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Characterization of Planar Cubic Alternative Curve

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Abstract The investigation on a planar cubic Alternative curve is carried out to determine the conditions for the existence of convex, loops, cusps and inflection points. First, the cubic curve is represented by a linear combination of three control points with the basis functions formulated with two shape parameters. Upon algebraic manipulation, the constraint of shape parameters are determined and sufficient conditions which ensure that the curve is either strictly convex or with loops, cusps and inflection point are identified resulting in a shape diagram.

Keywords Cubic Alternative curve; shape characterization; shape diagram.

1 Introduction

A generally accepted mathematical criterion for a curve to be fair is that it should have as few curvature extrema as possible, and it is often entailed upon the convexity of the curve. Convex segment have been derived as a segment that have neither inflection point, cusp, nor loop. Fair curves play an important role in the field of computer aided design (CAD) and computer aided geometric design (CAGD). The notion behind using fair curves for product design is that one would be able to shape a curve in such a way that it is visually pleasing. This criterion is a vital entity which ensures the success of a product.

Characterization of a curve is carried out to identify whether a curve has any inflection points, cusps, or loops. The characterization of the cubic curve has wide-ranging applications, for instance, in numerically controlled milling operations. In the design of highways, many of the algorithms rely on the fact that the trace of the curve or route is fair; an assumption that is violated if a cusp is present. Inflection points often indicate unwanted oscillations in applications such as the automobile body design and aerodynamics, and a surface that has a cross section curve possessing a loop cannot be manufactured.

Previous work in this area has been done by Su and Liu [7]; where they have presented a specific geometric solution for the Bezier basis function. By using the canonical curve, DeRose & Stone in [3] have characterized the parametric cubic curves. Walton and Meek [8] and Habib and Sakai [4] have presented results on the number and location of curvature extrema for cubic Bezier curves. All the stated researchers used discriminant method for the characterization process. Yang and Wang [9] have used the image of trochoids to investigate the occurrence of inflection and singularity of hybrid polynomial. For a Bezier-like curve, Azhar and Jamaludin [2] have characterized rational cubic Alternative representation by using the shoulder point methods and it is only restricted for trimmed shape parameters.

This paper deals with cubic Alternative curve, which is the linear combination of control points and basis functions that consist of two shape parameters. We apply the discriminant methods along with reparametrization methods to make a geometric characterization on untrimmed shape parameters of the basis functions.

The remaining part of this paper is organized as follows. In Section 2, we give a brief introduction of the cubic Alternative curve and some of its properties. The correspondence method and the usage of reparametrization are described in Section 3. The main result is discussed in section 4, 5 and 6. Sufficient condition of inflection and singularity for non-degenerate curve is presented in Section 4, as well as the degenerate curve in Section 6. Shape diagrams of inflection and singularity, as well as some examples of the corresponding curves are shown in Section 5.

2 Preliminaries

The planar cubic Alternative curve is a Bezier-like cubic curve, and it's defined as [6]:

$$Z(t) = F_0(t)P_0 + F_1(t)\ P_1 + F_2(t)P_2 + F_3(t)P_3;\ 0 \leq \mathrm{t} \leq 1 \tag{1}$$

where the basis functions are:

$$\begin{aligned} F_0(t) &= (1-t)^2(1+(2-\alpha)t) \\ F_1(t) &= \alpha(1-t)^2 t \\ F_2(t) &= \beta t^2(1-t) \\ F_3(t) &= t^2(1+(2-\beta)(1-t)) \end{aligned} \tag{2}$$

and the control points are denoted by $P_0, P_1,\ P_2,\ P_3$. The shape parameters are denoted by α and β, in which the untrimmed shape parameters used in this paper are $\alpha, \beta \in \mathrm{R}$. As Bezier curve, the cubic Alternative curve possesses the following properties:

- Endpoint Interpolation Property - $Z\,(0) = P_0$ and $Z\,(1) = P_3$.
- Endpoint Tangent Property - $Z'(0) = \alpha(P_1 - P_0)$ and $Z'(1) = \beta(P_3 - P_2)$.
- Invariance under Affine Transformations.

In general, cubic Alternative curve violates the convex hull property, however when $0 \leq \alpha, \beta \leq 3$, this property is satisfied [1]. The signed curvature $\kappa(t)$, of a plane curve $Z\,(t)$ is given by

$$\kappa\,(t) = \frac{Z'\,(t) \times Z''\,(t)}{\|Z'\,(t)\|^3} \tag{3}$$

where $Z'\,(t)$ and $Z''\,(t)$ are first and second derivation of $R(t)$ respectively. Notation "$\times$" is referring to the outer product of two plane vectors and it is scalar [8]. The signed radius of curvature at t is the reciprocal of $\kappa(t)$. $\kappa(t)$ is represented with a positive sign when the curve segment bends to left at t and it is negative if it bends to right at t.

3 Description of Method

A cubic Alternative curve can be represented as (using equation (1) and equation (2)):

$$Z(t) = (-1+t)^2 (1+2t) P_0 + (3-2t) t^2 P_3 + (-1+t)^2 t\alpha (P_1 - P_0) + (-1+t) t^2 \beta (P_3 - P_2). \tag{4}$$

Assigning $(P_1 - P_0) = T_0$ and $(P_3 - P_2) = T_1$, so equation (4) can be rewritten as a Hermite data of two points:

$$Z(t) = (-1+t)^2 (1+2t) P_0 + (3-2t) t^2 P_3 + (-1+t)^2 t\alpha T_0 + (-1+t) t^2 \beta T_1. \tag{5}$$

Letting $P_1 = P_2 = H$ therefore $\Delta Z = P_3 - P_0$ can be represented in the terms of T_0, T_1 as $\Delta Z = T_1 + T_0$.

Hence,

$$P_3 = T_1 + T_0 + P_0 \quad . \tag{6}$$

If T_0 and T_1 are linearly independent, i.e., $T_0 \times T_1 (= \Gamma) \neq 0$, substituting equation (6) into equation (5) yields

$$Z(t) = P_0 + t\left(t(3-2\alpha) + t^2(-2+\alpha) + \alpha\right) T_0 + t^2 (3 + t(-2+\beta) - \beta) T_1. \tag{7}$$

The first and second derivatives of equation (7) are:

$$Z'(t) = (-1+t)(3t(-2+\alpha) - \alpha) T_0 + t(6 + 3t(-2+\beta) - 2\beta) T_1 \tag{8}$$

$$Z''(t) = 2(3 + 3t(-2+\alpha) - 2\alpha) T_0 + 2(3 + t(-2+\beta) - \beta) T_1. \tag{9}$$

A straight forward calculation gives:

$$Z'(t) \times Z''(t) = \quad -2\Gamma\left(t^2(3\alpha(-1+\beta) - 3\beta) - 3t\alpha(-2+\beta) - 3\alpha + \alpha\beta\right). \tag{10}$$

Let us assign

$$\Phi(t) = \left(t^2(3\alpha(-1+\beta) - 3\beta) - 3t\alpha(-2+\beta) - 3\alpha + \alpha\beta\right). \tag{11}$$

Therefore, the discriminant of quadratic function $\Phi(t)$ is given as

$$-3\alpha\beta(12 - 4(\alpha+\beta) + \alpha\beta). \tag{12}$$

For a more spontaneous and convenient analysis, we reparametrize equation (10) by using $t = \frac{u}{1+u}$, as the results $t \in [0,1]$ will be mapped onto $u \in [0,\infty)$. Hence, equation (10) becomes a polynomial function in parameter u of degree2, so we can write

$$Z'(u) \times Z''(u) = -\frac{2\Gamma}{(1+u)^2}\left(u^2\alpha(-3+\beta) - u\alpha\beta + (-3+\alpha)\beta\right). \tag{13}$$

where

$$\Phi(u) = \quad u^2\alpha(-3+\beta) - u\alpha\beta + (-3+\alpha)\beta. \tag{14}$$

Consequently, the discriminant of $\Phi(u)$ is similar as shown by equation (12).

4 Characterization of the Non-degenerate Curve

The inflection point occurs if and only if $Z'(u) \times Z''(u) = 0$. If the solutions are denoted as r, s then the roots of equation(13) are:

$$r, s = \frac{-\alpha\beta \pm \sqrt{-3\alpha\beta\left(12 - 4\left(\alpha + \beta\right) + \alpha\beta\right)}}{2\alpha\left(-3 + \beta\right)}. \tag{15}$$

If we denote $I = 12 - 4\left(\alpha + \beta\right) + \alpha\beta$, then equation (15) can be simplified as follows:

$$r, s = \frac{\alpha\beta \pm \sqrt{-3\alpha\beta I}}{2\alpha\left(-3 + \beta\right)}. \tag{16}$$

From equation (16), it is obvious that there are at most two inflection points for planar cubic curve and the existence of inflection points and the number of points are dependent on α, β and I. The analyses of the roots are as follows.

Firstly, if r, s are complex roots then $\sqrt{-3\alpha\beta I}$ is not defined as $\alpha\beta I > 0$. This leads us to two conditions; if $\alpha\beta > 0$ then $I > 0$, and if $\alpha\beta < 0$ then $I < 0$. Second, if r, s are real numbers therefore one or two inflection points exist. By investigating equation (14), the sign of u is determined.

$$\left(u^2 - \frac{\beta}{\left(-3 + \beta\right)}u + \frac{\beta\left(-3 + \alpha\right)}{\alpha\left(-3 + \beta\right)}\right) = 0 \quad \Leftrightarrow \quad u^2 + Au + B = 0 \tag{17}$$

The two negative roots imply that no inflection point exists when A and B are both positive; $0 < \beta < 3$, $0 < \alpha < 3$. One positive root implies that one inflection point exists when B is negative and A is positive; $0 < \beta < 3$, $\alpha < 0$, $I > 0$ and $0 < \beta < 3$, $\alpha > 3$,$I < 0$. One positive root implies that one inflection point exists when both B and A are negative; $0 < \alpha < 3$, $\beta < 0$, $I > 0$ and $0 < \alpha < 3$, $\beta > 3$, $I < 0$. Finally, two positive roots imply that two inflection points exist when A is negative and B is positive which leads to $\beta > 3, \quad \alpha > 3, \quad I < 0$, and $\beta > 3, \alpha < 0, \quad I > 0$, and $\alpha > 3,\ \beta < 0,\ I > 0$.

The necessary condition for the occurrence of the inflection points along with the number of inflection points of cubic Alternative curve is stated in the following theorem.

Theorem 4.1. *For $Z\left(t\right)$ with untrimmed shape parameters, $\alpha, \beta \in \mathrm{R}$, as defined by (4), the presence of inflection points for the cubic curve is characterized by the sign of $I = 12 - 4(\alpha + \beta) + \alpha\beta$ with the following cases:*

Case 1 (2 inflection points): If $\beta > 3, \alpha > 3, I < 0$, and $\beta > 3, \alpha < 0, I > 0$, and $\alpha > 3, \beta < 0, I > 0$

Case 2 (1 inflection point): If $0 < \beta < 3, \alpha < 0, I > 0$, and $0 < \beta < 3, \alpha > 3, I < 0$, and $0 < \alpha < 3, \beta < 0, I > 0$, and $0 < \alpha < 3, \beta > 3, I < 0$.

Case 3 (no inflection point): If $\alpha\beta > 0, I > 0$, and $\alpha\beta < 0, I < 0$, and $0 < \alpha, \beta < 3, I < 0$.

Figure 1 shows the region for the numbers of inflection points $N_i, i = 0, 1, 2$ on restricted cubic Alternative curve, $0 \leq t \leq 1$. The shaded regions indicate two inflection points are in complex numbers (or no inflection points, N_0).

A necessary condition for the existence of cusps is given by the vanishing of the first

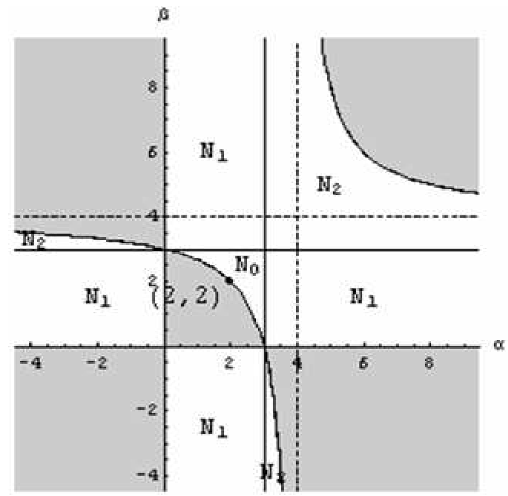

Fig 1: Region for Numbers of Inflection Points.

derivative vector $Z(t)$ in the given interval. Note that the quadratic polynomials $Z'(t) = (x'(t), y'(t))$ have the common zero(s). By using Sylvester's resultant [5] of the above quadratic equation (7), we obtain

$$R(x'(t), y'(t)) = -3\Gamma^2\alpha\beta I.$$

$-3\Gamma^2\alpha\beta I$ is equivalent to zero if and only if $x'(t)$ and $y'(t)$ have at least one common root. Hence, a cusp occurs if $I = 0$. A common zero is obtained by substituting

$$\alpha = \frac{4(-3+\beta)}{(-4+\beta)} \tag{18}$$

into equation (8). Therefore

$$t = \frac{2(-3+\beta)}{3(-2+\beta)} \tag{19}$$

where a cusp occurs at t. By reparametrizing t onto u using $t = \frac{1}{1+u}$, we get

$$u = \frac{\beta}{2(-3+\beta)}. \tag{20}$$

By analyzing equation (20) we obtain the following conditions:

$\beta < 0 \quad \Rightarrow \quad u$ is positive so there exist a cusp
$0 < \beta < 3 \quad \Rightarrow \quad u$ is negative, so no cusp
$3 < \beta < 4 \quad \Rightarrow \quad u$ is positive so there exist a cusp
$\beta > 4 \quad \Rightarrow \quad u$ is positive so there exist a cusp

Remark: Although it is said that a cusp occurs if $I = 0$ but it is not for $0 < \alpha, \beta \leq 3$. And it is clear that $Z(t)$ is a quadratic curve for $(\alpha, \beta) = (2, 2)$. This can be shown by substituting $\alpha = \beta = 2$ into equation (7).

A loop occurs if and only if there exist two roots of quadratic polynomials p and q, where $\frac{Z(p)-Z(q)}{p-q}=0$ and $p\neq q$ (from equation (7) where $Z(u)$ is defined with $u\geq 0$). It gives a homogeneous system of equations in $MT_0+NT_1=0$, where

$$M=-3p-p^2-3q-10pq-q^2-3pq^2+\left(p+q+3pq-p^2q^2\right)\alpha \tag{21}$$

$$N=-3p-p^2-3q-10pq-q^2-3pq^2+\left(-1+p^2q+pq\left(3+q\right)\right)\beta. \tag{22}$$

Since the matrix is nonsingular, we obtain $M=N=0$, and the roots are

$$p,q=\frac{\alpha\left(8+\alpha\left(-3+\beta\right)-3\beta\right)\beta\pm\left(\alpha+\beta-\alpha\beta\right)\sqrt{\alpha\beta I}}{2\alpha\left(\alpha+\left(-3+\beta\right)\beta\right)}. \tag{23}$$

The roots are defined when $\alpha\beta>0, I>0$ and $\alpha\beta<0, I<0$. We need to find the constraint of the loop by finding the relation between shape parameter if one of the intercept point occurs at $u=0$. From equation (23), we obtain

$$\left(\alpha-3\beta+\beta^2\right)\left(\beta-3\alpha+\alpha^2\right)=\quad 0 \tag{24}$$

The factors $\left(\alpha-3\beta+\beta^2\right)=0$ and $\left(\beta-3\alpha+\alpha^2\right)=0$ are parabola functions as shown in Figure 2 (represented by L_1 and L_2 respectively). The following theorem completes the investigation of the occurrence of singularities.

Theorem 4.2. *Let $Z(t)$, with untrimmed shape parameters, $\alpha,\beta\in\mathrm{R}$ be as defined by equation (4). The presence of singularity of the cubic Alternative curve is characterized by the sign of $I=12-4(\alpha+\beta)+\alpha\beta$*

Case 1 (Cusp): If $I=0$ and $\alpha,\beta\in\mathrm{R}-[0,3]$.

Case 2 (Loop): If $\alpha\beta>0, I>0$, and $\alpha<0,\beta>0, I<0, \left(\alpha-3\beta+\beta^2\right)<0$, and $\alpha>0,\beta<0, I<0, \left(\beta-3\alpha+\alpha^2\right)<0$.

Case 3 (Quadratic): If $I=0, (\alpha,\beta)=(2,2)$ no singularity, no inflection point.

5 Shape Diagram

Figure 2 represents the shape diagram of the values of shape parameters that consist in Alternative basis. Coincidently, our results (shape diagram) are similar to the shape diagram obtained in [9] for C-Bezier. The selection of the value of shape parameters to planar curves with different kind of characteristics namely convex curve, inflection point, cusp and loop are represented by the regions denoted as uppercasing letters (C, D, I, E, F, V, U, S and R).

Examples of these nine different shapes of cubic Alternative curves are shown in Figure 3. Details are as follows; (a) $(\alpha,\beta)\in$C, convex, (b) $(\alpha,\beta)\in$D, double inflections, (c) $(\alpha,\beta)\in$I (upper branch), cusp, (d) $(\alpha,\beta)\in$E, loop, (e) $(\alpha,\beta)\in$F, one inflection point, (f) $(\alpha,\beta)\in$V, convex, (g) $(\alpha,\beta)\in$U, loop, (h) $(\alpha,\beta)\in$S, one inflection point, and (i) $(\alpha,\beta)\in$R, convex.

6 Characterization of the Degenerate Curve

This section discusses on degenerate cubic which involves the parallel of the end tangent. The case when P_0, H, P_3 are collinear is omitted as this configuration yields a line segment. A cubic Alternative is given by equation (4). Let

$$P_1-P_0=\mu Z_0$$

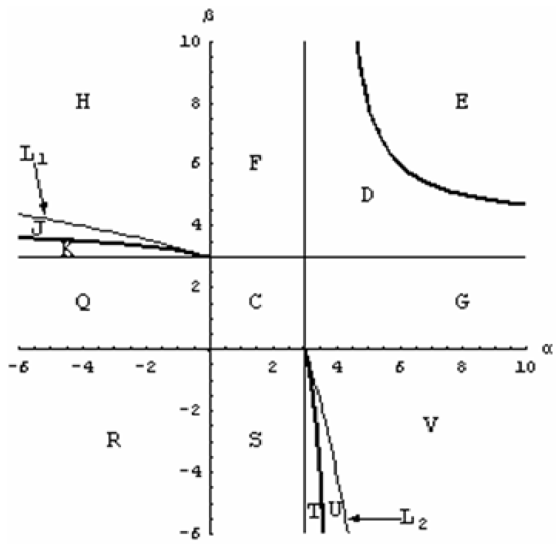

Fig. 2: Shape of Inflection and Singularity

$$P_3 - P_2 = -\nu Z_0$$
$$P_2 - P_1 = mZ_S \tag{25}$$

where $P_1 - P_0$ is parallel to $P_3 - P_2$, μ, ν, m are real arbitraries, Z_0 and Z_S denotes the unit tangent vector at the endpoint and vector unit along $P_2 - P_1$ respectively and P_3 defined in the terms of unit vector as:

$$P_3 = P_0 + (\mu\alpha - \nu\beta)\, Z_0 + mZ_S \tag{26}$$

If $a = \mu\alpha$, $b = \nu\beta$, substituting equation (25), (26) into equation (4) yields

$$Z\,(t) = P_0 + m\,(3 - 2t)\, t^2 Z_S + t\left(a\left(1 + t - t^2\right) + b\,(-2 + t)\, t\right) Z_0 \tag{27}$$

By reparametrization, t onto u, equation (27) can be rewritten as:

$$Z\,(u) = P_0 + \frac{m\,(1 + 3u)}{(1 + u)^3} Z_S + \frac{\left(-b\,(1 + 2u) + a\left(1 + 3u + u^2\right)\right)}{(1 + u)^3} Z_0 \tag{28}$$

Analogous to previous method (section non-degenerated curve), we obtain

$$Z'\,(u) \times Z''\,(u) = -\frac{6m\left(b + au^2\right)}{(1 + u)^2}\Upsilon, \tag{29}$$

where $\Upsilon = Z_0 \times Z_S$.

The inflection points occur if and only if $Z'(u) \times Z''(u) = 0$. Hence, the solutions are

$$u = \pm\frac{i\sqrt{b}}{\sqrt{a}}. \tag{30}$$

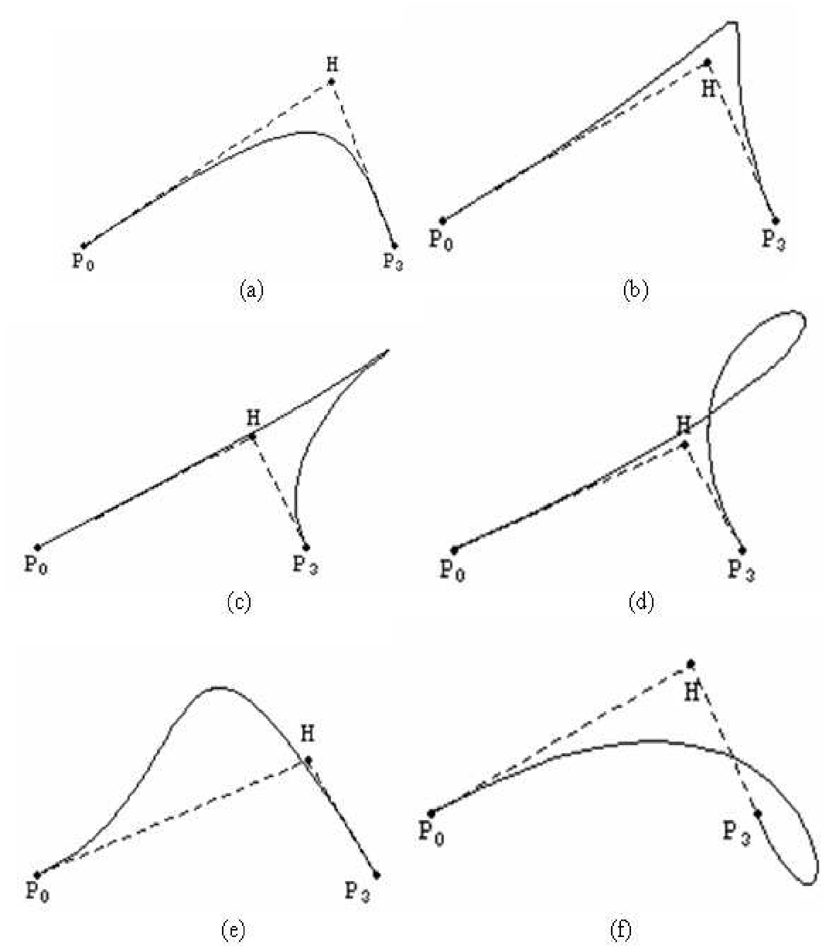

Figure 3: Examples of Cubic Alternative Curve

Observe that there is only one inflection point exist for $ab < 0$.

The cusp does not exist in this case as $Z'(t) = (x'(t), y'(t))$ yields

$$(x'(u), y'(u)) = -36abm^2\Upsilon^2 \neq 0.$$

Again, loop occurs if and only if there exist two roots of quadratic polynomials p and q, where $\frac{Z(p)-Z(q)}{p-q} = 0$ and $p \neq q$. A homogeneous system of equation in $MZ_0 + NZ_S = 0$ for this case is

$$\begin{aligned} M &= p^2\left(b + 2bq - a\left(1 + 3q + q^2\right)\right) + p\left(b\left(3 + 7q + 3q^2\right) - a\left(2 + 7q + 3q^2\right)\right) \\ &\quad - aq(2 + q) + b\left(1 + 3q + q^2\right) \\ N &= -mp^2(1 + 3q) - mp\left(3 + 10q + 3q^2\right) - mq(3 + q). \end{aligned} \tag{31}$$

A loop occurs when $M = N = 0$, therefore the roots of quadratic equations from equation (31) are

$$p, q = \frac{4ab \pm \sqrt{3}\sqrt{ab(a+b)^2}}{a(a-3b)}. \tag{32}$$

It is clear that p, q are not defined if $ab < 0$. Now let us consider $ab > 0$, such that a loop occurs if $p > 0$ and $q > 0$. By considering $(a - 3b) > 0$ from the denominator of equation(32), we can assign $a = 3\psi b$ where $\psi > 1$. Hence equation(32) can be rewritten as

$$p, q = \frac{4\psi \pm \sqrt{\psi(3\psi+1)^2}}{3\psi(\psi-1)}. \tag{33}$$

Therefore, if $\psi \to 1^+$ then $p \to \infty$, $q \to -1/3$, and if $\psi \to \infty$ we get $p \to 0$, $q \to 0$. In conclusion, since one of the roots is negative hence no loop is existed for this case.

7 Conclusion

We have successfully investigated the existence of inflection points, cusps, and loops for cubic Alternative curve by using the discriminant and reparametrization method and illustrated a simple shape diagram. To note, the conditions of shape parameters do not depend on the position of control points. As for future work, it would be interesting to find how the stated methods can be used to characterize the shapes of the curve in a more general case.

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