MATEMATIKA, 2009, Volume 25, Number 1, 39–51

Soliton Solutions of the Complex Ginzburg-Landau Equation

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Abstract The function transformed method is applied to an n-dimensional complex Ginzburg-Landau equation (CGLE), which is being transformed to physically significant well known nonlinear waves equations depending only on the function ξ, and these equations can be exactly solved. The solution of these equations in ξ is shown to lead to a general soliton solution of the CGLE.

Keywords Complex Ginzburg-Landau equation; nonlinear waves equation; soliton solution; function transformed method.

1 Introduction

The complex Ginzburg-Landau equation (CGLE) is one of the most studied nonlinear equations in the physics and applied mathematics communities (refer to Aranson & Kramer [1]). It describes vast variety phenomena from nonlinear waves to second order phase transitions, from superconductivity, superfluidity, and Bose-Einstein condensation to liquid crystals and strings theory (refer e.g. Kuramoto [6], Dangelmayr & Kramer [2], Pismen [11], Newell [7]).

The one dimensional Ginzburg–Landau equation (e.g. Nohara [9]), which has also been named the Newell-Whitehead equation (Newell & Whitehead [8]) or the Stewartson-Stuart equation (Stewartson & Stuart [12]), presents generally the time evolution of the amplitude of unstable wave with finite wave numbers near a critical unstable point. The Schrödinger equation is a special case of the Ginzburg–Landau equation for purely dispersive waves. The fact that the uniform solution for the nonlinear Schrödinger equation has the modulational instability is known, and also it is integrable so that the general exact solution can be presented. This modulational instability also occurs in the nonlinear Ginzburg–Landau equation (or CGLE). Nozaki and Bekki [10] and some other researchers have obtained particular solutions only.

Some studies of the initial value problem for CGLE have also been carried out. Originally, CGLE presents time evolution of the envelope of slightly unstable, nearly monochromatic waves, whose energy is almost concentrated in a wave number. Directional, nearly monochromatic waves have a fixed wave number but spread over some propagation area in propagating direction. The CGLE is the basic model, which describes these nonlinear phenomena far from equilibrium (Khater et al. [5]). It describes, for example, the open flow motion, travelling waves in binary fluid mixture, and spatially extended non equilibrium system.

In optics, it is useful in analyzing optical transmission lines passively mode-locked fiber lasers and spatial optical solitons. In this paper we apply the function transformed method (Wenhua [13]) to an n-dimensional CGLE, which is being transformed to physically significant nonlinear waves equations such as the sine–Gordon and sinh–Gordon equations, which depend only on one function, ξ, and can be exactly solved. The general solution of these equations in ξ leads to a general soliton solution of the n-dimensional CGLE.

This paper is organized as follows; the introduction is in section 1. In section 2, we discuss on the Ginzburg-Landau equation. In section 3 we obtain a sine-Gordon equation by resorting to a function transformation method on the CGLE, i.e. obtainable by applying certain functions and then we generate the solutions of the sine-Gordon equation in ξ. This then leads to a general soliton solution of the n-dimensional CGLE. In section 4 we obtain a sinh–Gordon equation by applying certain functions and we then obtain its solutions in ξ. Similarly we then derive the general soliton solution of the n-dimensional CGLE. Finally the paper ends with concluding remarks in section 5.

2 Ginzburg-Landau Equation

We consider the following Fourier integral representation, which is expanded directly from the stable plane travelling wave of a nearly monochromatic plane wave solution, i.e.

$$u(x,t) = \int_k S(k)e^{ikx}e^{[\delta(k,R)-iw(k,R)]t}dk \tag{1}$$

where $S(k)$ denotes the wave spectrum of the function of wave number k, and the growth rate δ and angular frequency w depends on R, the parameter used in the complex growth rate s. i is the imaginary unit.

If the growth rate δ is zero and $w = w(k)$, then it becomes purely dispersive waves and we have a critical parameter R_c. We assume that there is a critical wave number such that the real exponential growth rate $\delta(k,R)$ satisfies the equation (refer Haberman [4])

$$\left.\begin{aligned} \delta(k_c,R_c) &= 0 \\ \delta_k^{(1)}(k_c,R_c) &= 0 \\ \delta_k^{(2)}(k_c,R_c) &< 0 \\ \delta_R^{(1)}(k_c,R_c) &< 0 \end{aligned}\right] \tag{2}$$

near $k = k_c$and $R = R_c$. Here we have

$$\left.\begin{aligned} \delta_k^{(n)}(k_c,R_c) &= \frac{\partial^n\delta(k,R_c)}{\delta k^n}\Big|_{k=k_c} \\ \delta_R^{(n)}(k_c,R_c) &= \frac{\partial^n\delta(k,R_c)}{\delta k^n}\Big|_{R=R_c}. \end{aligned}\right] \tag{3}$$

We assume R is slightly greater than R_c, and this assumption means that there is a band of waves numbers near k_c in which $\delta > 0$. Nevertheless, the largest positive value of δ will be small so that the wave is slightly unstable.

In this case, there is a small band of unstable wave number near k_c. For the slightly unstable waves number of equation (1) let $k = k_c$ and $R = R_c$ and we take the nearly monochromatic assumption of wave solution of the following form

$$u(x,t) = A(x,t)e^{i\{x(k_c)-w(k_c,R_c)t\}}. \tag{4}$$

Equation (4) shows that most of the energy is concentrated in one wave number k_c and the amplitude $A(x,t)$ is not constant but varies slowly in space and time. The amplitude $A(x,t)$

acts as an envelope of the travelling wave. We shall now obtain the governing equation of the envelope.

First $A(x,t)$ is derived from equation (1) and (4) as follows,

$$A(x,t) = \int_k S(k) e^{ip(x,k)} e^{Q(t,k,R)} dk \tag{5}$$

where

$$p(x,k) = (k - k_c)x, \tag{6}$$

and

$$\begin{aligned} Q(t;k,R) &= [\delta(k,R) - i\{w(k,R) - w(k_c,R_c)\}]t \\ &= \{s(k,R) + iw(k_c,R_c)\}t. \end{aligned} \tag{7}$$

In equation (7), $s(k,R)$ is the complex growth rate as given by

$$s(k,R) = [\delta(k,R) - iw(k,R)]. \tag{8}$$

Subsequently we derive time and space derivatives of $A(x,t)$ and then combine these equations using the Taylor expansion of the dispersion relation of the complex growth rate.

The time derivative of of $A(x,t)$ is

$$\frac{\partial A(x,t)}{\partial t} = \int_R \{s(k,R) + iw(k_c,R_c)\} S(k) e^{ip(x,k)} e^{Q(t;k,R)} dk. \tag{9}$$

Moreover, the m-th spatial derivative of $A(x,t)$ is obtained as

$$\frac{\partial^m A(x,t)}{\partial x^m} = i^m \int_R (k - k_c)^m S(k) e^{ip(x,k)} e^{Q(t;k,r)} dk, \;\; m = 1,2,3\dots \tag{10}$$

In equation (9), we expand the complex growth rate $s(k,R)$ in Taylor series around $k = k_c$ and $R = R_c$, i.e.

$$s(k,R) = s(k_c,R_c) + \sum_{m=1}^{\infty} \frac{s_k^m(k_c,R_c)}{m!}(k-k_c)^m + \sum_{i=n}^{\infty} \frac{s_k^n(k_c,R_c)}{n!}(R-R_c)^n \tag{11}$$

where

$$\begin{aligned} s_k^{(n)}(k_c,R_c) &\triangleq \frac{\partial^n(k,R_c)}{\partial k^n}\Big|_{k=k_c} \\ s_R^{(n)}(k_c,R_c) &\triangleq \frac{\partial^n(k,R_c)}{\partial R^n}\Big|_{R=R_c}. \end{aligned} \tag{12}$$

The first equation of equation (2) and equation (8) yield the relations

$$s(k_c,R_c) = \delta(k_c,R_c) - iw(k_c,R_c) = iw(k_c,R_c) \tag{13}$$

$$s(k,R) + iw(k_c,R_c) = \sum_{m=1}^{\infty} \frac{s_k^m(k_c,R_c)}{m!}(k-k_c)^m + \sum_{n=1}^{\infty} \frac{s_k^n(k_c,R_c)}{n!}(R-R_c)^n. \tag{14}$$

Equation (9) can be rewritten by using (14) as follows,

$$\begin{aligned}
&\frac{\partial A(x,t)}{\partial t} \\
&= \int_R \left[\sum_{m=1}^{\infty} \frac{s_k^m(k_c,R_c)}{m!}(k-k_c)^m + \sum_{n=1}^{\infty} \frac{s_k^n(k_c,R_c)}{m!}(R-R_c)^n \right] S(k)e^{ip(x,k)}e^{Q(t;k,R)}dk \\
&= \int_R \sum_{m=1}^{\infty} \frac{s_k^m(k_c,R_c)}{m!}(k-k_c)^m S(k)e^{ip(x,k)}e^{Q(t;k,R)}dk \\
&\quad + \sum_{n=1}^{\infty} \frac{s_k^n(k_c,R_c)}{m!}(R-R_c)^n \int_R S(k)e^{ip(x,k)}e^{Q(t;k,R)}dk.
\end{aligned} \tag{15}$$

Substituting equation (10) into equation (15) leads to the following,

$$\frac{\partial A(x,t)}{\partial t} = \sum_{m=1}^{\infty} \frac{(-i)^m(k_c,R_c)}{m!}\frac{\partial^m A(x,t)}{\partial t^m} + \sum_{n=1}^{\infty} \frac{s_k^n(k_c,R_c)}{n!}(R-R_c)^n A(x,t). \tag{16}$$

Equation (16) represents the linear higher order governing equation which governs the amplitude of slightly unstable, nearly monochromatic wave namely the envelope.

Assuming $O(k-k)^2 = O(R-R_c)$ and neglecting the third and higher order of spatial derivatives and the second and the higher order of the coefficient of $A(x,t)$ in equation (16) (that is $m = 1$) and (n= 1), we obtain the linearized Ginzburg-Landau equation.

$$\frac{\partial A(x,t)}{\partial t} + w_k^{(1)}(k_c,R_c)\frac{\partial A(x,t)}{\partial x} = -\frac{s_k^2}{2!}(k_c,R_c)\frac{\partial^2 A(x,t)}{\partial x^2} + s(k_c,R_c)(R-R_c)A(x,t). \tag{17}$$

To derive equation (17), the following relation which is reduced from the second relation in equation (2) is used, i.e.

$$s_k^{(1)}(k_c,R_c) = \delta_k^{(1)}(k_c,R_c) - iw_k^{(1)}(k_c,R_c) = -iw_k^{(k)}(k_c,R_c). \tag{18}$$

Thus we can treat the nonlinear case of the dispersion relation of the complex growth rate as

$$s = s(k,R|A(x,t)|^2) = \delta(k,R,|A(x,t)|^2) - iw(k,R,|A(x,t)|^2). \tag{19}$$

We then expand s in Taylor series around $k = k_c$, $R = R_c$ and $|A(x,t)|^2 = 0$, i.e.

$$\begin{aligned}
s(k,R,|A(x,t)|^2) = s(K_c,R_c,0) &+ \sum_{m=1}^{\infty} \frac{s_k^m(k_c,R_c,0)}{n!}(k-k_c)^n \\
&+ \sum_{n=1}^{\infty} \frac{s_R^n(k_c,R_c,0)}{n!}(R-R_c)^n + \sum_{n=1}^{\infty} \frac{s_{|A|^2}^n(k_c,R_c,0)}{n!}|A(x,t)|^{2n}
\end{aligned} \tag{20}$$

where

$$s_{|A|^2}^n(k_c,R_c,0) = \frac{\partial^n s(k_c,R_c,|A|^2)}{\partial |A|^{2n}}\Big|_{|A|^2=0}. \tag{21}$$

Equation (20) can be applied to equation (1) and thus the following relation for $A(x,t)$ is obtained

$$\frac{\partial A(x,t)}{\partial t} = \sum_{m=1}^{\infty}(-1)^m \frac{s_k^m(k_c,R_c,0)}{m!} + \frac{\partial^m A(x,t)}{\partial x^m} + \sum_{n=1}^{\infty}\frac{s_k^m(k_c,R_c,0)}{n!}(R-R_c)A(x,t) + \sum_{n=1}^{\infty}\frac{s_R^m(k_c,R_c,0)}{n!}|A(x,t)|^{2n}A(x,t). \quad (22)$$

Neglecting the third and higher order of spatial derivatives and the second and higher order of the coefficient of $A(x,t)$ in equation (22) (that is we take $m = 1$, 2 and $n = 1$) we obtain the nonlinear complex Ginzburg-Landau equation (CGLE) as

$$\frac{\partial A(x,t)}{\partial t} + w_k^{(1)}(k_c,R_c,0)\frac{\partial A(x,t)}{\partial x} = -s_k^{(2)}(k_c,R_c,0)\frac{\partial^2 A(x,t)}{\partial x^2} + s_R^{(1)}(k_c,R_c,0)(R-R_c)A(x,t) + s_{|A|^2}^{(1)}(k_c,R_c,0)A(x,t)|A(x,t)|^2. \quad (23)$$

If we let $\delta = 0$ and $w = w(k)$ we obtain the linear Schrödinger equation

$$\frac{\partial A(x,t)}{\partial t} + w_k^{(1)}\frac{\partial A(x,t)}{\partial x} = -\frac{tw_k^{(2)}}{2!}\frac{\partial^2 A(x,t)}{\partial x^2}. \quad (24)$$

Moreover, the nonlinear Schrödinger equation is derived by considering the non linearity $w = w(k,|A|^2)$ and expanding w in Taylor series around $k = k_c$ and $|A(x,t)|^2 = 0$ to obtain

$$\frac{\partial A(x,t)}{\partial t} + w_k^{(1)}(k_c,0)\frac{\partial A(x,t)}{\partial x} = \frac{iw_k^{(2)}}{2!}\frac{(k_c,0)\partial^2 A(x,t)}{\partial x^2} - iw_{|A|^2}^{(1)}(k_c,0)|A(x,t)|^2A(x,t). \quad (25)$$

3 Sine-Gordon Equation

Consider an n-dimensional CGLE of the form

$$\partial_0 Z = \mu Z - Z|Z|^2 + \partial_i\partial_i Z \quad (26)$$

where $\partial_0 = \dfrac{\partial}{\partial_0} = \dfrac{\partial}{\partial_t}$, $\partial_i = \dfrac{\partial}{\partial x_i}$, μ is the diffusion coefficient.

We can find solutions of (26) in the form

$$Z = u(x)e^{(ic_\alpha x_\alpha)} \quad (27)$$

where $c_\alpha = c_\alpha^* =$ constant, $u(x) = u^*(x)$ where c_α^* stands for complex conjugate of c_α.

By substituting (27) into (26), one obtains

$$\partial_0 u - 2ic_i\partial_i u - \partial_i\partial_i u = \mu u - c_ic_iu - u^3 - ic_0u,$$

which can be rewritten in the form

$$\partial_0 u - 2ic_i\partial_i u - \partial_i\partial_i u = cu - u^3. \quad (28)$$

where $c = \mu - ic_0 - c_i c_i$.

Let us make a function transformation of the following form

$$u = \sqrt{c}\sin\left(\frac{\phi}{2}\right) \tag{29}$$

where ϕ is a function in x_α.

On splitting the first two terms, the third terms of the L.H.S and R.H.S of the equation (28) now become

$$\begin{aligned} \partial_0 u - 2ic_i\partial_i u &= \sqrt{c}\Big[\partial_0 \sin\left(\frac{\phi}{2}\right) - 2ic_i\partial_i \sin\left(\frac{\phi}{2}\right)\Big] \\ &= \frac{1}{2}\sqrt{c}\cos\left(\frac{\phi}{2}\right)[\partial_0\phi - 2ic_i\partial\phi] \end{aligned} \tag{30a}$$

$$\begin{aligned} \partial_i\partial_i u &= \frac{1}{2}\sqrt{c}\partial_i\Big[\cos\left(\frac{\phi}{2}\right)\partial_i\phi\Big] \\ &= \frac{1}{2}\sqrt{c}\cos\left(\frac{\phi}{2}\right)\Big[\partial_i\partial_i\phi - \frac{1}{2}\partial_i\phi\partial_i\phi\tan\left(\frac{\phi}{2}\right)\Big]. \end{aligned} \tag{30b}$$

$$\begin{aligned} cu - u^3 &= c\sqrt{c}\Big[\sin\left(\frac{\phi}{2}\right) - \sin^3\left(\frac{\phi}{2}\right)\Big] \\ &= \frac{c}{2}\sqrt{c}\cos\left(\frac{\phi}{2}\right)\sin\phi. \end{aligned} \tag{30c}$$

By substituting all equations (30a-c) into (28) we obtain

$$\begin{aligned} &\frac{1}{2}\sqrt{c}\cos\Big(\frac{\phi}{2}\Big)[\partial_0\phi - 2ic_i\partial\phi] - \frac{1}{2}\sqrt{c}\cos\left(\frac{\phi}{2}\right)\Big[\partial_i\partial_i\phi - \frac{1}{2}\partial_i\phi\partial_i\phi\tan\left(\frac{\phi}{2}\right)\Big] \\ &= \frac{c}{2}\sqrt{c}\cos\left(\frac{\phi}{2}\right)\sin\phi. \end{aligned} \tag{31a}$$

Dividing equation (31a) by $\sqrt{c}\cos\left(\frac{\phi}{2}\right)$ we obtain

$$\partial_0\phi - 2ic_i\partial_i\phi - \partial_i\partial_i\phi + \frac{1}{2}\partial_i\phi\partial_i\phi\tan\left(\frac{\phi}{2}\right) = c\sin\phi. \tag{31b}$$

Setting $\phi = \phi(\xi)$, which is a function of another function ξ only, we can find that

$$\partial_0\phi = \partial_0\xi\frac{d\phi}{d\xi},\ \partial_i\phi = \partial_i\xi\frac{d\phi}{d\xi} \tag{31c}$$

$$\partial_i\phi\partial_i\phi = \partial_i\xi\partial_i\xi\Big(\frac{d\phi}{d\xi}\Big)^2 \tag{31d}$$

$$\partial_i\partial_i\phi = \partial_i\xi\partial_i\xi\frac{d^2\phi}{d\xi^2} + \partial_i\partial_i\xi\frac{d\phi}{d\xi}. \tag{31e}$$

Substituting (31 a, c, d, e) into (31 b) we obtain

$$\partial_0\xi\frac{d\phi}{d\xi} - 2ic_i\partial_i\xi\frac{d\phi}{d\xi} - \partial_i\partial_i\xi\frac{d\phi}{d\xi} - \partial_i\xi\partial_i\xi\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\tan\left(\frac{\phi}{2}\right)\partial_i\xi\partial_i\xi\Big(\frac{d\phi}{d\xi}\Big)^2 = c\sin\phi$$

or

$$(\partial_0\xi - 2ic_i\partial_i\xi - \partial_i\partial_i\xi)\frac{d\phi}{d\xi} - \partial_i\xi\partial_i\xi\Big[\frac{d^2\phi}{d\xi^2} - \frac{1}{2}\tan\Big(\frac{\phi}{2}\Big)\Big(\frac{d\phi}{d\xi}\Big)^2\Big] = c\sin\phi. \tag{32}$$

Clearly, some solutions of (32) obey the following system of equations

$$\partial_0\xi - 2ic_i\partial_i\xi - \partial_i\partial_i\xi = 0,\ \partial_i\xi\partial_i\xi = 1, \tag{33}$$

$$\frac{d\phi}{d\xi} = \sqrt{2c}\cos\Big(\frac{\phi}{2}\Big),\ \frac{d^2\phi}{d\xi^2} = -\frac{c}{2}\sin\phi. \tag{34}$$

Equations (33) have a general solution in the form of (see Khater et al. [4])

$$\xi = F(\eta_j) + d_\alpha x_\alpha,\ \eta_j = b_{j\alpha}x_\alpha + \varepsilon_j = \text{constant}.$$

Equation (34) is equivalent to a sine-Gordon equation, and its solution is a well known soliton of the form (e.g. refer to Drazin & Johnson [3])

$$\phi = 4\tan^{-1} e^{(\sqrt{2c}\xi+\xi_0)} - \pi,\ \xi_0 = \text{constant}. \tag{35}$$

Inserting (35) into (29), we obtain the soliton solution of an n-dimensional CGLE (26) in the form of solution (36), i.e.

$$u = \sqrt{c}\sin[2e^{(\sqrt{2c}\xi+\xi_0)}] - \pi$$

or

$$u = \sqrt{c}\cos\big(2\tan^{-1} e^{(\sqrt{2c}\xi+\xi_0)}\big)$$

and

$$Z = -\sqrt{c}\cos\big(2\tan^{-1} e^{(\sqrt{2c}\xi+\xi_0)}\big)e^{ic_\alpha x_\alpha} \tag{36}$$

and graphically this solution emerges as in Figure 1, depicting a form of a breather-like solution.

Following the above-mentioned procedures, and applying another function transformation of equation (28) via

$$u = \sqrt{c}\cos\Big(\frac{\phi}{2}\Big). \tag{37}$$

then we have

$$\partial_0\phi - 2ic_i\partial_i\phi - \partial_i\partial_i\phi - \frac{1}{2}\partial_i\phi\cot\Big(\frac{\phi}{2}\Big) = -c\sin\phi. \tag{38}$$

Setting $\phi = \phi(\xi)$ and substituting (31 a, c, d, e) into (38) yield

$$\partial_0\frac{d\phi}{d\xi} - 2ic_i\partial_i\xi\frac{d\phi}{d\xi} - \partial_i\partial_i\xi\frac{d\phi}{d\xi} - \partial_i\xi\partial_i\xi\frac{d^2\phi}{d\xi^2} - \frac{1}{2}\cot\Big(\frac{\phi}{2}\Big)\partial_i\xi\partial_i\xi\Big(\frac{d\phi}{d\xi}\Big)^2 = -c\sin\phi.$$

or

$$(\partial_0\xi - 2ic_i\partial_i\xi - \partial_i\partial_i\xi)\frac{d\phi}{d\xi} - \partial_i\xi\partial_i\xi\Big[\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\cot\Big(\frac{\phi}{2}\Big)\Big(\frac{d\phi}{d\xi}\Big)^2\Big] = -c\sin\phi. \tag{39}$$

Explicitly, some solutions of (39) obey the following system of equations:

$$\partial_0\xi - 2ic_i\partial_i\xi - \partial_i\partial_i\xi = 0,\ \partial_i\xi\partial_i\xi = -1$$

$$\frac{d\phi}{d\xi} = \sqrt{2c}\sin\Big(\frac{\phi}{2}\Big),\ \frac{d^2\phi}{d\xi^2} - \frac{3c}{2}\sin\phi. \tag{40}$$

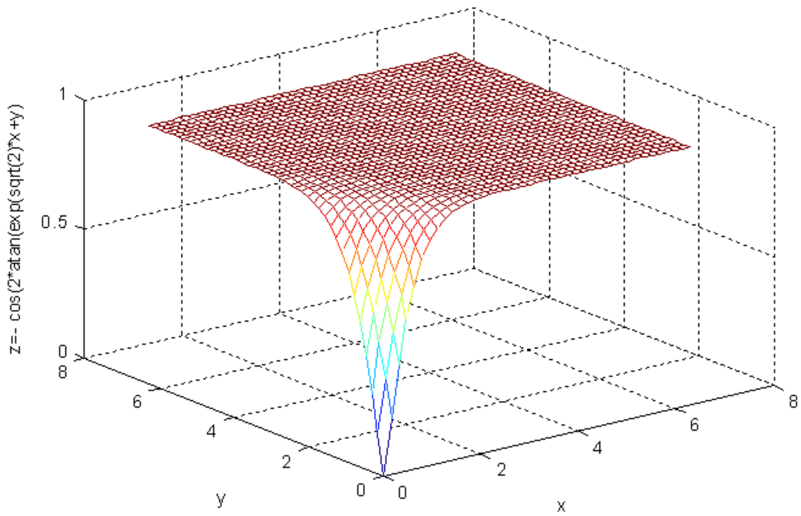

Figure 1: Soliton solution of the CGLE with $u = \sqrt{c}\cos\left(2\tan^{-1} e^{(\sqrt{2c}\xi+\xi_0)}\right)$

Similarly, equation (40) is equivalent to a sine-Gordon equation, where its solution is the well known soliton of the form

$$\phi = 4\tan^{-1} e^{(\sqrt{6c}\xi+\xi_0)} - \pi, \ \xi_0 = \text{constant}.$$

Inserting this into (29), we obtain the soliton solution of an n-dimensional CGLE (26) in the form of solution (41), i.e.

$$u = \sqrt{c}\cos\left(2\tan^{-1} e^{(\sqrt{6c}\xi+\xi_0)} - \frac{\pi}{2}\right)$$

or

$$u = \sqrt{c}\sin\left(2\tan^{-1} e^{(\sqrt{6c}\xi+\xi_0)}\right)$$

and

$$Z = \sqrt{c}\sin\left(2\tan^{-1} e^{(\sqrt{6c}\xi+\xi_0)}\right)e^{ic_\alpha x_\alpha} \tag{41}$$

and graphically from Figure 2, this solution portrays a similar positive breather-like solution form as in Figure 1.

The same steps can be taken by applying another function transformation of equation (28) via the exponential form

$$u = \sqrt{c}e^{\phi/2}. \tag{42}$$

Then we have (28) in the form

$$\partial_0\phi - 2ic_i\partial_i\phi - \partial_i\partial_i\phi - \frac{1}{2}\partial_i\phi\partial_i\phi = 2c[1 - e^{\phi}]. \tag{43}$$

Setting $\phi = \phi(\xi)$ and substituting (31 a, c, d, e) into (43) yield

$$(\partial_0\xi - 2ic_i\partial_i\xi - \partial_i\partial_i\xi)\frac{d\pi}{d\xi} - \partial_i\xi\partial_i\xi\left[\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2\right] = -2c[1 - e^{\phi}]. \tag{44}$$

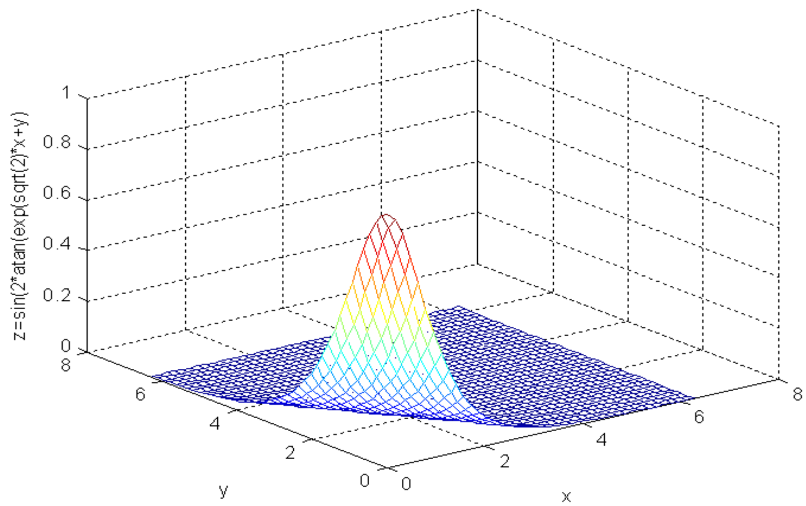

Figure 2: Soliton solution of the CGLE with $u = \sqrt{c}\sin\left(2\tan^{-1} e^{(\sqrt{6c}\xi+\xi_0)}\right)$

Then if

$$\partial_0\xi - 2ic_i\partial_i\xi - \partial_i\partial_i\xi = 0, \;\; \partial_i\partial_i\xi = 1$$

and

$$\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\Big(\frac{d\phi}{d\xi}\Big)^2 = -2c[1-e^{\phi}]. \tag{45}$$

Then the solution of equation (45) is

$$\phi = 2\ln[\sqrt{2}\text{sec}(\sqrt{c}\xi + \xi_0)], \;\; \xi_0 = \text{constant}.$$

Inserting this into (42), we obtain the soliton solution of an n-dimensional CGLE (26) in the form of solution (46), i.e.

$$Z = \sqrt{2c}\,\sec[\sqrt{c}\xi + \xi_0)]e^{ic_\alpha x_\alpha} \tag{46}$$

and graphically this solution takes a geometrical form as in Figure 3, portraying a kink-antikink solution.

4 Sinh-Gordon Equation

Following exactly the procedures as laid out in section 3, we obtain

$$\partial_0 u - 2ic_i\partial_i u - \partial_i\partial_i u = \acute{c}u - u^3, \tag{47}$$

where $\acute{c} = c_ic_i - \mu + ic_0$.

Let us make a function transformation

$$u = \sqrt{\acute{c}}\,\sinh\Big(\frac{\phi}{2}\Big) \tag{48}$$

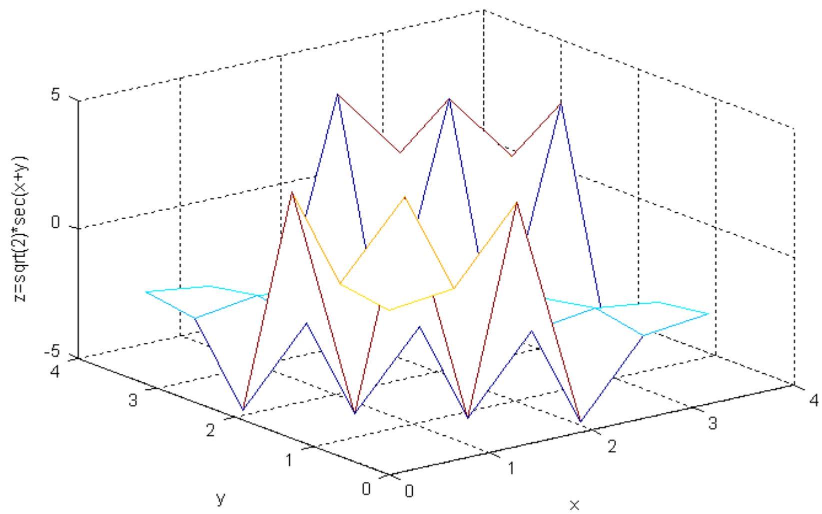

Figure 3: Soliton solution of the CGLE with $u = \sqrt{c}e^{\phi/2}$

then we have

$$\frac{1}{2}\sqrt{\acute{c}}\,\cosh\Big(\frac{\phi}{2}\Big)[\partial_0 - 2c_i\partial_i\phi] - \frac{1}{2}\sqrt{\acute{c}}\,\cosh\Big(\frac{\phi}{2}\Big)\Big[\partial_i\partial_i\phi + \frac{1}{2}\tanh\Big(\frac{\phi}{2}\Big)\partial_i\phi\partial_i\phi\Big]$$
$$= -\acute{c}\sqrt{\acute{c}}\,\sinh\Big(\frac{\phi}{2}\Big)\cosh^2\Big(\frac{\phi}{2}\Big)$$

or

$$\partial_0\phi - 2ic_i\partial_i\phi - \partial_i\partial_i\phi + \frac{1}{2}\,\tanh\Big(\frac{\phi}{2}\Big)\partial_i\phi\partial_i\phi = \acute{c}\,\sinh\phi. \tag{49}$$

Setting $\phi = \phi(\xi)$ and substituting (31 a, c, d, e) into (49) yield

$$\Big(\partial_0\xi - 2ic_i\partial_i\xi - \frac{1}{2}\partial_i\partial_i\xi\Big)\frac{d\phi}{d\xi} - \partial_i\xi\partial_i\xi\left[\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\tanh\Big(\frac{\phi}{2}\Big)\Big(\frac{d\phi}{d\xi}\Big)^2\right] = \acute{c}\,\sinh\phi. \tag{50}$$

Clearly again, we can set that some solutions of (50) to obey the following system of equations

$$\partial_0\xi - 2ic_i\partial_i\xi - \frac{1}{2}\partial_i\partial_i\xi = 0,\ \ \partial_i\xi\partial_i\xi = 1,$$
$$\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\,\tanh\Big(\frac{\phi}{2}\Big)\Big(\frac{d\phi}{d\xi}\Big)^2 = \acute{c}\,\sinh\phi$$

such that we have equation

$$\frac{d\phi}{d\xi} = \sqrt{2\acute{c}}\,\cosh\Big(\frac{\phi}{2}\Big),\ \frac{d^2\phi}{d\xi^2} = \frac{\acute{c}}{2}\,\sinh\phi. \tag{51}$$

Equation (51) is equivalent to the sinh-Gordon equation, where its solution is given by the well known soliton of the form (e.g. refer to Drazin & Johnson [3])

$$\phi = 4\,\tanh^{-1}\big[e^{\sqrt{2\acute{c}}\xi+\xi_0}\big] - \pi,\ \ \xi_0 = \text{constant}. \tag{52}$$

Substituting (52) and (47) into (27), we obtain the soliton solution of an n-dimensional CGLE (26) in the form

$$Z = \sqrt{\acute{c}}\,\sinh\Big[2\,\tanh^{-1}(e^{\sqrt{2\acute{c}}\xi+\xi_0}) - \frac{\pi}{2}\Big]e^{ic_\alpha x_\alpha} \tag{53}$$

and graphically Figure 4 depicts the geometrical form of the soliton solution.

Following the above-mentioned procedures, and applying another function transformation of equation (49) via

$$u = \sqrt{\acute{c}}\,\cosh\Big(\frac{\phi}{2}\Big) \tag{54}$$

and following the same steps taken above then we have

$$(-\partial_0\xi - 2ic_i\partial_i\xi + \partial_i\partial_i\xi)\frac{d\phi}{d\xi} - \partial_i\xi\partial_i\xi\left[\frac{d^2\phi}{d\xi^2} + \frac{1}{2}\cot\Big(\frac{\phi}{2}\Big)\Big(\frac{d\phi}{d\xi}\Big)^2\right] = \acute{c}\,\sinh\phi. \tag{55}$$

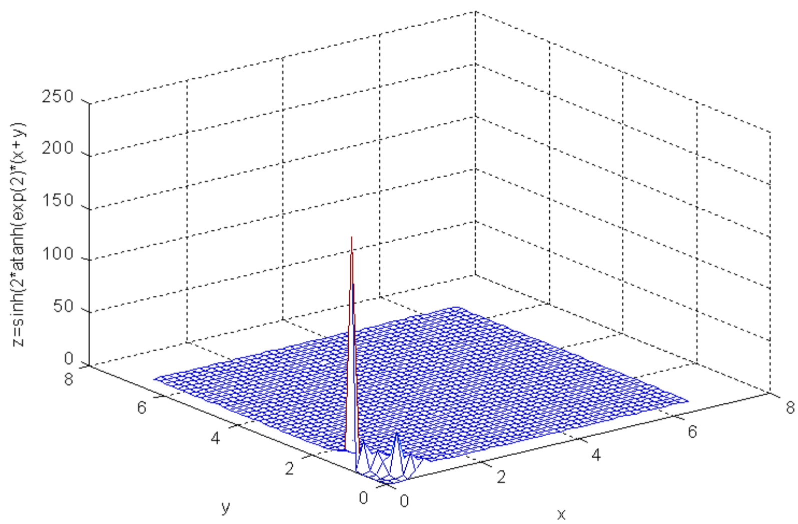

Figure 4: Soliton solution of the CGLE with $u = \sqrt{\acute{c}}\,\sinh\Big(\frac{\phi}{2}\Big)$

Then clearly, some solution of (54) obey the following system

$$\begin{gathered} -\partial_0\xi - 2ic_i\partial_i\xi + \partial_i\partial_i\xi = 0 \text{ and } \partial_i\xi\partial_i\xi = -1 \\ \frac{d\phi}{d\xi} = \sqrt{2\acute{c}}\,\sinh\Big(\frac{\phi}{2}\Big),\ \frac{d^2\phi}{d\xi^2} = \frac{\acute{c}}{2}\,\sinh\phi. \end{gathered} \tag{56}$$

Equation (56) is equivalent to the sinh-Gordon equation, where its solution is the well known soliton

$$\phi = -4\,\tanh^{-1}\Big[e^{\sqrt{2\acute{c}}\xi+\xi_0}\Big] - \pi,\ \ \xi_0 = \text{constant}.$$

Then soliton solution of solution of an n-dimensional CGLE (26) is in the form

$$Z = \sqrt{\acute{c}}\Big[\cosh\big(2\,\tanh^{-1}\big(e^{\sqrt{-2\acute{c}}\xi+\xi_0}\big)\big) - \frac{\pi}{2}\Big]e^{ic_\alpha x_\alpha}, \tag{57}$$

and graphically from Figure 5, this solution portrays a similar soliton form as in Figure 4.

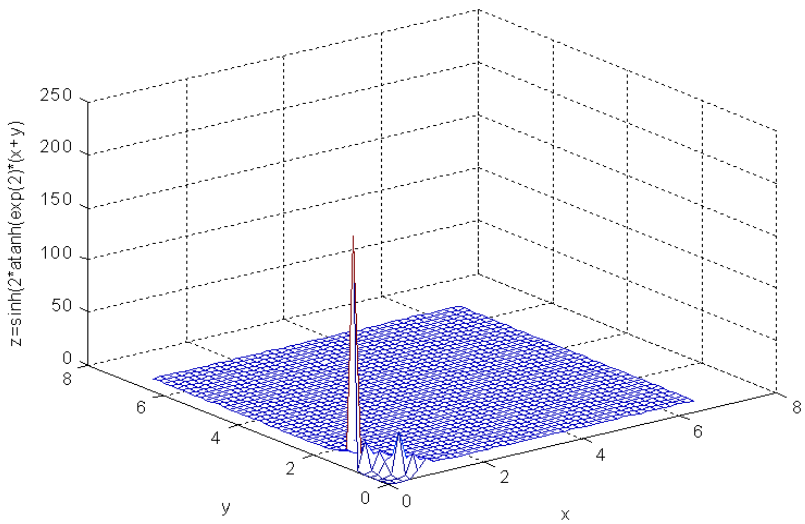

Figure 5: Soliton solution of the CGLE with $u = \sqrt{\acute{c}}\,\cosh\Big(\dfrac{\phi}{2}\Big)$

5 Concluding Remarks

We have applied the function transformation method to the n-dimensional CGLE, which is being transformed to physically significant well known nonlinear waves equations i.e. sine-Gordon and sinh-Gordon equations, which depend only on the function ξ, and typically can be exactly solved via the inverse scattering technique or the Hirota method (e.g. refer to Drazin & Johnson [3]). The general solution of these equations in ξ is shown to lead to a general soliton solution of n-dimensional CGLE. Interestingly we have tried to relate this scheme to other well known nonlinear waves equations such as the Korteweg-de Vries equation (KdV). Clearly this method is not applicable since KdV is of higher order than CGLE which is of second order, and similarly this rule applies to the other nonlinear waves equations.

Acknowledgment

Faisal is thankful to the Sudanese government for financial support.

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