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Positivity Preserving Interpolation of Positive Data by Cubic Trigonometric Spline

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Abstract For smooth and visualizing pleasant curve, we construct a positivity preserving interpolation by using alternative spline that is, cubic trigonometric spline with two shape parameters. In the description of the cubic trigonometric spline interpolant, positivity is preserved everywhere and has a unique representation for the positivity. We develop a constraints on the shape parameters to preserve the shape of the positive data in the view of smooth and pleasant positive curve by trigonometric interpolant. The degree of smoothness of the under discussion scheme is C^1.

Keywords Cubic Trigonometric Spline; Shape Preservation; Interpolation; Positivity; Positive Data; Shape parameters.

2010 Mathematics Subject Classification 65D05, 65D07, 65D17, 65D18, 65U07.

1 Introduction

Recently, many authors have been developed shape preserving interpolations of positive data for curves and surfaces because positivity preserving is a significance attribute of the shape. Different spline functions have been developed such as Natural Spline, Beta Spline, B-Spline, Trigonometric Spline, and Rational Spline for curves and surfaces in Computer Aided Geometric Design (CAGD), Computer Graphics (CG), Numerical Analysis, Computer Aided Design and Geometric Modeling. In this paper, we use a trigonometric spline in the description of positivity preserving of positive data. The significance of trigonometric Spline in other areas, such as electronic and medicine is acknowledged in Hoschek and Lasser [1]. For a long time, trigonometric splines have not acknowledged much interest in Computer Aided Geometric Design (CAGD). However, recently trigonometric polynomials and trigonometric splines have gained some interest within Computer Aided Geometric Design, especially in curve design. Schoenberg [3] has introduced the trigonometric B-Spline in 1964. The cubic trigonometric Bezier curve with two shape parameters introduced by Xi-An Han et al. [4], that is similar to the Cubic Bezier curve, if we put suitable conditions on shape parameters.

The development of interpolating and approximating schemes for shape preserving and shape control is a germane area of research in Computer Graphics and Data Visualization. Many researchers have worked in the shape preservation area. They started to build up models with geometric properties by using highly degree curves and surface in special Computer tasks. In generally speaking, many physical circumstances exist in everyday life where entities have only positive values. For instance, the presentation of probability distribution always is positive, the samples of population are always present in positive figure. During

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the chemical experiments gas discharge is also good example of positive data. Finally the importance of positive interpolant is always provided computationally very economical cost.

This paper will examined the problems of shape preserving of positive data set. For interpolating of a positive data, the cubic Hermite interpolation scheme ([2], pages. 102-106) is not better because the positive data in Table 1 is interpolated by that scheme, the curve looks like not positive as shown in Hussain et al. [5] of Figures 1 and Figures 2. To reduce this difficulty the authors use the rational function with four shape parameters. They preserved the positivity of positive data very well and looking very smooth as shown in the Figure 4 and Figure 6.

In this paper, we use the alternative cubic Trigonometric spline for development of positivity preserving interpolation with two families of the shape parameters. The trigonometric interpolant has many advantages aspects. It is easy to implement for 2D data and to compute very easily due to non rational form of the function. It constructs C^1 interpolant. There is no additional points are inserted. The scheme developed in this paper does not constrain interval length like in positivity preserving by cubic rational function. The developed scheme is applicable when the derivative is estimated by estimation technique which was given in Hussain [6]. The scheme of this paper is computationally less expensive than schemes in Hussain and Sarfraz [5], Sarfraz et al. [6] or Hussain et al. [7], by using piecewise cubic rational function for the same purpose.

The discussion in this paper is arranged as follow: the definition of cubic trigonometric function is rewritten in section 2. The determination of derivatives are discussed in Section 3. The positivity preserving trigonometric interpolation given in section 4. Some Numerical examples and conclusion of the paper are discussed in section 5 and section 6 respectively.

2 Cubic Trigonometric Function

In this section, the cubic trigonometric Bezier function with two shape parameters borrowed from Han, Xi-An et al. [4].The cubic trigonometric function with two shape parameters provides a guarantee to user for preservation of positivity of positive data if suitable constraints has developed on shape parameters. Let $\{(x_i, f_i), i = 0, 1, 2, 3, ...n\}$ be the given set of positive data points defined over the interval [a, b], where $a = x_0 < x_1 < x_2 <, ..., < x_n = b$. The C^1 cubic trigonometric function with two shape parameters u_i, v_i are defined over each sub interval $I_i = [x_i, x_{i+1}]$, i=0, 1, 2, 3,..., n-1 as:

$$S(x) \equiv S_i(x) = \sum_{i=0}^{3} P_i B_i \tag{1}$$

Where $B_i, i = 0, 1, 2, 3$ be the cubic trigonometric basis function [4] of the interpolant which can be define as:

$$\begin{aligned}
B_0(x) &= (1 - \sin\tfrac{\pi}{2}\theta)^2(1 - u_i \sin\tfrac{\pi}{2}\theta)\\
B_1(x) &= \sin\tfrac{\pi}{2}\theta(1 - \sin\tfrac{\pi}{2}\theta)(2 + u_i(1 - \sin\tfrac{\pi}{2}\theta))\\
B_2(x) &= \cos\tfrac{\pi}{2}\theta(1 - \cos\tfrac{\pi}{2}\theta)(2 + v_i(1 - \cos\tfrac{\pi}{2}\theta))\\
B_3(x) &= (1 - \cos\tfrac{\pi}{2}\theta)^2(1 - v_i \cos\tfrac{\pi}{2}\theta)
\end{aligned} \tag{2}$$

and

$$
\begin{aligned}
P_0 &= f_i \\
P_1 &= \frac{\pi f_i(2+u_i)+2h_i d_i}{\pi(2+u_i)} \\
P_2 &= \frac{\pi f_{i+1}(2+v_i)-2h_i d_{i+1}}{\pi(2+v_i)} \\
P_3 &= f_{i+1}
\end{aligned}
\tag{3}
$$

where $\theta = (x - x_i)/h_i \quad 0 \le \theta \le 1$ and $h_i = x_{i+1} - x_i$.
The $u_i, v_i > 0$ are shape parameters and P_0, P_1, P_2 and P_3 are control points.
The Equation (1) gives the guarantee of C^1 continuity, if the following conditions are satisfied:

$$S(x_i) = f_i \;, S(x_{i+1}) = f_{i+1} \;, S'(x_i) = d_i \;\text{ and }\; S'(x_{i+1}) = d_{i+1}, \tag{4}$$

where $S'(x)$ denotes the derivative with respect to x and d_i denotes derivative values estimated or given. The u_i, v_i are shape parameters preserved the positivity of the trigonometric interpolation with suitable constraints. If $u_i = v_i = 0$, then the basis functions reduce to Quadratic trigonometric polynomials.

3 Determination of Derivatives

The derivative values or parameters $\{d_i\}$ are determined either from the give data set (x_i, f_i), $i = 1, 2, 3, \ldots, n$ or by some other means (mathematically) because some time they are not give. In this paper, they are determined from the given data in such a way that the smoothness of the trigonometric interpolant (1) is maintained. These methods are the approximation based on various mathematical theories. The descriptions of such approximations are as follow:

3.1 Arithmetic Mean Method

This method is the three point difference approximation and borrowed from [6] which can be define as:

$$
\begin{aligned}
d_0 &= \Delta_0 + (\Delta_0 - \Delta_1)\frac{h_0}{h_0 + h_1} \\
d_1 &= \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\tfrac{h_{n-1}}{h_{n-1}+h_{n-2}} \\
d_i &= \frac{\Delta_i + \Delta_{i-1}}{2}, \quad i = 2, 3, 4, 5, \ldots, n-1
\end{aligned}
$$

where $\Delta_i = (f_{i+1} - f_i)/h_i$.

4 Positivity Preserving Trigonometric Spline Interpolation

Positivity is very significant aspect of the shape in the Computer Graphics (CG) and Computer Aided Geometric Design (CAGD). The cubic hermite interpolation discussed in [2] has many deficiencies to interpolate the positive data. As far as the positivity of positive data is concerned. For instance, it is clearly shown from the Figure 1 and Figure 2 that the curves does not preserve the shape of positive data. To remove this unwanted behavior which can be observed in Figure 1 and Figure 2 of the positive data set shown in Table 1 and Table 2 taken from [7].

It is required to assign suitable values to the shape parameters. In this regard Hussain et al. [5] has developed a scheme, they assign appropriate values to the different shape parameters, then the curve preserves the positivity of positive data as shown in Figure 3, Figure 4, Figure 5 and Figure 6 of given same positive data set by using piecewise rational cubic function. The rational cubic function involves four family of parameters are arranged in such a way that two of them are constrained for presentation of positive curves through positive data while the other two provide extra freedom to the user to modify the shape of curve as desired. So we are interested to preserve the shape of same data by any other source. We are introducing a cubic trigonometric spline scheme to preserve the shape of positive data. For this purpose, it is required some mathematical treatment to achieve a shape preserving curve for positive data. The two ways are discussed next.

One way, is a game to play with shape parameters for achievement of positivity preserving trigonometric interpolation on trial and error basis, in those region where shape is not preserved. This approach may result in a required demonstration of the data but this is not a comfortable and perfect way to manipulate the desired shape preserving curve.

Secondly, this is more accurate and useful for the objective of this paper, is the automated generation of positivity preserving curve which gives the automated computation of accurate shape parameters and derivatives values. To proceed this method, some mathematical treatment is required which will be given in the following paragraph.

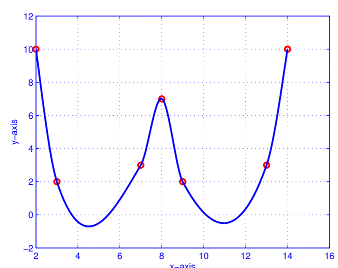

Figure 1: Cubic Hermite Spline Interpolation

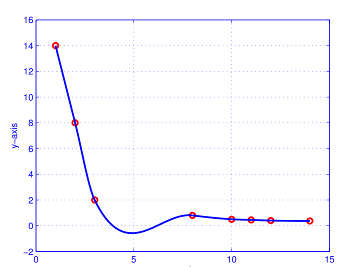

Figure 2: Cubic Hermite Spline Interpolation

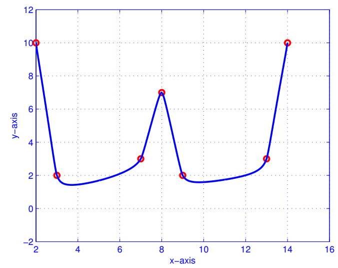

Figure 3: Positive Rational Cubic Spline [5] with $r_i = w_i = 0.1$

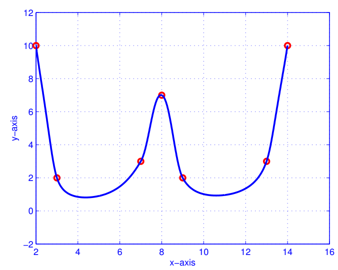

Figure 4: Positive Rational Cubic Spline [5] with $r_i = w_i = 0.5$

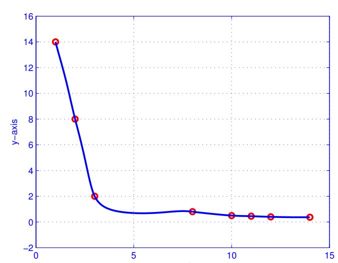

Figure 5: Positive Rational Cubic Spline [5] with $r_i = w_i = 0.5$

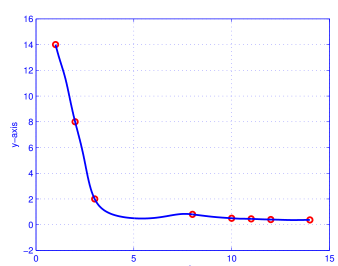

Figure 6: Positive Rational Cubic Spline [5] with $r_i = w_i = 2.0$

Let us assume positive data set for the simplicity of the presentation (x_i, f_i), $i = 1, 2, 3, \ldots, n$ so that

$$x_1 < x_2 < x_3 <, ..., < x_n \tag{5}$$

and

$$f_i > 0, i = 1, 2, 3, ..., n \tag{6}$$

In this paper, we shall develop sufficient conditions on cubic trigonometric function (1) which C^1 positive interpolant preserved. The idea, to preserve positivity using S(x), is to assign suitable automated values u_i, v_i in each interval. As $u_i, v_i > 0$, the interpolant (1) will be positive if the cubic trigonometric function gives the guarantee of positivity. The problem reduces to the determination of suitable values of u_i, v_i for which the trigonometric function is positive. So interpolant (1) can be written as

$$S(x) \equiv S_i(x) = P_0 B_0(x) + P_1 B_1(x) + P_2 B_2(x) + P_3 B_3(x)$$

where $B_i(x), i = 0, 1, 2, 3$ are Bezier-like cubic trigonometric basis function. The basis functions $B_1(x), B_2(x)$ are monotonically increasing and $B_0(x), B_4(x)$ are monotonically decreasing in this interpolant. So for the positivity of cubic trigonometric function (1), the coefficients $P_i, i = 0, 1, 2, 3$ of the trigonometric function are only positive.

It is obvious that $P_0 > 0$ and $P_3 > 0$ from (3) and if $P_1 > 0$ then it will give the following constraints on u_i

$$u_i > -\frac{2h_i d_i}{\pi f_i} \tag{7}$$

and if $P_2 > 0$ then it will give the following constraints on v_i

$$v_i > \frac{2h_i d_{i+1}}{\pi f_{i+1}}. \tag{8}$$

The results in Equations (7) and (8) can be summarized as in the following theorem.

Theorem 1 *The cubic Trigonometric function $S_i(x)$ defined over the interval $[a, b]$, is positive if in each sub interval $I_i = [x_i, x_{i+1}]$, the following sufficient conditions are satisfied*

$$u_i > \max\left\{0, -\frac{2h_i d_i}{\pi f_i}\right\}$$

and

$$v_i > \max\left\{0, \frac{2h_i d_{i+1}}{\pi f_{i+1}}\right\}.$$

The above result can also be written as

$$u_i = r_i + \max\left\{0, -\frac{2h_i d_i}{\pi f_i}\right\}, \quad r_i > 0$$

$$v_i = w_i + \max\left\{0, \frac{2h_i d_{i+1}}{\pi f_{i+1}}\right\}, \quad w_i > 0$$

5 Numerical Examples

Example 1 The positive data in Table 1 is obtained when we taken a well known volume of Sodium hydro oxide (NaOH) in a beaker then its conductivity was determined. we was added a Hydro Caloric acid (HCl) in steps (drop by drop) in the burette. After each iteration, the conductance (f) was observed. The data was obviously positive. Visualization of this positive data is shown in Figure 7 by using developed trigonometric scheme in section 4 preserves the shape of positive data. We use the trial and error method for the control of shape which showed in Figure 8 is not accurate. If we assign any negative value to the shape parameter in every piece of the interval then the shape parameter does not give a guarantee to preserve the shape of positive data as shown in Figure 9.

Table 1: Positive Data Set

i	1	2	3	4	5	6	7
x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	3	10

Example 2 The second example relates to a number of computers which were installed in University Computer Lab. Due to latest technology in the market, the computer prices were depreciated. Due to Depreciation and continuous usage of computers. The valuation of market prices was compared at different stages of time, which is displayed in Table 2. At the end of first year, the price was evaluated as \$140000 and it depreciated to \$3700 after 14 years. One can observe that all data values are positive. Visualization of this positive data is shown in Figure 10 by using developed trigonometric scheme in section 4 preserve the shape of positive data which is nicely remove the obscurity occurred by cubic hermite spline scheme. We use the trial and error method shown in Figure 11 is not accurate as compared to Figure 10. If we assign any negative value to the shape parameter in every subinterval then the shape parameter does not give a guarantee to preserve the shape of positive data as shown in Figure 12.

Table 2: Positive Data Set

i	1	2	3	4	5	6	7	8
x_i	1	2	3	8	10	11	12	14
f_i	14	8	3	0.8	0.5	0.45	0.40	0.37

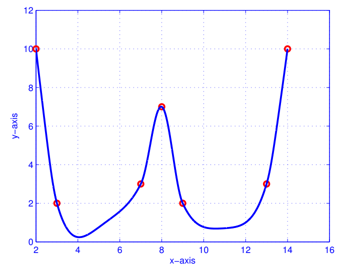

Figure 7: Cubic Trigonometric Spline Interpolation

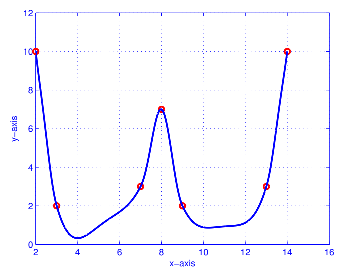

Figure 8: Trial and Error Method with $u_i = v_i = 1$

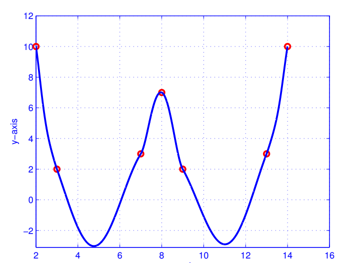

Figure 9: Trial and Error Method with $u_i = v_i = -1$

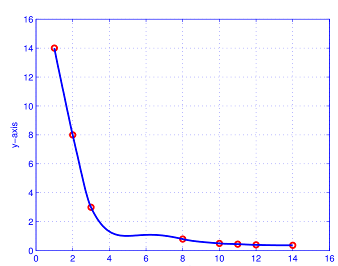

Figure 10: Cubic Trigonometric Spline Interpolation

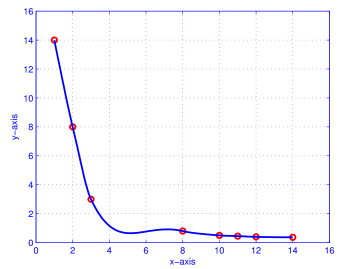

Figure 11: Trial and Error Method with $u_i = v_i = 1$

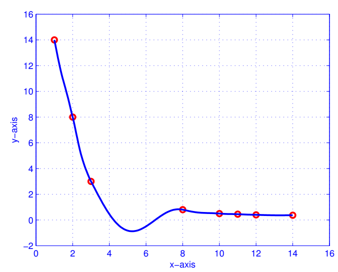

Figure 12: Trial and Error Method with $u_i = v_i = -1$

6 Conclusion

Many C^1 piecewise rational cubic interpolant with different shape parameters have been developed for the positivity preserving of positive data, to overcome the difficulty of the curve when it lost the positivity by using the cubic hermite spline scheme. In this regard, we are using C^1 the cubic trigonometric spline (in non rational form) instead of C^1 piecewise cubic rational function because that function is very easy to compute and implement for 2D data. Positivity preserving trigonometric interpolation is more economically and accurate than Hussain and Sarfraz [5]. We think that our scheme is useful in some applications, for example in font design or in surface designing. The degree of smoothness attained is C^1. Our method of manipulation is forceful.

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References

[1] Hoschek, J. and Lasser, D. *Fundamentals of Computer Aided Geometric Design, translated by L.L. Schumaker*. Massachusetts: A K Peters, Wellesley. 1993.

[2] Farin, G. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide.* 5$^{\text{th}}$ Edition. San Diego New York: Academic Press, Inc. 1993.

[3] Schoenberg, I. J. On trigonometric spline interpolation. *J. Math. Mech.* 1994. 13: 795–825.

[4] Han, Xi-An., Yi Chen M., and Huang, X. The cubic trigonometric Bzier curve with two shape parameters. *Applied Math. Letters*. 2009. 22: 226–231.

[5] Hussain, M. Z. and Sarfraz, M. Positivity preserving interpolation of positive data by rational cubics. *Journal of Comput. and applied Math.* 2008. 218: 446–458.

[6] Sarfraz, M., Hussain, M. Z. and Nasar, A. Positive data modeling using spline function. *Applied Mathematics and Computation*. 2010. 216: 2036-2049

[7] Hussain, M. Z. and Jamaludin, Md. Ali. Positivity preserving piecewise rational cubic interpolation. *MATEMATIKA*. 2006. 22(2): 147-153.