MATEMATIKA, 2008, Volume 24, Number 1, 31–42

Convexity-Preserving Scattered Data Interpolation

[1]**Abd. Rahni Mt. Piah**, [2]**Azizan Saaban** & [3]**Ahmad Abd. Majid**
School of Mathematical Sciences, Universiti Sains Malaysia 11800 USM Pulau Pinang, Malaysia
e-mail: [1]arahni@cs.usm.my, [2]azizan.s@uum.edu.my, [3]majid@cs.usm.my

Abstract This study deals with constructing a convexity-preserving bivariate C^1 interpolants to scattered data whenever the original data are convex. Sufficient conditions on lower bound of Bézier points are derived in order to ensure that surfaces comprising cubic Bézier triangular patches are always convex and satisfy C^1 continuity conditions. Initial gradients at the data sites are estimated and then modified if necessary to ensure that these conditions are satisfied. The construction is local and easy to be implemented. Graphical examples are presented using several test functions.

Keywords Scattered data, interpolation, convexity, continuity.

1 Introduction

In Computer Aided Geometric Design, we often need to consider designing interpolating surfaces with convexity or concavity properties. Several authors ([4], [5], [2], [10]) have derived sufficient conditions to ensure the convexity of either triangular or tensor-product Bernstein-Bézier surfaces. Examples of the methods used to construct convexity-preserving interpolating surfaces are given in ([1], [6], [10], [12], [13], [15]). [1] derived the convexity-preserving conditions for piecewise quadratic C^1 interpolation based on Powell-Sabin splits but are not complete as shown by a counter example in [13]. The complete set of conditions on convexity-preserving interpolation by piecewise quadratic and cubic C^1 functions was then derived by [13] by combining local convexity conditions with an optimization scheme which takes into account the global nature of the problem. [10] used bivariate C^1 cubic splines to deal with convexity-preserving scattered data interpolation. A necessary and sufficient condition on Bézier-Bernstein polynomials were derived. The problem of convexity-preserving is set into quadratically constraints in a quadratic programming problem. Then the quadratic constraints were replaced by three linear constraints and the problem was formulated into linearly constraint quadratic programming. [12] described a C^1 convexity-preserving scattered data interpolation. The method consists of constructing a triangulation of the nodes for which the triangular-based piecewise linear interpolant is convex, computes a set of nodal gradients for which a convex Hermite interpolant exists, and constructs a smooth convex surface that interpolates the nodal values and gradients. It involves two data-dependent triangulations with a straight-line dual of each. [6] defined new triangular macro-elements based on Clough-Tocher cubic elements which allow the construction of a C^1 surface connecting patches originally of non-equal degree. The main tool in this construction relies on the degree elevation process associated with careful modification of the Bézier nets. In addition, [6] proposed a shape-preserving criterion for automatic selection of the tension parameters, which permits preserving monotonicity and/or convexity of the data in domains containing, and along directions parallel to the edges of the triangulation.

Motivated by sufficient condition for convexity of Bernstein-Bézier surfaces over triangles given in [2] and [10], this paper will propose sufficient conditions by imposing a lower bound on Bézier points in order to ensure that surfaces comprising cubic Bézier triangular patches are always convex if given data are convex and satisfy C^1 continuity conditions. Each triangular patch of the interpolating surface is formed as a convex combination of three cubic Bézier triangular patches. This method is local and the convexity-preserving interpolating surfaces are controlled by certain free parameters. In Section 2, we present some technical prerequisites concerning convex data, the Bernstein-Bézier representation of polynomials, the convexity conditions of triangular Bézier surfaces and the sufficient conditions for C^1 convexity-preserving interpolant. An outline of the surface construction process is given in Section 3, while Section 4 presents some examples. Finally, the conclusion will be given in Section 5.

2 Background

2.1 Convex Data

Scattered data, $(x_i, y_i, z_i), i = 1, 2, ..., N$, are called strictly convex if there exists a strictly convex function F such that $F(x_i,\ y_i)\ =\ z_i$ and

$$F(\mu p + (1-\mu)q) = \mu F(p) + (1-\mu)F(q), \quad \forall \mu \in\ [0,\, 1] \tag{1}$$

for every pair of points $p,\ q \in\ D$ where D is the convex hull of the nodes. Equivalently, the data set is convex if there is a triangulation of the nodes for which the triangular-based piecewise linear interpolant is convex (Such a triangulation is unique and the data are said to be strictly convex if no four data points are coplanar [12]). An algorithm to construct suitable triangulation T for convex data is given in [16].

2.2 Triangular Bézier Patch

A triangular Bernstein-Bézier surface of degree n over triangle $T =< V_1, V_2, V_3 >$ is defined as

$$P(x, y) = \sum_{\substack{i+j+k=n \\ i\geq 0,\, j\geq 0,\, k\geq 0}} b_{ijk} B^n_{ijk}(x, y) \tag{2}$$

where b_{ijk} are the Bézier ordinates or control points of P and

$$B^n_{ijk}(x, y) = \frac{n!}{i!j!k!} r^i s^j t^k \tag{3}$$

with $(x,\ y)\ =\ rV_1 +\ sV_2\ +\ tV_3$ and r, s, t , are barycentric coordinates (i.e. $r + s + t = 1$).

2.3 Convexity Conditions

The convexity of a polynomial can be conveniently characterized through its Bézier representation. There are many sufficient conditions for convexity-preserving triangle in the literature, such as in [1], [2], [4], [5], [10]. For our study, we use convexity-preserving conditions in [10]. For any direction given as vector $d = (d_1,\, d_2,\, -(d_1 + d_2))$, the second order directional derivative is given by

$$D_d^2 P(x,\, y) \;=\; n(n-1) \sum_{i+j+k=n-2} C^m_{i,j,k}(d) B^{n-2}_{ijk}(x,y) \tag{4}$$

where $B^n_{ijk}(x,\, y)\,)$ is defined in (3) and

$$C^m_{i,j,k}(d) \;=\; d_1(C^{m-1}_{i+1,j,k}(d) - C^{m-1}_{i,j,k+1}(d)\,) \;+\; d_2(C^{m-1}_{i,\,j+1,k}(d) \;-\; C^{m-1}_{i,j,k+1}(d)\,) \tag{5}$$

for $m \geq 1$, with $C^0_{i,j,k}(d) \;=\; b_{ijk},\;\; i+j+k = n.$
Using (4) and (5) on a triangular patch represented by (2), we obtain

$$\begin{aligned} D_d^2 P(x,\, y) \;&=\; n(n-1) \sum_{i+j+k=n-2} (d_1^2(b_{i+2,\,j,\,k} \;+\; b_{i,j,k+2} \;-\; 2b_{i+1,\,j,\,k+1}) \\ &\quad +2d_1 d_2(b_{i,\,j,\,k+2} \;+\; b_{i+1,\,j+1,\,k} \;-\; b_{i+1,\,j,\,k+1} \;-\; b_{i,\,j+1,\,k+1}) \\ &\quad +\, d_2^2(b_{i,\,j+2,\,k} + b_{i,j,k+2} - 2b_{i,j+1,k+1}))B^{n-2}_{ijk}(x,y) \\ &=\; n(n-1) \sum_{i+j+k=n-2} (d_1\; d_2) Q(i,j,k) \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} B^{n-2}_{ijk}(x,y) \end{aligned}$$

with $Q(i,j,k)$, a 2 x 2 matrix defined by

$$Q(i,j,k) \;=\; \begin{pmatrix} \Delta_1 b_{ijk} & \Delta_2 b_{ijk} \\ \Delta_2 b_{ijk} & \Delta_3 b_{ijk} \end{pmatrix} \tag{6}$$

where difference operators, $\Delta_l b_{ijk},\;\; l \;=\; 1,\, 2,\, 3$ is defined as

$$\Delta_1 b_{ijk} \;=\; b_{i+2,j,k} +\; b_{i,j,k+2} \;-\; 2b_{i+1,\,j,\,k+1}\,,$$

$$\Delta_2 b_{ijk} \;=\; b_{i,j,k+2} +\; b_{i+1,j+1,k} \;-\; b_{i+1,\,j,\,k+1} \;-\; b_{i,\,j+1,\,k+1}$$

and $\Delta_3 b_{ijk} \;=\; b_{i,j+2,k} +\; b_{i,j,k+2} \;-2\, b_{i,\,j+1,\,k+1}$ for $i+j+k \;=\; n-2,\; i,\; j,\; k \;\geq\; 0$. Note that $P(x,y)$ is convex on T if and only if $D_d^2 P(x,\, y) \;\geq 0$ for all $(x,\, y) \in T$ for all direction d. $P(x,y)$ is convex when $Q(i,j,k)$ in (6) is positive semi-definite for all $i,\; j,\; k \;\geq\; 0$ with $i+j+k = n-2$ which is equivalent to the conditions

$$\left.\begin{aligned} &\Delta_1 b_{ijk} \;\geq\; 0, \quad \Delta_3 b_{ijk} \;\geq\; 0 \\ &\text{and} \\ &(\Delta_1 b_{ijk}\,)(\Delta_3 b_{ijk}) \;\geq\; (\Delta_2 b_{ijk})^2 \end{aligned}\right\} . \tag{7}$$

2.4 C^1 Cubic Convexity-Preserving Interpolants

A cubic Bézier triangular patch P on T is defined as,

$$\begin{aligned}P(x,y) &= \sum_{\substack{i+j+k=3\\ i\geq 0,\, j\geq 0,\, k\geq 0}} b_{ijk} B^3_{ijk}(x,y)\\ &= b_{300}r^3 + b_{030}s^3 + b_{003}t^3 + 3b_{210}r^2s + 3b_{201}r^2t + 3b_{120}rs^2 + 3b_{102}rt^2\\ &\qquad +3b_{021}s^2t + 3b_{012}st^2 + 6b_{111}rst \end{aligned} \tag{8}$$

where b_{ijk} are the Bézier ordinates of P as shown in Figure 1.

Given the Bézier ordinates at vertices i.e. $b_{300}, b_{030}, b_{003}$, the sufficient conditions on the remaining Bézier ordinates will be derived to ensure the convexity of the entire patch. Let $A = b_{300}$, $B = b_{030}$, and $C = b_{003}$, $A,\, B,\, C \in \Re$. Note that in this paper we will only focus on positive and convex data i.e. for the case of $A > 0$, $B > 0$ and $C > 0$. Our approach is to find lower bounds on the remaining Bézier ordinates, so that second directional derivative, $D^2_d P(x,\, y) \geq 0$ for all $(x,\, y) \in T$ for any direction d. Let Bézier ordinates $b_{210}, b_{201}, b_{120}, b_{021}, b_{012}, b_{102}$ have the same value $-p$ and $b_{111} = -q$, with $p,\, q > 0$.

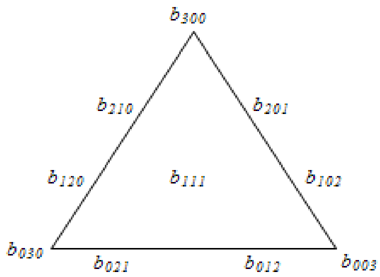

Figure 1. Relative locations of Bezier ordinates for $P(x,y)$

Using (7), we obtain the following set of conditions for convexity-preserving on each triangular patch,

$$-p + q \geq 0, \tag{9}$$

$$p \geq -A, \quad p \geq -B, \quad p \geq -C, \tag{10}$$

$$(-p+q)(2A - q + 3p) \geq 0, \tag{11}$$

$$(-p+q)(2B - q + 3p) \geq 0, \tag{12}$$

$$(-p+q)(2C - q + 3p) \geq 0. \tag{13}$$

Since $A > 0, B > 0, C > 0$, equations (9) – (13) clearly hold when $p = q = 0$.
From (10), we obtain

$$p \geq \max(-A, -B, -C). \tag{14}$$

Thus, the lower bound of p is given by

$$p_0 = \alpha \max(-A, -B, -C) \tag{15}$$

where α is a free parameter.We need $\alpha < 0$ in order for $p_0 > 0$. If (9) holds, then conditions (11), (12) and (13) will just reduce to $q \leq \min(3p + 2A, 3p + 2B, 3p + 2C)$.
It is then easy to show that $\min(3p + 2A, 3p + 2B, 3p + 2C) \geq p$. Thus

$$p \leq q \leq \min(3p + 2A, 3p + 2B, 3p + 2C). \tag{16}$$

From (15), when p_0 is a lower bound of p, then (16) becomes

$$p_0 \leq q_0 \leq \min(3p_0 + 2A, 3p_0 + 2B, 3p_0 + 2C). \tag{17}$$

The lower bound of q can then be written as

$$q_0 = \beta p_0 + (1 - \beta) \min(3p_0 + 2A, 3p_0 + 2B, 3p_0 + 2C) \tag{18}$$

where $0 \leq \beta \leq 1$ is a free parameter.

Proposition
Consider the cubic Bézier triangular patch $P(x, y)$ with $b_{300} = A, b_{030} = B, b_{003} = C$. If $b_{021}, b_{012}, b_{201}, b_{102}, b_{210}, b_{120} \geq -p_0$, and $b_{111} \geq -q_0$,where p_0, q_0 are given by (15) and (18) respectively then $P(x, y)$ is convex on a given convex domain $D \in \Re^2$.

3 Construction of Convexity-Preserving Interpolating Surface

We want to construct a C^1 convexity-preserving functional surface $F(x, y)$ which interpolates a given positive and convex scattered data, $(x_i, y_i, z_i), i = 1, 2, ..., N,\ z_i > 0$. The surface comprises cubic Bézier triangular patches, each of which is guaranteed to remain convex. We use Delaunay triangulation to triangulate the convex hull of the data points. Estimations of the first order partial derivatives of F with respect to x and y are obtained using the methods of quadratic approximation and least squares i.e.

$$\min \sum_i \left(ax_i^2 + by_i^2 + cx_i + dy_i + e - z_i \right)^2$$

where a, b, c, d and e are coefficients to be determined.

Then first order partial derivatives at vertex V_i can be estimated as $F_x = 2ax + c$ and $F_y = 2by + d$. Note that, in this paper we use a quadratic function without cross-product term because it is suitable for convex data. Let $V_i = (x_i, y_i), i = 1, 2, 3$ be vertices of a triangle, such that $F(V_i) = z_i$, and the first partial derivatives, $F_x(V_i)$ and $F_y(V_i)$. For each triangular patch P as in (8), the derivative along edge e_{jk} joining (x_j, y_j) to (x_k, y_k) is given by

$$\frac{\partial P}{\partial e_{jk}} = (x_k - x_j)\frac{\partial F}{\partial x} + (y_k - y_j)\frac{\partial F}{\partial y}.$$

The given data and estimated derivative values at (x_i, y_i), enable us to determine all the Bézier ordinates b_{rst} except for b_{111}. For example, we will have:

$$b_{300} = F(V_1), b_{210} = F(V_1) + \frac{1}{3}\frac{\partial F}{\partial e_{12}}(V_1) \text{ and } b_{201} = F(V_1) - \frac{1}{3}\frac{\partial F}{\partial e_{31}}(V_1).$$

However, the initial estimate for each edge ordinate may not satisfy the convexity conditions for P. In view of the Proposition, we need these Bézier ordinates to be greater or equal to $-p_0$. If they are not, then the magnitudes of F_x, F_y at the vertices need to be reduced so that the convexity conditions are satisfied. The modification of these partial derivatives at a vertex V_i, is achieved by multiplying each derivative at that vertex, by a scaling factor $0 < \gamma < 1$. The smallest value of γ is obtained by considering all triangles that meet at vertex V_i, which satisfy the convexity conditions of all the triangles. For example,

$$(b_{210})_j = F(V_1) + \gamma\frac{\partial F/\partial e_{12}}{3} \geq (y_3)_j$$

and

$$(b_{201})_j = F(V_1) - \gamma\frac{\partial F/\partial e_{31}}{3} \geq (y_2)_j,$$

where subscript j represents a quantity which corresponds to triangle j. Having adjusted these derivatives, if necessary, the Bézier ordinates are recalculated using the formulae above. The above process is repeated at all the nodes, Vi (see [3] for further details). For each triangle, the inner Bézier ordinate b_{111}, remains to be calculated, in such a way to guarantee the preservation of convexity and to ensure C^1 continuity across patch boundaries. Let T_1 and T_2 be two adjacent cubic Bézier triangular patches with a common boundary curve (see Figure 2). C^1 continuity conditions between adjacent patches are given by

$$c_{102} = \mu_1 b_{120} + \mu_2 b_{030} + \mu_3 b_{021} \tag{19}$$

$$c_{111} = \mu_1 b_{111} + \mu_2 b_{021} + \mu_3 b_{012} \tag{20}$$

$$c_{120} = \mu_1 b_{102} + \mu_2 b_{012} + \mu_3 b_{003} \tag{21}$$

where (μ_1, μ_2, μ_3) is the barycentric coordinates of W_3 with respect to triangle T_1. (19) and (21) are automatically satisfied since the Bézier ordinates b_{201}, $b_{210}, b_{012}, b_{102}, b_{120}$, and b_{021} have already been obtained.

We shall use similar method as in [9] to determine initial value of inner Bézier ordinate, b^i_{111} which is C^1 across boundary $e_i, i = 1, 2, 3$ where e_i is the edge of the triangle opposite vertex V_i. Local scheme P_i is defined by replacing b_{111} in (8) with b^i_{111}. The values of inner Bézier ordinates b^1_{111}, b^2_{111} and b^3_{111} for local schemes P_1, P_2, and P_3 are given by

$$b^1_{111} = \frac{1}{2}\left(b_{120} + b_{102} + \frac{1}{2}(b_{012} + b_{021} - b_{003} - b_{030})\right),$$

$$b^2_{111} = \frac{1}{2}\left(b_{210} + b_{012} + \frac{1}{2}(b_{102} + b_{201} - b_{003} - b_{300})\right)$$

and

$$b^3_{111} = \frac{1}{2}\left(b_{021} + b_{201} + \frac{1}{2}(b_{120} + b_{210} - b_{300} - b_{030})\right)$$

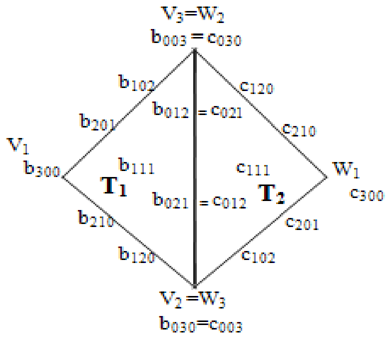

Figure. 2. Adjacent cubic triangular patches

respectively. Initial estimate of the inner Bézier ordinates as above in each triangle may not satisfy the convexity conditions of $P(x, y)$ as stated in the Proposition i.e. $b_{111} \geq -q_0$. For example, when e_1 is a common edge to two triangles as in Figure 2, and the initial value b^1_{111} or c^1_{111} is less than $-q_0$, then these ordinates are both modified to ensure that b^1_{111}, $c^1_{111} \geq -q_0$, and C^1 continuity condition along the common boundary curve of adjacent patches is satisfied. When e_1 is on the boundary of the domain, and b^1_{111} is less than $-q_0$ then b^1_{111} is reset to be equal to q_0. The ordinates b^2_{111} and b^3_{111} of P_2 and P_3 are adjusted accordingly using a similar approach (see [3] for further details). The interpolating surface P on triangle T is then defined as a convex combination of all the local schemes such that sufficient conditions on all sides of the triangles are satisfied, i.e.

$$P(u, v, w) = c_1 P_1(u, v, w) + c_2 P_2(u, v, w) + c_3 P_3(u, v, w)$$

or

$$P(u,\, v,\, w) \;=\; \sum_{\substack{i+j+k=3 \\ i\neq 1, j\neq 1, k\neq 1}} b_{ijk} B^3_{ijk}(u, v, w) \;+\; 6uvw(c_1 b^1_{111} \;+\; c_2 b^2_{111} \;+\; c_3 b^3_{111})$$

with $c_1 = \dfrac{vw}{vw + vu + uw}$, $c_2 = \dfrac{uw}{vw + vu + uw}$ and $c_3 = \dfrac{vu}{vw + vu + uw}$, and u, v, w are the barycentric coordinates.

4 Examples

We will illustrate our interpolating scheme using the following convex test functions:

$$f(x, y) \;=\; x^4 \;+\; y^4 \text{ and } g(x,\, y) \;=\; x^3 + 5(y - 0.6)^2 \;+\; 1.$$

The first data set consists of eight points of convex triangulation defined on rectangular grid $[-1, 1] \times [-1, 1]$ are taken from [15] with data values taken from function f. Figure 3 shows a triangular domain of the data set. The corresponding linear interpolant is in Figure 4 and the convexity-preserving interpolating surface of our method is given in Figure 5. The second data set of nine data points are defined on rectangular grid $[0, 1] \times [0, 1]$ ([13]) with data values taken from function g. Figure 6 shows the triangular domain, Figure 7 is the corresponding linear interpolant while Figure 8 shows the interpolating surface using our convexity-preserving scheme.

The final data set consists of 26 scattered points from [11] with data values taken from function f. Figures 9 to 11 show a triangular domain, linear interpolant and convexity-preserving interpolating surface respectively. Note that for the first and second data set, we choose $\alpha = -0.028$ and $\beta = 0.2$, while for the third data set we take $\alpha = -0.3$ and $\beta = 0.1$ as parameters to ensure convexity-preserved of the given functions.

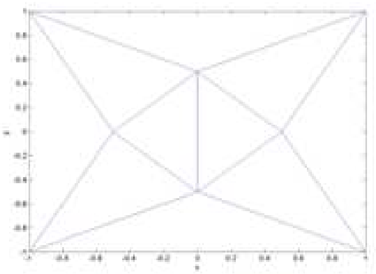

Figure 3. Triangular domain of data set 1

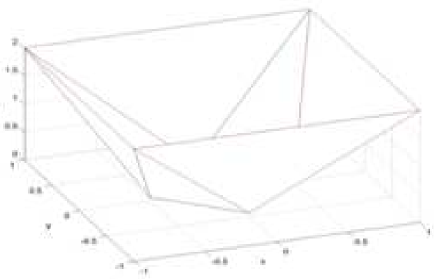

Figure 4. Linear convexity-preserving interpolation

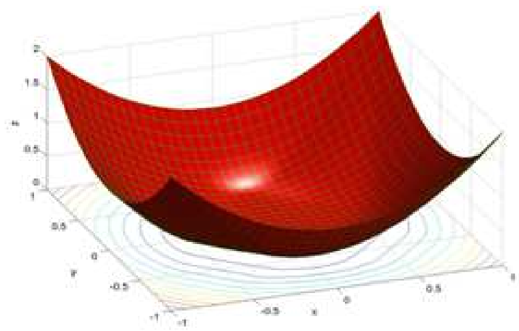

Figure 5. C^1 convexity-preserving interpolation of our scheme

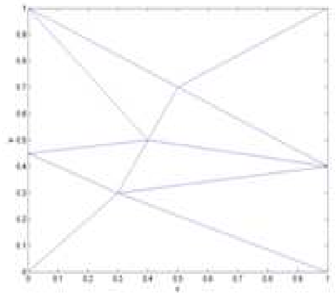

Figure 6. Triangular domain of data set 2

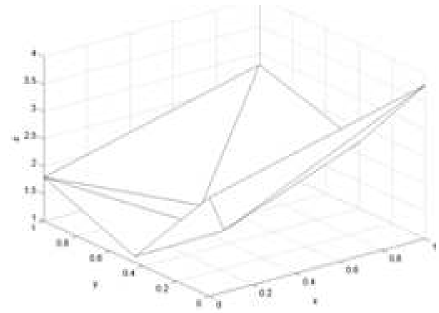

Figure 7. Linear convexity-preserving interpolation

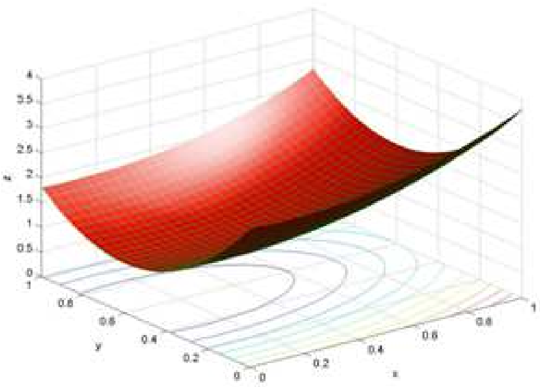

Figure 8. C^1 convexity-preserving interpolation of our scheme

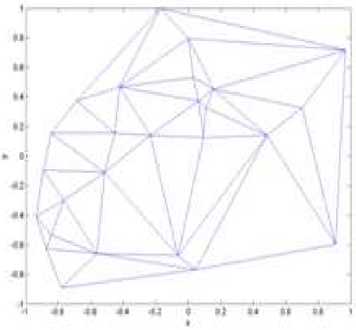

Figure 9. Triangular domain of data set 3

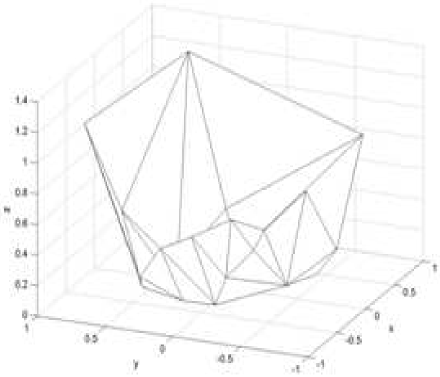

Figure 10. Linear convexity-preserving interpolation

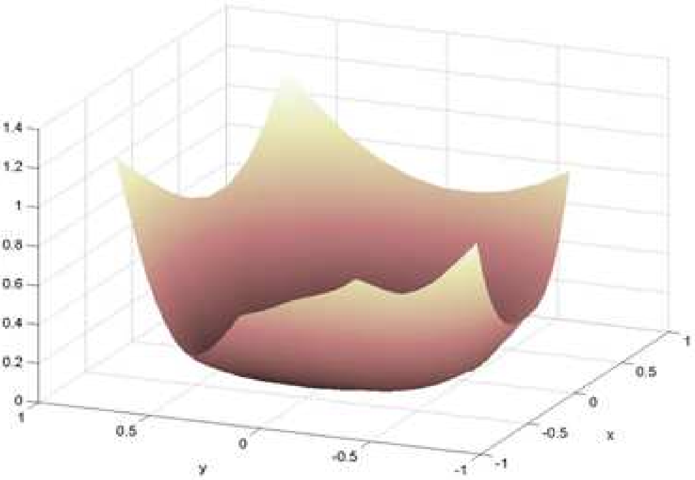

Figure 11. C^1 convexity-preserving interpolation of our scheme

5 Conclusion

We have considered generating non-parametric surfaces which interpolate positive and convex scattered data using convex combination of cubic triangular patches. We have imposed lower bounds on Bézier points in order to ensure that surfaces comprising cubic Bézier triangular patches satisfy C^1 continuity conditions and are always convex when the given data are convex.

Acknowledgements The authors are very grateful to the Ministry of Higher Education and Universiti Sains Malaysia for providing us with the fundamental research grant scheme (FRGS) account number 203/PMATHS/671040 to enable us to pursue this research.

References

[1] J.M. Carnicer & W. Dahmen, *Convexity preserving interpolation and Powell-Sabin elements*, Computer Aided Geometric Design 9 (1992), 279-289.

[2] J.M. Carnicer, M.S. Floater & J.M. Pena, *Linear convexity conditions for rectangular and triangular Bernstein-Bézier surfaces*, Computer Aided Geometric Design 15 (1997), 27-38.

[3] E.S. Chan & B.H. Ong, *Range restricted scattered data interpolation using convex combination of cubic Bézier triangles*, J. Comp. Appl. Math. 136 (2001), 135-147.

[4] G. Chang & P.J. Davis, *The convexity of Bernstein polynomials over triangles*, J. Approx. Theory 40 (1984), 11-28.

[5] G. Chang & Y.Y. Feng, *An improved condition for the convexity of Bernstein-Bézier surfaces over triangles*, Computer Aided Geometric Design 1(1984), 279-283.

[6] P. Costantini & C. Manni , *A local shape-preserving interpolation scheme for scattered data*, Computer Aided Geometric Design 16 (1999), 385-405

[7] T.P. Fang & L.A. Piegl, *Algorithm for Delaunay triangulation and convex-hull computation using a sparse matrix.* Computer Aided Design 24 (1992), 425-436.

[8] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design-A Practical Guide 4th Edition*, Academic Press, San Diego (1996).

[9] T.N.T. Goodman & H.B. Said, *C^1 triangular interpolant suitable for scattered data interpolation*, Communications in Applied Numerical Methods 7 (1991), 479-485.

[10] M.J. Lai, *Convex preserving scattered data interpolation using bivariate C^1 cubic splines*, J. Comput. Applied Math. 119 (2000), 249-258.

[11] C.L. Lawson, *Software for C^1 surface interpolation*, in Mathematical Software III, J. R. Rice (ed)., Academic Press, New York (1977), 161-194.

[12] N.K. Leung & R.J. Renka, *C^1 convexity-preserving interpolation of scattered data*, SIAM J. Sci. Comput. 20(5) (1999), 1732-1752.

[13] A. Li, *Convexity preserving interpolation*, Computer Aided Geometric Design 16 (1999), 127-147.

[14] A.R.M. Piah, T.N.T. Goodman & K. Unsworth, *Positivity-preserving scattered data interpolation*. In Mathematics of Surfaces 2005, LNCS 3604, R. Martin et. al. (eds), Springer-Verlag Berlin Heidelberg (2005) , 336-349.

[15] R.J. Renka, *Algorithm 833: CSRFPACK-Interpolation of Scattered Data with a C^1 Convexity-Preserving Surface*, ACM Transactions on Mathematical Software 30(2) (2004), 200-211.

[16] D.S. Scott, *The complexity of interpolating given data in three space with a convex function of two variables*, J. Approx. Theory 42 (1984), 52-63.