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2-Exponents of Two-Coloured Lollipops

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Abstract This paper shows that for an asymmetric primitive two-coloured (n, s)-lollipop on n vertices with $s \leq n$, its 2-exponent is at most $(s^2 - 1)/2 + (s + 1)(n - s)$. The (n, s)-lollipops whose 2-exponents achieving the bound is characterised and for any asymmetric primitive two-coloured (n, s)-lollipop, a simple algorithm to find its exponent is presented.

Keywords Two-coloured digraphs; primitive; 2-exponents; (n, s)-lollipops.

1 Digraphs and Two-Coloured Digraphs

In this paper we discuss 2-exponents of a special type of primitive asymmetric two-coloured digraphs. We follow notations and terminologies for digraphs in [3] and for two-coloured digraphs in [10]. In particular a *directed walk* of length m from a vertex u to a vertex v is a sequence of m arcs of the form

$$(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$$

where $v_0 = u$ and $v_m = v$. We denote a directed walk w from u to v by (u, v)-walk or w_{uv} and its length is denoted by $\ell(w_{uv})$. A (u, v)-walk is *closed* provided $u = v$ and is *open* otherwise. A *directed* path from u to v is a directed walk with no repeated vertices except possibly $u = v$. A *directed cycle* is a closed directed path. A *loop* is a closed directed cycle of length 1. The *distance* of distinct vertices u and v in a digraph $D, d(u, v)$, which length is represent the shortest (u, v)-path in D.

A digraph D is *strongly connected* provided for each pair of vertices u and v in D there is a (u, v)-walk and a (v, u)-walk of length exactly k. The smallest of such positive integer k is the *exponent* of D and is denoted by $\exp(D)$. A strongly connected digraph D is primitive if and only if the greatest common divisor of lengths of all directed cycles in D is 1 [3]. A *symmetric* digraph D is a digraph such that the arc (u, v) is in D whenever the arc (v, u) is in D. Since a symmetric digraph must have a directed cycle of length 2, a symmetric digraph is primitive if and only if it contains a directed cycle of odd length. A *directed (n, s)-lollipop* on n vertices is a symmetric strongly connected digraph consisting of directed cycles $(1, 2), (2, 3), \ldots, (s - 1, s), (s, 1)$ and $(1, s), (s, s - 1), \ldots, (3, 2), (2, 1)$ of length s and directed paths $(s, s+1), (s+1, s+2), \ldots, (n-1, n)$ and $(n, n-1), (n-1, n-2), \ldots, (s+1, s)$ of length $(n - s)$. Figure 1 shows a (9,5)-lollipop.

Research on exponent of digraph is initiated by Wielandt [9, 15]. Wielandt shows that for primitive digraphs on n vertices the $\exp(D) \leq (n - 1)^2 + 1$ and then characterized

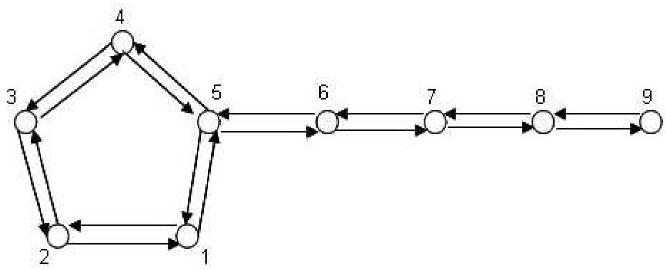

Figure 1: A (9,5)-lollipop

the digraph whose exponent attains the bound. Since then exponents of various classes of primitive digraphs have been obtained (see [3]). In particular, Dulmage and Mendelsohn [4] gave more general bound on exponents of primitive digraphs in term of the length of the smallest cycle. They have shown that if D is a primitive digraph on n vertices with the smallest cycle of length s, then $\exp(D) \leq n + s(n-2)$. Beasley and Kirkland [1] discussed a bound on exponent of primitive digraphs in terms of the exponent of its primitive subdigraphs. For primitive symmetric digraphs on n vertices, Shao [11] proved that the $\exp(D) \leq 2n-2$ and showed that the exponent $2n-2$ is attained if and only if the digraph D is isomorphic to a symetric digraph that consists of the directed paths $(1,2),(2,3),\ldots,(n-1,n)$ and $(n,n-1),(n-1,n-2),\ldots,(2,1)$ of length $(n-1)$ and the loop (1,1). Let D be a loopless primitive symmetric digraph on n vertices. Liu et.al [8] show that the $\exp(D) \leq 2n-4$ and prove that the upper bound is achieved if an only if D is isomorphic to a directed $(n,3)$-lollipop. Dulmage and Mendelsohn [4], Suwilo and Mardiningsih [13] have shown that exponents of symmetric primitive digraphs are bounded above by $(s-1)+2\ell$ where s is the length of smallest directed cycle in the digraph, and ℓ is the length of the longest path connecting a vertex in the smallest directed cycle and a vertex in D but not in the smallest directed cycle. In particular they have shown that the exponent of a symmetric directed (n,s)-lollipop is $2n-s-1$ which then implies that the exponent of a symmetric directed cycle of odd length n is $(n-1)$.

A *two-coloured digraph,* or a *2-digraph* for short, is a digraph in which each of its arcs is coloured by either red or blue. In a two-coloured digraph we differentiate a walk by how many red and blue arcs it contains. By an $(h,k)^T$-walk from u to v we mean a (u,v)-walk of length $h+k$ consisting of h red arcs and k blue arcs. So a $(1,2)^T$-walk could be a walk of length 3 whose all arcs are blue except the first arc, or all arcs are blue except the second arc, or all arcs are blue except the third arc. In general by a $(r(w),b(w))^T$-walk w we mean a walk w consisting of $r(w)$ red arcs and $b(w)$ blue arcs. The vector $(r(w),b(w))^T$ is called the composition of the walk w. A two-coloured digraph D is *strongly connected* provided that its underlying digraph, the digraph obtained from D by ignoring its arc color, is strongly connected. An *asymmetric two-coloured digraph* is a symmetric two-coloured digraph for which an arc (u,v) is coloured red whenever the arc (v,u) is coloured blue and vice versa. A strongly connected two-coloured digraphs is *primitive* provided there are nonnegative integers h and k such that for each pair of vertices u and v there is an $(h,k)^T$-walk from u to v. The smallest nonnegative integer $h+k$ among all such nonnegative integers h and k is called the 2-*exponent* of D, denoted by $\exp_2(D)$.

Let D be a two-coloured digraph and let $\gamma = \{\gamma_1,\gamma_2,\ldots,\gamma_t\}$ be the set of all directed cycles in D. A *cycle matrix* M of D is a 2 by t matrix whose ith column is the composition of the cycle $\gamma_i, i = 1, 2, \ldots, t$. That is

$$M = \begin{bmatrix} r(\gamma_1) & r(\gamma_2) & \cdots & r(\gamma_t) \\ b(\gamma_1) & b(\gamma_2) & \cdots & b(\gamma_t) \end{bmatrix}.$$

The content of M is defined to be 0 if the rank$(M) = 1$ and is the greatest common divisor of the 2 by 2 minors of M, otherwise. The following result, due to Fornasini and Valcher [5], gives necessary and sufficient conditions for primitive two-coloured digraph.

Theorem 1.1. *Let D be a strongly connected two-coloured digraph with at least one arc of each color. The two-coloured digraph D is primitive if and only if the content of its cycle matrix is 1.*

Let D be a strongly connected two-coloured digraph on n vertices. Let $R = (r_{ij})$ be the n by n (0,1)-matrix with $r_{ij} = 1$ if and only if the arc (i, j) in D is a red arc. Similarly, let $B = (b_{ij})$ be the n by n (0,1)-matrix with $b_{ij} = 1$ if and only if the arc (i, j) in D is a blue arc. The matrices R and B are the red and the blue adjacency matrices of D, respectively. Conversely for any pair (A, B) of nonnegative matrices we can find a two-coloured digraph D on n vertices associated with (A, B) as follows. The arc (i, j) in D is a red arc if and only if the entry $a_{ij} > 0$ and the arc (i, j) in D is a blue arc if and only if the entry $b_{ij} > 0$.

For any pair of nonnegative matrices (A, B) and nonnegative integers h and k define the (h, k)-Hurwitz product, $(A, B)^{(h,k)}$, of A and B to be the sum of all matrices that are a product of h A's and k B's. For example,

$$(A, B)^{(2,0)} = A^2, (A, B)^{(0,3)} = B^3$$

and

$$\begin{aligned}(A, B)^{(3,2)} = {} & A^3B^2 + A^2BAB + A^2B^2A + ABA^2B + ABABA \\ & + AB^2A^2 + BA^3B + BA^2BA + BABA^2 + B^2A^3.\end{aligned}$$

An (h, k)-Hurwitz product of matrices A and B can be computed using recurrence relation as follows:

For any nonnegative integers $h \geq 1$ and $k \geq 1$, $(A, B)^{(h,0)} = A^h$, $(A, B)^{(0,k)} = B^k$, and

$$(A, B)^{(h,k)} = A(A, B)^{(h-1,k)} + B(A, B)^{(h,k-1)}.$$

Using induction on $h + k$ one can show the following result:

Lemma 1.2. *Let D be a two-coloured digraph on n vertices and let R and B, respectively, be the red and the blue adjacency matrices of D. Then the (i, j)-entry of $(R, B)^{(h,k)}$ is the number of (h,k)T-walk from i to j.*

We note that by Lemma 1.2, the 2-exponent of a primitive two-coloured digraph can be obtained by finding the smallest positive integer $h + k$ such that all entries of the matrix $(R, B)^{(h,k)}$ are positive for some nonnegative integers h and k.

Research on 2-exponents of two-coloured digraphs is initiated by Shader and Suwilo [10]. They prove that the largest 2-exponent of primitive two-coloured digraphs on n vertices lies on the interval $[(n^3-5n^2)/3, (3n^3+2n^2-2n)/2]$. Since then many papers have been published on the subject (see [2, 6, 7, 14]). Suwilo [12] shows that for an asymmetric complete two-coloured digraph D on n vertices, the 2-exponent of D lies on interval [2,4] and proves that for each integer $k, 2 \leq k \leq 4$, there is an asymmetric complete two-coloured digraph whose 2-exponent is exactly k.

The purpose of this paper is to discuss bound on 2-exponent of asymmetric two-coloured (n, s)-lollipops. For such two-coloured digraphs, in Section 2, we show the 2-exponent is bounded above by $(s^2 - 1)/2 + (s + 1)(n - s)$ and the bound can be obtained by a $(t, t)^T$-walk where $t = (s^2 - 1)/4 + (s + 1)(n - s)/2$. Furthermore, in Section 3, we characterize asymmetric two-coloured digraphs whose 2-exponents attain the bound, and finally we give a formula for finding 2-exponent of any asymmetric two-coloured (n, s)-lollipop in Section 4.

2 Bound for 2-Exponents of Two-coloured Lollipops

Let D be an asymmetric primitive two-coloured (n, s)-lollipop. Since D is primitive, then s must be odd. Furthermore, since D is asymmetric, then D has directed cycles of length 2 with composition $(1, 1)^T$, and has two directed cycles γ_1 and γ_2 of length s, say the cycles

$$(1, 2), (2, 3), \ldots, (s - 1, s), (s, 1) \text{ and } (1, s), (s, s - 1), \ldots, (3, 2), (2, 1).$$

Hence, the compositions of the directed cycles of D are of the form $(1, 1)^T, (a, s - a)^T$, or $(s - a, a)^T$ for some nonnegative integer $a \geq 0$. This implies the cycle matrix of D is of the form

$$M = \begin{bmatrix} a & s-a & 1 & 1 & \cdots & 1 \\ s-a & s & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Since D is primitive, by Theorem 1.1 the content of M is 1. Hence

$$1 = \gcd(s(2a - s), s - 2a, 2a - s) = \pm(s - 2a).$$

This implies either $a = (s + 1)/2$ or $a = (s - 1)/2$, so without loss of generality we may assume that

$$M = \begin{bmatrix} (s-1)/2 & (s+1)/2 & 1 & 1 & \cdots & 1 \\ (s+1)/2 & (s-1)/2 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We note that since D is asymmetric, every vertex of D lies on a $(1, 1)^T$-walk. This implies each $(h, k)^T$-walk in D can be extended to a $(h + t, k + t)^T$-walk for each positive integer $t \geq 1$.

Theorem 2.1. *Let D be an asymmetric primitive two-coloured (n, s)-lollipop. Then*

$$\exp_2(D) \leq (s2 - 1)/2 + (s + 1)(n - s).$$

Proof. For each pair of vertices u and v we show that there is a $(t, t)^T$-walk with

$$t = (s^2 - 1)/4 + (s + 1)(n - s)/2.$$

Since D is asymmetric it suffices to show that for each pair of vertices u and v in D there is an $(e, e)^T$-walk with $e \leq t$. Since D is asymmetric, for each vertices u in D there is a closed $(1, 1)^T$-walk from u to itself. Let u and v be two distinct vertices in D and let p_{uv}

be the shortest path from u to v. Then $\ell(p_{uv}) \leq (s-1)/2 + (n-s)$. If $r(p_{uv}) = b(puv)$, then $r(p_{uv}), b(p_{uv}) \leq (s^2-1)/4 + (s+1)(n-s)/2$ and we are done. So we assume that $r(p_{uv}) \neq b(p_{uv})$ and without loss of generality we assume that $r(p_{uv}) > b(p_{uv})$.

We first assume that the path p_{uv} does not intersect the cycle γ_1 (or γ_2). In this case, the walk that starts at u, moves to s along the path p_{us}, moves $(r(p_{uv}) - b(p_{uv}))$ times around the cycle γ_1 and back to s, and finally moves to v along the path p_{sv} is the shortest $(e,e)^T$-walk from u to v. The composition of this walk is

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell_{uv} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}$$

where $\ell_{uv} = d(u,s)$ if $d(u,s) < d(v,s)$ and $\ell_{uv} = d(v,s)$ otherwise. We note that

$$r(p_{uv}) + b(p_{uv}) + \ell_{uv} \leq n - s.$$

Hence

$$\begin{aligned} \begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} &\leq \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell_{uv} \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} \\ &= (r(p_{uv}) - b(p_{uv}) + \ell_{uv}) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} \leq (n-s) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} \\ &\leq \begin{bmatrix} (s^2-1)/4 + (n-1)(s+1)/2 \\ (s^2-1)/4 + (n-1)(s+1)/2 \end{bmatrix}. \end{aligned}$$

We now assume that the path p_{uv} has vertices in common with the cycle γ_1 (or cycle γ_2). The walk that starts at u, follows the path p_{uv} to v, and long the way moves $(r(p_{uv}) - b(p_{uv}))$ times around the cycle γ_1 is an $(e,e)^T$-walk with composition

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} = (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix}.$$

We note in this cases that since the path p_{uv} is the shortest path form u to v,

$$(r(puv) + b(puv))((n-s) + \tfrac{1}{2}(s-1).$$

This implies $(r(p_{uv}) - b(p_{uv})) \leq r(p_{uv})((n-s) + (s-1)/2$. Therefore, we now have

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} \leq \begin{bmatrix} (s^2-1)/4 + (n-1)(s+1)/2 \\ (s^2-1)/4 + (n-1)(s+1)/2 \end{bmatrix}.$$

Now using $(1,1)^T$-walks, we can extend the walk w_{uv} into a $(t,t)^T$-walk with

$$t = (s^2-1)/4 + (s+1)(n-s)/2.$$

Since for every of vertices u and v one can find a $(t,t)^T$-walk from u to v with

$$t = (s^2-1)/4 + (s+1)(n-s)/2,$$

then

$$\exp_2(D) \leq (s^2-1)/2 + (n-s)(s+1).$$

In the proof of Theorem 2.1, we assume that $r(p_{uv}) > b(p_{uv})$. If $b(p_{uv}) > r(p_{uv})$, then the composition of the $(e,e)^T$-walk w_{uv} can be chosen as follows.

- If p_{uv} has vertices in common with the cycle γ_1 (or γ_2), then

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + (b(p_{uv}) - r(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}.$$

- If p_{uv} has no vertices in common with the cycle γ_1 (or γ_2), then

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell_{uv} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b(p_{uv}) - r(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}.$$

On both cases we move $(b(p_{uv}) - r(p_{uv}))$ times around the directed cycle γ_2. For the rest of the paper for a (u,v)-path puv we assume that $r(p_{uv}) > b(p_{uv})$.

Corollary 2.2. *Let D be an asymmetric primitive two-coloured cycle on n vertices. Then* $\exp_2(D) \leq (n^2-1)/2$.

Proof. Notice that a two-coloured cycle can be thought of as a two-coloured (n,s)-lollipop with $s = n$.

We end this section by setting up an upper bound for 2-exponent of asymmetric two-coloured (n,s)-lollipop in term of n, the number of vertices in D.

Corollary 2.3. *Let D be an asymmetric primitive two-coloured (n, s)-lollipop. Then*

$$\exp_2(D) \leq \begin{cases} (n^2-1)/2, & \text{if } n \text{ is odd} \\ n^2/2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $f(s) = (s^2-1)/2 + (s+1)(n-s)$. Notice that f achieves its global optima at $s = n-1$. Since $f(s)$ is quadratic and s is odd, $f(s)$ has global optima at $s = n-1$ when n is even, and has global optimal at $s = n$ or at $s = n-2$ when n is odd. This implies

$$\exp_2(D) \leq \begin{cases} (n^2-1)/2, & \text{if } n \text{ is odd} \\ n^2/2, & \text{if } n \text{ is even.} \end{cases}$$

3 Two-coloured (n, s)-lollipops Achieving the Bound

In this section, we discuss classes of asymmetric primitive two-coloured (n,s)-lollipops whose 2-exponents are exactly $(s^2-1)/2 + (s+1)(n-s)$.

Theorem 3.1. *Let D be a primitive asymmetric two-coloured (n,s)-lollipop. If D has a red path of length $(s+1)/2 + (n-s)$, then $exp_2(D) = (s^2-1)/2 + (s+1)(n-s)$.*

Proof. Let ${p'}_{uv}$ be a red path of length $(s+1)/2 + (n-s)$ in D. Since there are only $(n-s)$ vertices not in γ_1 (or γ_2), one of the vertices u or v must lie on the cycle γ_1 (or γ_2). Since D is asymmetric D has a blue path of length $(s+1)/2 + (n-s)$. Since the length of the cycle γ_1 (or γ_2) is odd, there is a red shortest path p_{uv} from u to v in D of length $q = (s-1)/2 + (n-s)$. Again since D is asymmetric, this implies there is a blue shortest path p_{vu} from v to u of length $q = (s-1)/2 + (n-s)$. Let w_{uv} and w_{vu}, respectively, be the walk from u to v and the walk from v to u that have the same composition. Hence the composition of w_{uv} and w_{vu}, respectively, are of the form

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + x_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{3.1}$$

for some nonnegative integers $x_1, x2, x3$, and $\varepsilon = 0, 1$, and

$$\begin{bmatrix} r(w_{vu}) \\ b(w_{vu}) \end{bmatrix} = \begin{bmatrix} 0 \\ q \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + y_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{3.2}$$

for some nonnegative integers y_1, y_2, y_3, and $\delta = 0, 1$. We note that if $\varepsilon = 0$, then we use the red path of length $q = (s+1)/2 + (n-s)$ in constructing the walk w_{uv}. If $\varepsilon = 1$, we use the red path of length $(s+1)/2 + (n-s)$ in constructing the walk w_{uv}. Similar argument applies for δ.

If $\varepsilon = \delta = 0$, setting Equation (3.1) and Equation (3.2) equals we have

$$\begin{bmatrix} -q \\ q \end{bmatrix} = (x_1 - y_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (x_2 - y_2) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + (x_3 - y_3) \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}. \tag{3.3}$$

Subtracting the second by the first component of Equation 3.3 we have $(x_2 - y_2) + (y_3 - x_3) \geq 2q$. This implies $x_2 + y_3 > 2q$ and hence $x_2 \geq q$ or $y_3 \geq q$.

It is not hard to see the following results. If $\varepsilon = 1$ and $\delta = 0$ or if $\varepsilon = 0$ and $\delta = 1$, then $x_2 \geq q$ or $y_3 \geq q$. Finally if $\varepsilon = \delta = 1$, then $x_2 \geq q+1$ and $y_3 \geq q+1$.

Therefore, in each case we have

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} \geq \begin{bmatrix} (s^2-1)/4 + (s+1)(n-s)/2 \\ (s^2-1)/4 + (s+1)(n-s)/2 \end{bmatrix}.$$

Hence we now have that $\exp_2(D) \geq (s^2-1)/2 + (s+1)(n-s)$. This and Theorem 2.1 imply that the 2-exponent of D is $\exp_2(D) = (s^2-1)/2 + (s+1)(n-s)$.

Example 3.2. Let D be a primitive asymmetric two-coloured (n, s)-lollipop with the coloring as follows. Color the path pns from vertex n to vertex s by red and color the arcs $(s, 1), (1, 2), \ldots, ((s-1)/2, (s+1)/2)$ by red. Notice that D has a red path of length $(s+1)/2 + (n-s)$, namely the directed path

$$(n, n-1), (n-1, n-2), \ldots, (s+1, s), (s, 1), (1, 2), \ldots, ((s-1)/2, (s+1)/2)$$

from n to $(s+1)/2$. This implies the directed path

$$(n, n-1), (n-1, n-2), \ldots, (s+1, s), (s, s-1), (s-1, s-2), \ldots, ((s+3)/2, (s+1)/2)$$

is a red path from n to $(s+1)/2$ of length exactly $(s-1)/2 + (n-s)$. Theorem 3.1 implies that $\exp_2(D) = (s^2-1)/2 + (s+1)(n-s)$.

As a consequence of Theorem 3.1 and Corollary 2.3 we have (n,s)-lollipop with the largest 2-exponent as follows.

Corollary 3.3. *Let D be a primitive asymmetric two-coloured (n, s)-lollipop with a red path of length $(s+1)/2 + (n-s)$.*

(a) If n is odd and $s = n$ or $s = n-2$, then $\exp(D) = (n^2-1)/2$.

(b) If n is even and $s = n-1$, then $\exp_2(D) = n^2/2$.

4 Formula for 2-exponents of Two-coloured (n, s)-lollipops

We have discussed that Lemma 1.2 together with the definition of (h, k)-Hurwitz product can be used to find the 2-exponent of an asymmetric primitive two-coloured digraph. In this section, we discuss a way of finding the 2-exponent of asymmetric primitive two-coloured (n, s)-lollipop without using the (h, k)-Hurwitz product. First, we introduce some notations necessary for our result. Let u and v be any vertices in D and let p_{uv} be a (u, v)-path. Notice that for each pair of vertices u and v in D, except if both u and v lie on the path p_{ns}, there are two paths from u to v. One passes through the cycle γ_1 and the other passes through the cycleγ_2. Let u and v be vertices in D such that u lies on cycle and v does not lie on cycle. Then a path p_{uv} from u to v consists of a path p_{us} from u to s and the path p_{sv} from s to v. Assume that the composition of the path p_{us} that passes through the cycle γ_1 is $(r_1, b_1)^T$ and the composition of the path p_{sv} is $(r, b)^T$. The composition of the path p_{uv} that passes through the cycle γ_1 and the composition of the path p_{vu} that passes through the cycle γ_2, respectively, are

$$\begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix}_{\gamma_1} = \begin{bmatrix} r_1 + r \\ b_1 + b \end{bmatrix} \text{ and } \begin{bmatrix} r(p_{vu}) \\ b(p_{vu}) \end{bmatrix}_{\gamma_2} = \begin{bmatrix} b_1 + b \\ r_1 + r \end{bmatrix}.$$

This implies the composition of the path puv that passes through the cycleγ_2 and the composition of the path p_{vu} that passes through the cycleγ_1 are

$$\begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix}_{\gamma_2} = \begin{bmatrix} (s+1-2b_1)/2 + r \\ (s-1-2r_1)/2 + b \end{bmatrix} \text{ and } \begin{bmatrix} r(p_{vu}) \\ b(p_{vu}) \end{bmatrix}_{\gamma_1} = \begin{bmatrix} (s-1-2r_1) + b \\ (s+1-2b_1) + r \end{bmatrix},$$

respectively. Note that if both vertices u and v lie on the cycle then the composition of paths p_{uv} and p_{vu} depend only on r_1 and b_1. Similarly if both vertices u and v lie on the path p_{ns}, then there is only one path p_{uv} and only one path p_{vu} and their composition depend only on $r(p_{uv})$ and $b(p_{uv})$.

For any pair of vertices u and v, let w'_{uv} be the shortest $(e, e)^T$-walk from u to v. We note that the composition of w'_{uv} is of the form

$$\begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell_{uv} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}$$

for some path puv such that $r(p_{uv}) \geq b(p_{uv})$, or

$$\begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell_{uv} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b(p_{uv}) - r(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}$$

for some path p_{uv} with $b(p_{uv}) > r(p_{uv})$. We also note that

$$\ell_{uv} = \begin{cases} d(v, s), & \text{if } d(v, s) < d(u, s) \\ d(u, s), & \text{if } d(u, s) < d(v, s) \\ 0, & \text{if } r(p_{uv}) = b(p_{uv}) \text{ or at least one of } u \text{ or } v \text{ is on the cycle.} \end{cases}$$

Define

$$\begin{bmatrix} h_{\max} \\ k_{\max} \end{bmatrix} = \max_{u,v \in V} \left\{ \begin{bmatrix} r(w'_{uv}) \\ b(w'_{uv}) \end{bmatrix} \right\}.$$

Then we have the following formula for 2-exponent of an (n, s)-lollipop.

Theorem 4.1. *Let D be an asymmetric primitive two-coloured (n,s)-lollipop. Then*

$$exp_2(D) = h_{max} + k_{max}.$$

Proof. Let u and v be any vertices in D and let w'_{uv} be a shortest $(e, e)^T$-walk from u to v. Notice by the definition of

$$\begin{bmatrix} h_{\max} \\ k_{\max} \end{bmatrix}$$

for any shortest $(e, e)^T$-walk w'_{uv} from u to v we have

$$\begin{bmatrix} r(w'_{uv}) \\ b(w'_{uv}) \end{bmatrix} \leq \begin{bmatrix} h_{\max} \\ k_{\max} \end{bmatrix}.$$

Since $r(w'_{uv}) = b(w'_{uv})$ and $h_{\max} = k_{\max}$, using $(1, 1)^T$-walks we can extend the walk w'_{uv} to a walk wuv such that

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} h_{max} \\ k_{max} \end{bmatrix}.$$

This implies for any pair of vertices u and v in D there is a $(h_{\max}, k_{\max})^T$-walk from u to v. Hence $\exp_2(D) \leq h_{\max} + k_{\max}$. It remains to show that $exp_2(D) \geq h_{max} + k_{max}$.

Let $u_0$and v_0 be the vertices in D such that the shortest $(e, e)^T$-walk $w'_{u_0v_0}$ in D has composition $(h_{\max}, k_{\max})^T$. Let $p'_{u_0v_0}$ be the path from u_0 to v_0 contained in the walk $w'_{u_0v_0}$. Let $w_{u_0v_0}$ and $w_{v_0u_0}$, respectively, be the walk from u_0 to v_0 and from v_0 to u_0 that have the same composition. We consider four cases.

Case 1. The path $p'_{u_0v_0}$ passes through γ_1 and the path $p'_{v_0u_0}$ passes through γ_2.

The composition of the walk $w_{u_0v_0}$ is of the form

$$\begin{bmatrix} r_1 + r \\ b_1 + b \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + x_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.1}$$

for some nonnegative integers x_1, x_2 and x_3. The composition of the walk $w_{v_0u_0}$ is of the form

$$\begin{bmatrix} b_1 + b \\ r_1 + r \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + y_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.2}$$

for some nonnegative integers y_1, y_2 and y_3. Since $w_{u_0v_0}$ and $w_{v_0u_0}$ have the same composition, from Equation (4.1) and Equation (4.2) we have

$$\begin{bmatrix} (b_1 + b) - (r_1 + r) \\ (r_1 + r) - (b_1 + b) \end{bmatrix} = (x_1 - y_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (x_2 - y_2) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + (x_3 - y_3) \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}. \tag{4.3}$$

From the last equation, subtracting the second component by the first component, we have that

$$(x_2 - y_2) + (y_3 - x_3) = 2((r_1 + r) - (b_1 + b)).$$

This implies $x_2 + y_3 \geq 2((r_1 + r) - (b_1 + b))$ and hence $x_2 \geq (r_1 + r) - (b_1 + b)$ or $b_3 \geq (r_1 + r) - (b_1 + b)$.

Case 2. The path $p'_{u_0v_0}$ passes through γ_1 and the path $p'_{v_0u_0}$ passes throughγ_1.

The composition of the walk $w_{u_0v_0}$ is of the form

$$\begin{bmatrix} r_1+r \\ b_1+b \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + x_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.3}$$

for some nonnegative integers x_1, x_2 and x_3. The composition of the walk $w_{v_0u_0}$ is of the form

$$\begin{bmatrix} (s-1-2r_1)+b \\ (s+1-2b_1)+r \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + y_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.4}$$

for some nonnegative integers y_1, y_2 and y_3. From Equation (4.3) and Equation (4.4) we have $(x_2+y_3) \geq 2((r_1+r)-(b_1+b))+1$, hence $x_2 \geq (r_1+r)-(b_1+b)$ or $y_3 \geq (r_1+r)-(b_1+b)$.

Case 3. The path $p'_{u_0v_0}$ passes through γ_2 and the path $p'_{v_0u_0}$ passes through γ_1.

The composition of the walk $w_{u_0v_0}$ is of the form

$$\begin{bmatrix} (s+1-2b_1)/2+r \\ (s-1-2r_1)/2+b \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + x_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.5}$$

for some nonnegative integers x_1, x_2 and x_3. The composition of the walk is $w_{v_0u_0}$ of the form

$$\begin{bmatrix} b_1+b \\ r_1+r \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + y_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.6}$$

for some nonnegative integers y_1, y_2 and y_3. Since $w_{u_0v_0}$ and $w_{v_0u_0}$ have the same composition, Equations (4.5) and (4.6) imply $x_2 + y_3 \geq 2((r_1+r)-(b_1+b))+1$, and hence $x_2 \geq (r_1+r)-(b_1+b)$ or $y_3 \geq (r_1+r)-(b_1+b)$.

Case 4. The path $p'_{u_0v_0}$ passes through γ_2 and the path $p'_{v_0u_0}$ passes throughγ_2.

The composition of the walk $w_{u_0v_0}$ is of the form

$$\begin{bmatrix} (s+1-2b_1)/2+r \\ (s-1-2r_1)/2+b \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + x_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.7}$$

for some nonnegative integers x_1, x_2 and x_3. The composition of the walk $w_{v_0u_0}$ is of the form

$$\begin{bmatrix} (s-1-2r_1)+b \\ (s+1-2b_1)+r \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + y_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \tag{4.8}$$

for some nonnegative integers y_1, y_2 and y_3. Equating Equations (4.7) and (4.8) we have $x_2 + y_3 \geq 2((r_1+r)-(b_1+b))+2$ and hence $x_2 \geq (r_1+r)-(b_1+b)+1$ or $y_3 \geq (r_1+r)-(b_1+b)+1$.

Therefore, in all cases we have $x_2 \geq (r_1+r)-(b_1+b)$ which then implies

$$\begin{bmatrix} r(w_{u_0v_0}) \\ b(w_{u_0v_0}) \end{bmatrix} \geq \begin{bmatrix} r({p'}_{u_0v_0}) \\ b({p'}_{u_0v_0}) \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r({p'}_{u_0v_0}) - b({p'}_{u_0v_0})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} \\ + x_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}.$$

Notice that by definition

$$\begin{bmatrix} h_{\max} \\ k_{\max} \end{bmatrix} = \min_{p'_{u_0v_0}} \left\{ \begin{bmatrix} r(p'_{u_0v_0}) \\ b(p'_{u_0v_0}) \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p'_{u_0v_0}) - b(p'_{u_0v_0})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} \right\},$$

where the minimum is taken over all paths $p'_{u_0v_0}$. Hence we have that

$$\begin{bmatrix} r(w_{u_0v_0}) \\ b(w_{u_0v_0}) \end{bmatrix} \geq \begin{bmatrix} h_{max} \\ k_{max} \end{bmatrix},$$

and consequently $\exp_2(D) \geq h_{max} + k_{max}$.

Theorem 4.1 actually guarantees that the following way of determining the 2-exponent of an asymmetric primitive two-coloured (n, s)-lollipop works.

- **Step 1.** For each pair of vertices u and v find the shortest $(e, e)^T$-walk w'_{uv} from u to v for some positive integer $e \geq 1$.
- **Step 2.** Among all walks w'_{uv} in Step 1, find the longest walk w_{uv}.
- **Step 3.** The 2-exponent of D is the length of the walk w_{uv}.

Example 4.2. Let D be the asymmetric primitive two-coloured (9,5)-lollipop as follows. Color the arcs (5,1), (2,1), (2,3), (4,3), (4,5), (5,6), (7,6), (8,7) and (9,8) with red and color the others with blue. One can check that the longest $(e, e)^T$-walk w_{uv} in Step 2 is the $(10, 10)^T$-walk from 6 to 9. Hence the 2-exponent of D is 20.

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