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On Some New Generalized Difference Double Sequence Spaces Defined By Orlicz Functions

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Abstract In this article, the author defines the generalized difference double paranormed sequence spaces $c^2(\Delta^m, M, p, q, s)$, $c_o^2(\Delta^m, M, p, q, s)$ and $l_\infty^2(\Delta^m, M, p, q, s)$ defined over a seminormed sequence space (X, q). The author also studies their properties and inclusion relations between them.

Keywords P-convergent; difference sequence; modulus function.

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1 Introduction

Let l_∞, c and c_o be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$. Kizmaz [14] introduced the notion of difference sequence spaces as follows:

$$X(\Delta) = \{x = (x_k) : \ (\Delta x_k) \in X\}$$

for $X = l_\infty$, c and c_o. Later on, the notion was generalized by Et and Çolak [15] as follows:

$$X(\Delta^m) = \{x = (x_k) : \ (\Delta^m x_k) \in X\}$$

for $X = l_\infty$, c and c_o, where $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right), \Delta^0 x = x$ and also this generalized difference notion has the following binomial representation:

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \ \text{ for all } k \in \mathbb{N}.$$

Subsequently, difference sequence spaces were studied by Esi [4], Esi and Tripathy [5], Tripathy et.al [10] and many others.

An *Orlicz function* M is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \le M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [16]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$.

Remark 1 An Orlicz function satisfies the inequality $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage, different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta [6], Esi [1,2], Esi and Et [3], Parashar and Choudhary [7] and many others.

Let w^2 denote the set of all double sequences of complex numbers. By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$ [8]. We shall describe such an $x = (x_{k,l})$ more briefly as "*$P-$convergent*". We shall denote the space of all *$P-$convergent* sequences by c^2. The double sequence $x = (x_{k,l})$ is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. We shall denote all bounded double sequences by l_∞^2.

2 Definitions and Results

In this presentation our goal is to extend a few results known in the literature from ordinary (single) difference sequences to difference double sequences. Some studies on double sequence spaces can be found in [11–13].

Definition 1 Let M be an Orlicz function and $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers and $s \geq 0$ is a real number. Let X be a seminormed space over the complex field $\mathbb{C}$ with the seminorm q. We now define the following new generalized difference sequence spaces:

$$c^2(\Delta^m, M, p, q, s) = \left\{ \begin{array}{c} x = (x_{k,l}) \in w^2 : \ P - \lim_{k,l} (kl)^{-s} \left[M\left(\frac{q(\Delta^m x_{k,l} - L)}{\rho}\right)\right]^{p_{k,l}} = 0, \\ \text{for some } \rho > 0, \ L \text{ and } s \geq 0 \end{array} \right\},$$

$$c_o^2(\Delta^m, M, p, q, s) = \left\{ \begin{array}{c} x = (x_{k,l}) \in w^2 : \ P - \lim_{k,l} (kl)^{-s} \left[M\left(\frac{q(\Delta^m x_{k,l})}{\rho}\right)\right]^{p_{k,l}} = 0, \\ \text{for some } \rho > 0 \text{ and } s \geq 0 \end{array} \right\},$$

and

$$l_\infty^2(\Delta^m, M, p, q, s) = \left\{ \begin{array}{c} x = (x_{k,l}) \in w^2 : \ \sup_{k,l} (kl)^{-s} \left[M\left(\frac{q(\Delta^m x_{k,l})}{\rho}\right)\right]^{p_{k,l}} < \infty, \\ \text{for some } \rho > 0 \text{ and } s \geq 0 \end{array} \right\},$$

where $\Delta^m x = (\Delta^m x_{k,l}) = \left(\Delta^{m-1}x_{k,l} - \Delta^{m-1}x_{k,l+1} - \Delta^{m-1}x_{k+1,l} + \Delta^{m-1}x_{k+1,l+1}\right)$, $(\Delta^1 x_{k,l}) = (\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1})$, $\Delta^0 x = (x_{k,l})$ and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{k,l} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{k+i,l+j}.$$

Some double spaces are obtained by specializing M, p, q, s and m. Here are some examples:

(i) If $M(x) = x$, $m = s = 0$, $p_{k,l} = 1$ for all $k,l \in \mathbb{N}$, and $q(x) = |x|$, then we obtain ordinary double sequence spaces c^2, c_o^2 and l_∞^2.

(ii) If $M(x) = x$, $m = s = 0$ and $q(x) = |x|$, then we obtain new double sequence spaces as follows:

$$c^2(p) = \left\{x = (x_{k,l}) \in w^2 :\ P - \lim_{k,l}\left(|x_{k,l} - L|\right)^{p_{k,l}} = 0, \text{ for some } L\right\},$$

$$c_o^2(p) = \left\{x = (x_{k,l}) \in w^2 :\ P - \lim_{k,l}\left(|x_{k,l}|\right)^{p_{k,l}} = 0\right\},$$

and

$$l_\infty^2(p) = \left\{x = (x_{k,l}) \in w^2 :\ \sup_{k,l}\left|x_{k,l}\right|^{p_{k,l}} < \infty\right\}.$$

(iii) If $m = 0$ and $q(x) = |x|$, then we obtain new double sequence spaces as follows:

$$c^2(M,p,s) = \left\{\begin{array}{c} x = (x_{k,l}) \in w^2 :\ P - \lim_{k,l}(kl)^{-s}\left[M\left(\frac{|x_{k,l}-L|}{\rho}\right)\right]^{p_{k,l}} = 0, \\ \text{for some } \rho > 0,\ L \text{ and } s \geq 0 \end{array}\right\},$$

$$c_o^2(M,p,s) = \left\{\begin{array}{c} x = (x_{k,l}) \in w^2 :\ P - \lim_{k,l}(kl)^{-s}\left[M\left(\frac{|x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} = 0, \\ \text{for some } \rho > 0 \text{ and } s \geq 0 \end{array}\right\},$$

and

$$l_\infty^2(M,p,s) = \left\{\begin{array}{c} x = (x_{k,l}) \in w^2 :\ \sup_{k,l}(kl)^{-s}\left[M\left(\frac{|x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty, \\ \text{for some } \rho > 0 \text{ and } s \geq 0 \end{array}\right\},$$

(iv) If $m = 1$ and $q(x) = |x|$, then we obtain new double sequence spaces as follows:

$$c^2(\Delta,M,p,s) = \left\{\begin{array}{c} x = (x_{k,l}) \in w^2 :\ P - \lim_{k,l}(kl)^{-s}\left[M\left(\frac{|\Delta x_{k,l}-L|}{\rho}\right)\right]^{p_{k,l}} = 0, \\ \text{for some } \rho > 0, \text{ L and } s \geq 0 \end{array}\right\},$$

$$c_o^2(\Delta,M,p,s) = \left\{\begin{array}{c} x = (x_{k,l}) \in w^2 :\ P - \lim_{k,l}(kl)^{-s}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} = 0, \\ \text{for some } \rho > 0 \text{ and } s \geq 0 \end{array}\right\},$$

and

$$l_\infty^2(\Delta,M,p,s) = \left\{\begin{array}{c} x = (x_{k,l}) :\ \sup_{k,l}(kl)^{-s}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty, \\ \text{for some } \rho > 0 \text{ and } s \geq 0 \end{array}\right\},$$

where $(\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1})$.

3 Main Results

Theorem 1 *Let $p = (p_{k,l})$ be bounded. The classes of $c^2(\Delta^m, M, p, q, s)$, $c_o^2(\Delta^m, M, p, q, s)$ and $l_\infty^2(\Delta^m, M, p, q, s)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof We give the proof only $l_\infty^2(\Delta^m, M, p, q, s)$. The others can be treated similarly. Let $x = (x_{k,l})$, $y = (y_{k,l}) \in l_\infty^2(\Delta^m, M, p, q, s)$. Then we have

$$\sup_{k,l} (kl)^{-s} \left[M\left(\frac{q(\Delta^m x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} < \infty, \text{ for some } \rho_1 > 0 \text{ and } s \geq 0 \tag{1}$$

and

$$\sup_{k,l} (kl)^{-s} \left[M\left(\frac{q(\Delta^m y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}} < \infty, \text{ for some } \rho_2 > 0 \text{ and } s \geq 0. \tag{2}$$

Let $\alpha, \beta \in \mathbb{C}$ be scalars and $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing convex function, we have

$$\begin{aligned}\left[M\left(\frac{q(\Delta^m(\alpha x_{k,l} + \beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}} &\leq D\left\{\left[M\left(\frac{q(\Delta^m x_{k,l})}{2\rho_1}\right)\right]^{p_{k,l}} + \left[M\left(\frac{q(\Delta^m y_{k,l})}{2\rho_2}\right)\right]^{p_{k,l}}\right\} \\ &\leq D\left\{\left[M\left(\frac{q(\Delta^m x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} + \left[M\left(\frac{q(\Delta^m y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}\right\},\end{aligned}$$

where $D = \max(1, 2^H)$, $H = \sup_{k,l} p_{k,l} < \infty$. Now, from (1) and (2), we have

$$\sup_{k,l} (kl)^{-s} \left[M\left(\frac{q(\Delta^m(\alpha x_{k,l} + \beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}} < \infty.$$

Therefore $\alpha x + \beta y \in l_\infty^2(\Delta^m, M, p, q, s)$. Hence $l_\infty^2(\Delta^m, M, p, q, s)$ is a linear space.

Theorem 2 *The double sequence spaces $c^2(\Delta^m, M, p, q, s)$, $c_o^2(\Delta^m, M, p, q, s)$ and $l_\infty^2(\Delta^m, M, p, q, s)$ are seminormed spaces, seminormed by*

$$f((x_{k,l})) = \sum_{k=1}^{m} q(x_{k,1}) + \sum_{l=1}^{m} q(x_{1,l}) + \inf\left\{\rho > 0 : \; sup_{k,l} M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right) \leq 1\right\}.$$

Proof Since q is a seminorm, so we have $f((x_{k,l})) \geq 0$ for all $x = (x_{k,l})$; $f(\theta^2) = 0$ and $f((\lambda x_{k,l})) = |\lambda| f((x_{k,l}))$ for all scalars λ.

Now, let $x = (x_{k,l})$, $y = (y_{k,l}) \in c_o^2(\Delta^m, M, p, q, s)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k,l} M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_1}\right)\right) \leq 1 \text{ and } \sup_{k,l} M\left(q\left(\frac{\Delta^m y_{k,l}}{\rho_2}\right)\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have,

$$\begin{aligned}\sup_{k,l} M\left(q\left(\frac{\Delta^m(x_{k,l} + y_{k,l})}{\rho}\right)\right) &\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \sup_{k,l} M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_1}\right)\right) \\ &\quad + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right) \sup_{k,l} M\left(q\left(\frac{\Delta^m y_{k,l}}{\rho_2}\right)\right) \leq 1.\end{aligned}$$

Since $\rho_1,\ \rho_2 > 0$, so we have

$$\begin{aligned}
f\left((x_{k,l})+(y_{k,l})\right) =& \sum_{k=1}^{m} q\left(x_{k,1}+y_{k,1}\right)+\sum_{l=1}^{m} q\left(x_{1,l}+y_{1,l}\right) \\
&+\inf\left\{\rho=\rho_1+\rho_2>0: \sup_{k,l} M\left(q\left(\frac{\Delta^m\left(x_{k,l}+y_{k,l}\right)}{\rho}\right)\right)\leq 1\right\} \\
\leq& \sum_{k=1}^{m} q\left(x_{k,1}\right)+\sum_{l=1}^{m} q\left(x_{1,l}\right)+\inf\left\{\rho_1>0: \sup_{k,l} M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_1}\right)\right)\leq 1\right\} \\
&+\sum_{k=1}^{m} q\left(y_{k,1}\right)+\sum_{l=1}^{m} q\left(y_{1,l}\right)+\inf\left\{\rho_2>0: \sup_{k,l} M\left(q\left(\frac{\Delta^m y_{k,l}}{\rho_2}\right)\right)\leq 1\right\} \\
=& f\left((x_{k,l})\right)+f\left((y_{k,l})\right).
\end{aligned}$$

Therefore f is a seminorm.

Theorem 3 *Let (X,q) be a complete seminormed space. Then the spaces $c^2\left(\Delta^m, M, p, q, s\right)$, $c_o^2\left(\Delta^m, M, p, q, s\right)$ and $l_\infty^2\left(\Delta^m, M, p, q, s\right)$ are complete seminormed spaces seminormed by f.*

Proof We prove the theorem for the space $c_o^2\left(\Delta^m, M, p, q, s\right)$. The other cases can be establish following similar technique.

Let $x^i=\left(x^i_{k,l}\right)$ be a Cauchy sequence in $c_o^2\left(\Delta^m, M, p, q, s\right)$. Let $\varepsilon>0$ be given and for $r>0$, choose x_o fixed such that $M\left(\frac{rx_o}{2}\right)\geq 1$ and there exists $m_o\in\mathbb{N}$ such that

$$f\left(\left(x^i_{k,l}-x^j_{k,l}\right)\right)<\frac{\varepsilon}{rx_o},\ \text{for all } i,j\geq m_o$$

By definition of seminorm, we have

$$\sum_{k=1}^{m} q\left(x^i_{k,1}\right)+\sum_{l=1}^{m} q\left(x^j_{1,l}\right)+\inf\left\{\rho>0:\ \sup_{k,l} M\left(q\left(\frac{\Delta^m x^i_{k,l}-\Delta^r x^j_{k,l}}{\rho}\right)\right)\leq 1\right\}<\frac{\varepsilon}{rx_o} \quad (3)$$

This shows that $q\left(x^i_{k,1}\right)$ and $q\left(x^j_{1,l}\right)$ $(k,l\leq r)$ are Cauchy sequences in (X,q). Since (X,q) is complete, so there exists $x_{k,1},x_{1,l}\in X$ such that

$$\lim_{i\to\infty} q\left(x^i_{k,1}\right)=x_{k,1} \text{ and } \lim_{j\to\infty} q\left(x^j_{1,l}\right)=x_{1,l} \quad (k,l\leq m).$$

Now from (3), we have

$$M\left(q\left(\frac{\Delta^m\left(x^i_{k,l}-x^j_{k,l}\right)}{f\left(\left(x^i_{k,l}-x^j_{k,l}\right)\right)}\right)\right)\leq 1\leq M\left(\frac{rx_o}{2}\right),\ \text{for all } i,j\geq m_o. \quad (4)$$

This implies

$$q\left(\Delta^m\left(x^i_{k,l}-x^j_{k,l}\right)\right)\leq\frac{rx_o}{2}.\frac{\varepsilon}{rx_o}=\frac{\varepsilon}{2},\ \text{for all } i,j\geq m_o.$$

So, $q\left(\Delta^m\left(x^i_{k,l}\right)\right)$ is a Cauchy sequence in (X,q). Since (X,q) is complete, there exists $x_{k,l}\in X$ such that $\lim_i \Delta^m\left(x^i_{k,l}\right)=x_{k,l}$ for all $k,l\in\mathbb{N}$. Since M is continuous, so for $i\geq m_o$, on taking limit as $j\to\infty$, we have from (4),

$$M\left(q\left(\frac{\Delta^m\left(x^i_{k,l}\right)-\lim_{j\to\infty}\Delta^r x^j_{k,l}}{\rho}\right)\right)\leq 1\Rightarrow M\left(q\left(\frac{\Delta^m\left(x^i_{k,l}\right)-x_{k,l}}{\rho}\right)\right)\leq 1.$$

On taking the infimum of such $\rho's$, we have

$$f\left(\left(x^i_{k,l}-x_{k,l}\right)\right)<\varepsilon,\text{ for all } i\geq m_o.$$

Thus $\left(x^i_{k,l}-x_{k,l}\right)\in c^2_o\left(\Delta^m,M,p,q,s\right)$. By linearity of the space $c^2_o\left(\Delta^m,M,p,q,s\right)$, we have for all $i\geq m_o$,

$$\left(x_{k,l}\right)=\left(x^i_{k,l}\right)-\left(x^i_{k,l}-x_{k,l}\right)\in c^2_o\left(\Delta^m,M,p,q,s\right).$$

Thus $c^2_o\left(\Delta^m,M,p,q,s\right)$ is a complete space.

Proposition 1 **(a)** $c^2\left(\Delta^m,M,p,q,s\right)\subset l^2_\infty\left(\Delta^m,M,p,q,s\right)$,
(b) $c^2_o\left(\Delta^m,M,p,q,s\right)\subset l^2_\infty\left(\Delta^m,M,p,q,s\right)$.
The inclusions are strict.

Proof It is easy, so omitted.

To show that the inclusions are strict, consider the following example.

Example 1 Let $M(x)=x^p$, $p\geq 1$, $m=1$, $s=0$, $q(x)=|x|$, $p_{k,l}=2$ for all $k,\ l\in\mathbb{N}$ and consider the double sequence

$$x_{k,l}=\begin{cases}0 & , \text{ if } k+l \text{ is odd}\\ k & , \text{ otherwise}\end{cases}$$

Then

$$\Delta^m x_{k,l}=\begin{cases}2k+1 & , \text{ if } k+l \text{ is even}\\ -2k-1 & , \text{ otherwise}\end{cases}.$$

Here $x=\left(x_{k,l}\right)\in l^2_\infty\left(\Delta^m,M,p,q,s\right)$, but $x=\left(x_{k,l}\right)\notin c^2\left(\Delta^m,M,p,q,s\right)$.

Theorem 4 *The double spaces* $c^2\left(\Delta^m,M,p,q,s\right)$ *and* $c^2_o\left(\Delta^m,M,p,q,s\right)$ *are nowhere dense subsets of* $l^2_\infty\left(\Delta^m,M,p,q,s\right)$.

Proof The proof is obvious in view of Theorem 3 and Proposition 1.

Theorem 5 *Let* $m\geq 1$, *then for all* $0<i\leq m$, $Z^2\left(\Delta^i,M,p,q,s\right)\subset Z^2\left(\Delta^m,M,p,q,s\right)$, *where* $Z^2=c^2$, c^2_o *and* l^2_∞. *The inclusions are strict.*

Proof We establish it for only $c_o^2\left(\Delta^{m-1}, M, p, q, s\right) \subset c_o^2\left(\Delta^m, M, p, q, s\right)$. Let $x=(x_{k,l}) \in c_o^2\left(\Delta^{m-1}, M, p, q, s\right)$. Then

$$P-\lim_{k,l}(kl)^{-s}\left[M\left(\frac{q\left(\Delta^{m-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}}=0, \text{ for some } \rho>0 \text{ and } s\geq 0 \tag{5}$$

Thus from (5) we have

$$P-\lim_{k,l}(kl)^{-s}\left[M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l+1}}=0,$$

$$P-\lim_{k,l}(kl)^{-s}\left[M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k+1,l}}=0,$$

and

$$P-\lim_{k,l}(kl)^{-s}\left[M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k+1,l+1}}=0.$$

Now for

$$\Delta^m x=(\Delta^m x_{k,l})=\left(\Delta^{m-1}x_{k,l}-\Delta^{m-1}x_{k,l+1}-\Delta^{m-1}x_{k+1,l}+\Delta^{m-1}x_{k+1,l+1}\right),$$

we have

$$\begin{aligned}
&(kl)^{-s}\left[M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}\\
&\leq (kl)^{-s}\left[M\left(q\left(\frac{\Delta^{m-1}x_{k,l}}{\rho}\right)+q\left(\frac{\Delta^{m-1}x_{k,l+1}}{\rho}\right)\right.\right.\\
&\quad\left.\left.+q\left(\frac{\Delta^{m-1}x_{k+1,l}}{\rho}\right)+q\left(\frac{\Delta^{m-1}x_{k+1,l+1}}{\rho}\right)\right)\right]^{p_{k,l}}\\
&\leq D^2\,(kl)^{-s}\left\{\left[M\left(q\left(\frac{\Delta^{m-1}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}+\left[M\left(q\left(\frac{\Delta^{m-1}x_{k+1,l}}{\rho}\right)\right)\right]^{p_{k,l}}\right.\\
&\quad\left.+\left[M\left(q\left(\frac{\Delta^{m-1}x_{k,l+1}}{\rho}\right)\right)\right]^{p_{k,l}}+\left[M\left(q\left(\frac{\Delta^{m-1}x_{k+1,l+1}}{\rho}\right)\right)\right]^{p_{k,l}}\right\}\\
&\leq D^2\left\{\left[(kl)^{-s}M\left(q\left(\frac{\Delta^{m-1}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}+\left[(kl)^{-s}M\left(q\left(\frac{\Delta^{m-1}x_{k+1,l}}{\rho}\right)\right)\right]^{p_{k+1,l}}\right.\\
&\quad\left.+\left[(kl)^{-s}M\left(q\left(\frac{\Delta^{m-1}x_{k,l+1}}{\rho}\right)\right)\right]^{p_{k,l+1}}+\left[(kl)^{-s}M\left(q\left(\frac{\Delta^{m-1}x_{k+1,l+1}}{\rho}\right)\right)\right]^{p_{k+1,l+1}}\right\}
\end{aligned}$$

from which it follows that $x=(x_{k,l}) \in c_o^2\left(\Delta^m, M, p, q, s\right)$ and hence $c_o^2\left(\Delta^{m-1}, M, p, q, s\right) \subset c_o^2\left(\Delta^m, M, p, q, s\right)$. On applying the principle of induction, it follows that $c_o^2\left(\Delta^i, M, p, q, s\right) \subset c_o^2\left(\Delta^m, M, p, q, s\right)$ for $i=0,1,2,...,m-1$. The proof for the rest cases are similar. To show that the inclusions are strict, consider the following example.

Example 2 Let $M(x)=x^p$, $s=0$, $m=1$, $q(x)=|x|$, $p_{k,l}=1$ for all k odd and for all $l \in \mathbb{N}$ and $p_{k,l}=2$ otherwise. Consider the sequence $x=(x_{k,l})$ defined by $x_{k,l}=k+l$ for all $k,l \in \mathbb{N}$. We have $\Delta^m x_{k,l}=0$ for all $k,l \in \mathbb{N}$. Hence $x=(x_{k,l}) \in c_o^2\left(\Delta, M, p, q, s\right)$ but $x=(x_{k,l}) \notin c_o^2\left(\Delta^m, M, p, q, s\right)$.

Theorem 6 ***(a)*** *If* $0 < \inf_{k,l} p_{k,l} \leq p_{k,l} < 1$, *then* $Z^2(\Delta^m, M, p, q, s) \subset Z^2(\Delta^m, M, q, s)$,
(b) *If* $1 < p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$, *then* $Z^2(\Delta^m, M, q, s) \subset Z^2(\Delta^m, M, p, q, s)$, *where* $Z^2 = c^2$, c_o^2 *and* l_∞^2.

Proof The first part of the result follows from the inequality

$$(kl)^{-s} M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right) \leq (kl)^{-s}\left[M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}$$

and the second part of the result follows from the inequality

$$(kl)^{-s}\left[M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}} \leq (kl)^{-s} M\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right).$$

Theorem 7 *Let* M_1 *and* M_2 *be Orlicz functions satisfying* Δ_2*-condition. If* $\beta = \lim_{t\to\infty} \frac{M_2(t)}{t} \geq 1$, *then* $Z^2(\Delta^m, M_1, p, q, s) = Z^2(\Delta^m, M_2 o M_1, p, q, s)$, *where* $Z^2 = c^2$, c_o^2 *and* l_∞^2.

Proof We prove it for $Z^2 = c^2$ and the other cases will follows on applying similar techniques. Let $x = (x_k) \in c^2(\Delta^m, M_1, p, q, s)$, then

$$P - \lim_{k,l} (kl)^{-s}\left[M_1\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}} = 0.$$

Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $M_2(t) < \varepsilon$ for $0 \leq t < \delta$. Let

$$y_{k,l} = M_1\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)$$

and consider

$$[M_2(y_{k,l})]^{p_{k,l}} = [M_2(y_{k,l})]^{p_{k,l}} + [M_2(y_{k,l})]^{p_{k,l}} \tag{6}$$

where the first term is over $y_{k,l} \leq \delta$ and the second is over $y_{k,l} > \delta$. From the first term in (6), using the Remark

$$(kl)^{-s}[M_2(y_{k,l})]^{p_{k,l}} < (kl)^{-s}[M_2(2)]^H [(y_{k,l})]^{p_{k,l}} \tag{7}$$

On the other hand, we use the fact that

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}.$$

Since M_2 is non-decreasing and convex, it follows that

$$M_2(y_{k,l}) < M_2\left(1 + \frac{y_{k,l}}{\delta}\right) < \frac{1}{2}M_2(2) + \frac{1}{2}M_2\left(\frac{2y_{k,l}}{\delta}\right).$$

Since M_2 satisfies Δ_2-condition, we have

$$M_2(y_{k,l}) < \frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2) = K\frac{y_{k,l}}{\delta}M_2(2).$$

Hence, from the second term in (6), it follows that

$$(kl)^{-s}\left[M_2\left(y_{k,l}\right)\right]^{p_{k,l}} \leq \max\left(1, \left(KM_2\left(2\right)\delta^{-1}\right)^H\right)(kl)^{-s}\left[\left(y_{k,l}\right)\right]^{p_{k,l}} \tag{8}$$

By the inequalities (7) and (8), taking limit in the Pringsheim sense, we have $x = (x_k) \in c^2\left(\Delta^m, M_2 o M_1, p, q, s\right)$. Observe that in this part of the proof we did not need $\beta \geq 1$. Now, let $\beta \geq 1$ and $x = (x_k) \in c^2\left(M_1, \Delta^r, q, p\right)$. Since $\beta \geq 1$ we have $M_2(t) \geq \beta t$ for all $t \geq 0$. It follows that $x = (x_k) \in c^2\left(\Delta^m, M_2 o M_1, p, q, s\right)$ implies $x = (x_k) \in c^2\left(\Delta^m, M_1, p, q, s\right)$. This implies $c^2\left(\Delta^m, M_2 o M_1, p, q, s\right) = c^2\left(\Delta^m, M_1, p, q, s\right)$.

Theorem 8 *Let M, M_1 and M_2 be Orlicz functions, q, q_1 and q_2 be seminorms and s, s_1 and s_2 be positive real numbers. Then*

(i) $Z^2\left(\Delta^m, M_1, p, q, s\right) \cap Z^2\left(\Delta^m, M_2, p, q, s\right) \subset Z^2\left(\Delta^m, M_1 + M_2, p, q, s\right)$,

(ii) $Z^2\left(\Delta^m, M, p, q_1, s\right) \cap Z^2\left(\Delta^m, M, p, q_2, s\right) \subset Z^2\left(\Delta^m, M, p, q_1 + q_2, s\right)$,

(iii) *If q_1 is stronger than q_2, then* $Z^2\left(\Delta^m, M, p, q_1, s\right) \subset Z^2\left(\Delta^m, M, p, q_2, s\right)$,

(iv) *If $s_1 \leq s_2$, then* $Z^2\left(\Delta^m, M, p, q, s_1\right) \subset Z^2\left(\Delta^m, M, p, q, s_2\right)$,

where $Z^2 = l^2_\infty, c^2$ and c^2_o.

Proof **(i)** We establish it for only $Z^2 = c^2_o$. The rest cases are similar. Let $x = (x_{k,l}) \in c^2_o\left(\Delta^m, M_1, p, q, s\right) \cap c^2_o\left(\Delta^m, M_1, p, q, s\right)$. Then

$$P - \lim_{k,l}(kl)^{-s}\left[M_1\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_1}\right)\right)\right]^{p_{k,l}} = 0 \text{ for some } \rho_1 > 0,$$

$$P - \lim_{k,l}(kl)^{-s}\left[M_2\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_2}\right)\right)\right]^{p_{k,l}} = 0 \text{ for some } \rho_2 > 0.$$

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the following inequality

$$(kl)^{-s}\left[(M_1 + M_2)\left(q\left(\frac{\Delta^m x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}$$

$$\leq D\left\{(kl)^{-s}\left[M_1\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_1}\right)\right)\right]^{p_{k,l}} + (kl)^{-s}\left[M_2\left(q\left(\frac{\Delta^m x_{k,l}}{\rho_2}\right)\right)\right]^{p_{k,l}}\right\}.$$

The proofs of (ii), (iii) and (iv) follow obviously.

The proof of the following result is also routine work.

Proposition 2 For any Orlicz function, if $q_1 \approxeq$ (equivalent to) q_2, then $Z^2\left(\Delta^m, M, p, q_1, s\right) = Z^2\left(\Delta^m, M, p, q_2, s\right)$, where $Z^2 = l^2_\infty$, c^2 and c^2_o.

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