total horizontal wave number of the disturbance and s is a complex growth rate with a real part representing the growth rate of the instability and the imaginary part representing its frequency. At marginal stability, the growth rate s of perturbation is zero and the real part of s, $\Re(s) > 0$ represents unstable modes while $\Re(s) < 0$ represents stable modes. Substituting equation (8) into equations (3) - (5) and neglecting terms of the second and higher orders in the perturbations we obtain the corresponding linearized equations involving only the z-dependent parts of the perturbations to the temperature and the z-components of the velocity denoted by T and w, respectively,

$$\left(D^2 - a^2\right)\left[\left(D^2 - a^2\right) w - sP_r^{-1}\right] = 0, \tag{9}$$

$$\left(D^2 - a^2 - s\right) T + \left[1 - Q\,(1 - 2z)\right] w = 0, \tag{10}$$

subject to

$$sf - w\,(1) = 0, \tag{11}$$

$$C_{\mathrm{r}}\left[\left(D^2 - 3a^2 - sP_r^{-1}\right) Dw\,(1)\right] - a^2\left(a^2 + B_{\mathrm{o}}\right) f = 0, \tag{12}$$

$$\left(D^2 + a^2\right) w(1) + a^2 M\,(T(1) - (1 + Q)\,f) \;\; = 0, \tag{13}$$

$$DT\,(1) + B_{\mathrm{i}}\,(T\,(1) - (1 + Q)\,f) = 0, \tag{14}$$

$$w(0) = 0, \tag{15}$$

$$Dw(0) = 0, \tag{16}$$

and

$$T(0) + KT(1) = 0.. \tag{17}$$

On the lower rigid boundary $z = 0$. The operator $D = d/dz$ denotes the differentiation with respect to the vertical coordinate z. The variables w, T and f denote respectively the vertical variation of the z-velocity, temperature and the magnitude of the free surface deflection of the linear perturbation to the basic state with total wave number a in the horizontal x-y plane and complex growth rates.

4 Results and Discussion

We use the symbolic algebra package MAPLE 10 running on a Pentium PC to carry out much of the tedious algebraic manipulations needed in the course of finding analytical solutions. Closed form analytical expressions can be obtained for the Marginal stability curves for the onset of steady convection. By substituting the general solution of equations (9) and (10) into the boundary conditions (11) - (17) and requiring the existence of nontrivial solutions, we obtain the expression for Marangoni number M in terms of a, Q, C_r, B_o and B_i

on the marginal curve which can be conveniently written in the form of

$$M = \frac{\left(a^2 + B_o\right) f_1\left(a, B_i, K\right)}{\left(a^2 + B_o\right)\left[f_2\left(a\right) + Q\, f_3\left(a\right)\right] + C_r\left(1 + Q\right) f_4\left(a, K\right)}, \tag{18}$$

where

$$\begin{aligned} f_1\left(a, B_i, K\right) &= 24a^2\left[aK + aC + B_i S\right]\left(SC - a\right) \\ f_2\left(a\right) &= 3a\left(S^3 - a^3 C\right) \\ f_3\left(a\right) &= 3aS^3 + 12aS + 4a^3 S - 6a^2 C - 6CS^2 - a^4 C \\ f_4\left(a, K\right) &= 24a^6\left(K + C\right) \end{aligned}$$

where we have define $C = \cosh a$ and $S = \sinh a$. When we set $K = 0$, the equation (18) reduces to the expression given by Wilson [12] and when $Q = 0$, the equation (18) reduces to the expression given by Bau [2].

Figure 2 shows the Marangoni number as a function of the wave number, a, for controller gains: $K = 0, 5$ and 20. As a increases, the Marangoni number decreases, attains a minimum at some critical wave number, and increases again. In the absence of the controller, $K = 0$, we reproduce numerical result obtained by Boeck and Thess [3]. As the controller gain, K, increases, the marginal stability curves shift upwards, shows that the controller stabilizes the no-motion state for all wave numbers. The critical Marangoni number, M, increases monotonically as the controller gain K increases. In the case of nondeformable free surface ($C_r = 0$), the controller can suppress the modes and maintain a no-motion state, but this situation is significantly different if the free surface is deformable. When C_r becomes large the long-wavelength instability sets in as a primary one and the critical Marangoni numbers are at $a = 0$.

Figure 3 shows the Marangoni number as a function of the wave number, a, in the case $C_r = 0.001$ when $B_i = 0$ and $B_o = 0$ for a range of values of controller gains, K. The situation here is significantly different than the case of $C_r = 0$ (Figure 2). At $a = 0$ (long wavelengths), the critical Marangoni number is zero and a no-motion conductive state does not exist. It is shown that the controller is not effective at the wave number $a = 0$.

5 Conclusions

The effect of control on the onset of Marangoni-Bénard convection in a horizontal layer of fluid with internal heat generation heated from below and cooled from above is investigated. The explicit analytical expressions for the critical Marangoni-Bénard number in the presence of the effect of control have been obtained. We have shown numerically that the effect of the controller gain, K is always to stabilize the layer in the case of a nondeforming surface. However, the controller gain is not effective in the case of a deforming surface.

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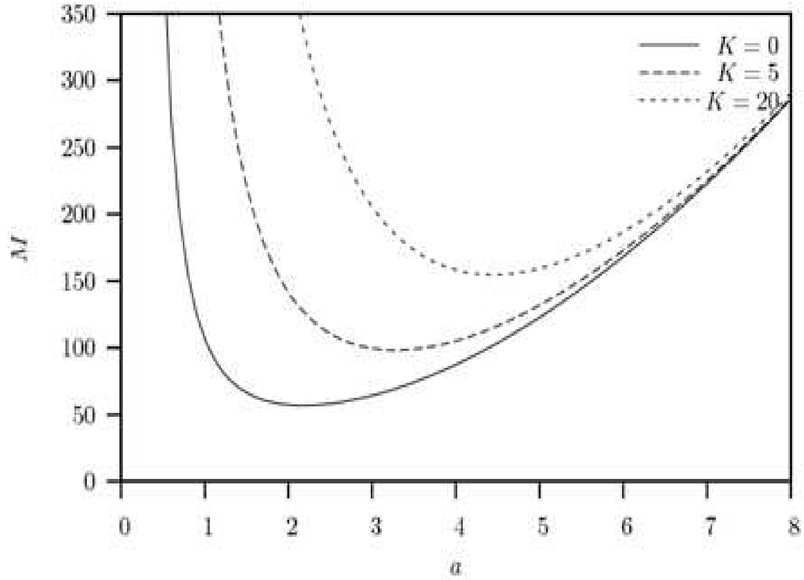

Figure 2: Marginal stability curves when $B_i = 0$, $B_o = 0$, and $C_r = 0$ for a range of values of controller gains K

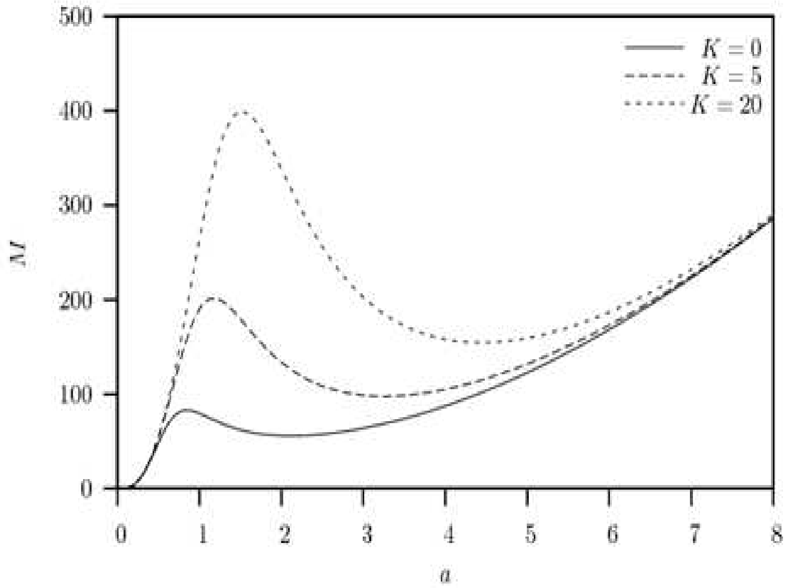

Figure 3: Marginal stability curves in the case $C_r = 0.001$ when $B_i = 0$ and $B_o = 0$, for a range of values of controller gains, K

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