MATEMATIKA, 2011, Volume 27, Number 1, 59–78

A New Class of Rational Multistep Methods for the Numerical Solution of First Order Initial Value Problems

[1]Teh Yuan Ying, [2]Nazeeruddin Yaacob and [3]Norma Alias

[1]School of Quantitative Sciences, College of Arts and Sciences,
Universiti Utara Malaysia, 06010 UUM Sintok, Kedah Darul Aman, Malaysia

[2]Department of Mathematics, Faculty of Science,
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor Darul Ta'zim, Malaysia

[3]Ibnu Sina Institute for Fundamental Science Studies,
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor Darul Ta'zim, Malaysia

e-mail: [1]yuanying@uum.edu.my, [2]ny@mel.fs.utm.my, [3]norma@ibnusina.utm.my

Abstract In this paper, we have developed a new class of 2-step rational multistep methods (RMMs) in second to fifth order of accuracy. We have presented the developments of these RMMs, as well as the local truncation error and stability analysis for each RMM that we have developed. Numerical experiments have shown that all RMMs presented in this paper are suitable to solve initial value problem of various dimensions and also stiff problems.

Keywords Rational functions; rational multistep methods; initial value problems; problems whose solutions posses singularities; stiff problems.

2010 Mathematics Subject Classification 65L06, 41A20.

1 Introduction

Let us consider the initial value problem given by

$$\begin{aligned} &y' = f(x,y),\ y(a) = \eta, \\ &y,\ f(x,y) \in \mathbb{R},\ x \in [a,b] \subset \mathbb{R}, \end{aligned} \tag{1}$$

where f is assumed to satisfy all the conditions in order that (1) has a unique solution. Conventional linear multistep method given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j},$$

is based on the local representation of a polynomial of the theoretical solution to (1). If a linear multistep method was used to pursue the numerical solutions possess singularities, then it fails woefully near the singular points, [1], [2] and [3]. This is because a linear multistep method is formulated on the basis that (1) satisfies the existence and uniqueness theorem, so that polynomial interpolation can be applied quite successfully in the formulation, [3]. Therefore, a natural step would appear to be the replacement of the polynomial function for a linear multistep method, by a rational function due to its smooth behaviour in the neighbourhood of singularities, [3]. We have addressed multistep methods that based on rational interpolant as rational multistep methods or in brief as RMMs. In this paper, we have developed a new class of RMMs that based on the rational interpolants introduced by Van Niekerk [2]. The discussions of these new RMMs are presented as follows.

Suppose that we have solved (1) numerically up to a point x_n and have obtained a value y_n as an approximation of $y(x_n)$, which is the theoretical solution of (1). Following Lambert [1] and Lambert [4], we assume that no previous truncation errors have been made i.e. $y_n = y(x_n)$, we are interested in obtaining y_{n+2} as the approximation of $y(x_{n+2})$. For that purpose, we suggest an approximation to the theoretical solution $y(x_{n+2})$ of (1) given by

$$y_{n+2} = a_0 + \cfrac{a_1 h}{1 + \cfrac{a_2 h}{1 + \cfrac{a_3 h}{1 + \cdots \atop {\vdots \atop \frac{a_k h}{1 + a_{k+1} h}}}}}, \tag{2}$$

where a_i for $i = 0, 1, \ldots, k, k+1$ are parameters that may contain approximations of $y(x_n)$ and higher derivatives of $y(x_n)$.

RMM (2) is defined as 2-step p-th order RMM2 or in brief as RMM2(2,p) with $p = 2, 3, \ldots$. Before establishing the difference operator for (2), we need to simplify the right-hand side of (2). The simplified version of (2) can be written in the form of

$$y_{n+2} = a_0 + \frac{P(a_j, h)}{Q(a_j, h)}, \tag{3}$$

where $P(a_j, h)$ and $Q(a_j, h)$ are functions that contain the parameters

$$a_j \quad \text{for} \quad j = 1, 2, \ldots, k, k+1 \ \text{ and } \ k \geq 1.$$

With the RMM2 of the form in (3), we associate the difference operator L defined by

$$L[y(x); h]_{\text{RMM2}} = (y(x+2h) - a_0) \times Q(a_j, h) - P(a_j, h), \tag{4}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $x \in [a, b] \subset \mathbb{R}$. Expanding $y(x+2h)$ as Taylor series and collecting terms in (4) gives the following general expression:

$$L[y(x); h]_{\text{RMM2}} = C_0 h^0 + C_1 h^1 + \cdots + C_k h^k + C_{k+1} h^{k+1} + \cdots. \tag{5}$$

We note that the "C" in (5) contain corresponding parameters that need to be determined in the derivation processes. Therefore, the order and local truncation errors of RMM2 based on (2) are defined as follows.

Definition 1 *The difference operator* (4) *and the associated rational multistep method* (2) *are said to be of order* $p = k+1$ *if, in* (5), $C_0 = C_1 = \cdots = C_{k+1} = 0$, $C_{k+2} \neq 0$.

Definition 2 *The local truncation error at* x_{n+2} *of* (2) *is defined to be the expression* $L[y(x_n); h]_{\text{RMM2}}$ *given by* (4), *when* $y(x_n)$ *is the theoretical solution of the initial value problem* (1) *at a point* x_n. *The local truncation error of* (2) *is then*

$$L[y(x_n); h]_{\text{RMM2}} = C_{k+2} h^{k+2} + O\left(h^{k+3}\right). \tag{6}$$

2 2-step Second Order RMM2

In order to derive a second order RMM2, we have to take $k = 1$ in (2) and express in the form of (3). Next, from (4), expand $y(x + 2h)$ into series, and the following expression is obtained:

$$\begin{aligned} &L[y(x);h]_{\text{RMM2}} \\ &= -a_0 + y(x) + h(-a_1 - a_0a_2 + a_2y(x) + 2y'(x)) + h^2(2a_2y'(x) + 2y''(x)) \\ &\quad + h^3\left(2a_2y''(x) + \tfrac{4}{3}y'''(x)\right) + O\left(h^4\right). \end{aligned} \tag{7}$$

Following Definition 1 and (5), it is readily deduced that

$$\begin{aligned} &\{C_0 = -a_0 + y(x), C_1 = -a_1 - a_0a_2 + a_2y(x) + 2y'(x), C_2 = 2a_2y'(x) + 2y''(x), \\ &C_3 = 2a_2y''(x) + \tfrac{4}{3}y'''(x)\}. \end{aligned}$$

With $C_0 = C_1 = C_2 = 0$, we obtain a system of three simultaneous equations which have the following solutions:

$$\left\{a_0 = y(x), a_1 = 2y'(x), a_2 = -\frac{y''(x)}{y'(x)}\right\}. \tag{8}$$

Substituting the parameters in (8) into C_3, we obtain

$$C_3 = -\frac{2y''(x)^2}{y'(x)} + \frac{4}{3}y'''(x). \tag{9}$$

When $y(x)$ is now taken as the theoretical solution of the initial value problem (1) at a point x_n i.e. $y(x) = y(x_n)$, (8) may be written as

$$\left\{a_0 = y_n, a_1 = 2y'_n, a_2 = -\frac{y''_n}{y'_n}\right\}, \tag{10}$$

where $y_n = y(x_n)$ and $y_n^{(m)} = y^{(m)}(x_n)$, $m = 1, 2$ by the localizing assumption. By taking $k = 1$, (2) becomes

$$y_{n+2} = a_0 + \frac{a_1h}{1 + a_2h}, 1 + a_2h \neq 0. \tag{11}$$

We indicate (11) based on (10) as RMM2(2,2) is given as

$$y_{n+2} = y_n + \frac{2h\left(y'_n\right)^2}{y'_n - hy''_n}, \tag{12}$$

provided $|y'_n| + |y''_n| \neq 0$, as to ensure that the denominator of the rational expression in (12) does not equal to zero. From Definition 2 and (9), the local truncation error (in brief as LTE) of RMM2(2,2) becomes

$$\text{LTE}_{\text{RMM2(2,2)}} = h^3\left(-\frac{2\left(y''_n\right)^2}{y'_n} + \frac{4}{3}y'''_n\right) + O\left(h^4\right), \tag{13}$$

where $y_n^{(m)} = y^{(m)}(x_n)$, $m = 1, 2, 3$ by the localizing assumption. This LTE analysis has confirmed that RMM2(2,2) is a second order method. If we apply RMM2(2,2) to the Dahlquist's test equation $y' = \lambda y$, $\mathrm{Re}(\lambda) < 0$, yielding the difference equation

$$y_{n+2} = \frac{1 + h\lambda}{1 - h\lambda} y_n. \tag{14}$$

Setting $z = h\lambda$, $y_{n+2} = \xi^2$ and $y_n = \xi^0 = 1$ in (14) yield the characteristic equation

$$\xi^2 - \frac{1+z}{1-z} = 0. \tag{15}$$

The roots of (15) are given by

$$\xi_{(15,1)} = -\frac{\sqrt{1+z}}{\sqrt{1-z}} \text{ and } \xi_{(15,2)} = \frac{\sqrt{1+z}}{\sqrt{1-z}}.$$

By taking $z = x + \mathrm{i}y$ in the roots of (15), we obtain the region of absolute stability of RMM2(2,2) as shown in Figure 1.

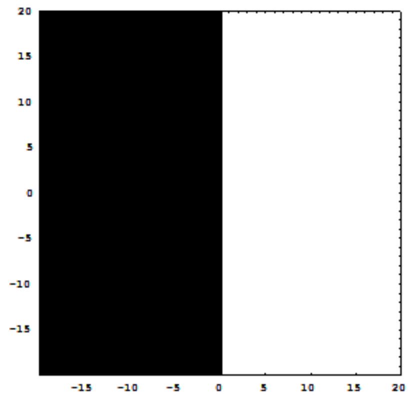

Figure 1: Stability Region of RMM2(2,2)

The shaded region in Figure 1 is the region of absolute stability of RMM2(2,2), where the conditions: $\left|\xi_{(15,1)}\right| \leq 1$ and $\left|\xi_{(15,2)}\right| \leq 1$ are satisfied. From Figure 1, we can see that the region of absolute stability of RMM2(2,2) contains the whole left-hand half plane, which show that RMM2(2,2) is A-stable.

3 2-step Third Order RMM2

In order to derive a third order RMM2, we have to take $k = 2$ in (2) and express in the form of (3). Next, from (4), expand $y(x + 2h)$ into series, and the following expression is obtained:

$$\begin{aligned} &L\left[y\left(x\right);h\right]_{\text{RMM2}} \\ &= -a_0 + y\left(x\right) + h\left(-a_1 - a_0a_2 - a_0a_3 + a_2y\left(x\right) + a_3y\left(x\right) + 2y'\left(x\right)\right) \\ &\quad + h^2\left(-a_1a_3 + 2a_2y'\left(x\right) + 2a_3y'\left(x\right) + 2y''\left(x\right)\right) \\ &\quad + h^3\left(2a_2y''\left(x\right) + 2a_3y''\left(x\right) + \tfrac{4}{3}y'''\left(x\right)\right) + h^4\left(\tfrac{4}{3}a_2y'''\left(x\right) + \tfrac{4}{3}a_3y'''\left(x\right) + \tfrac{2}{3}y^{(4)}\left(x\right)\right) \\ &\quad + O\left(h^5\right). \end{aligned} \tag{16}$$

Following Definition 1 and (5), it is readily deduced that

$$\begin{aligned} &\{C_0 = -a_0 + y\left(x\right), C_1 = -a_1 - a_0a_2 - a_0a_3 + a_2y\left(x\right) + a_3y\left(x\right) + 2y'\left(x\right), \\ &C_2 = -a_1a_3 + 2a_2y'\left(x\right) + 2a_3y'\left(x\right) + 2y''\left(x\right), C_3 = 2a_2y''\left(x\right) + 2a_3y''\left(x\right) + \tfrac{4}{3}y'''\left(x\right), \\ &C_4 = \tfrac{4}{3}a_2y'''\left(x\right) + \tfrac{4}{3}a_3y'''\left(x\right) + \tfrac{2}{3}y^{(4)}\left(x\right)\}. \end{aligned}$$

With $C_0 = C_1 = C_2 = C_3 = 0$, we obtain a system of four simultaneous equations which have the following solutions:

$$\left\{a_0 = y\left(x\right), a_1 = 2y'\left(x\right), a_2 = -\frac{y''\left(x\right)}{y'\left(x\right)}, a_3 = \frac{3y''\left(x\right)^2 - 2y'\left(x\right)y'''\left(x\right)}{3y'\left(x\right)y''\left(x\right)}\right\}. \tag{17}$$

Substituting the parameters in (17) into C_4, we obtain

$$C_4 = -\frac{8y'''\left(x\right)^2}{9y''\left(x\right)} + \frac{2}{3}y^{(4)}\left(x\right). \tag{18}$$

When $y\left(x\right)$ is now taken as the theoretical solution of the initial value problem (1) at a point x_n i.e. $y\left(x\right) = y\left(x_n\right)$, (17) may be written as

$$\left\{a_0 = y_n, a_1 = 2y'_n, a_2 = -\frac{y''_n}{y'_n}, a_3 = \frac{3\left(y''_n\right)^2 - 2y'_ny'''_n}{3y'_ny''_n}\right\}, \tag{19}$$

where $y_n = y\left(x_n\right)$ and $y_n^{(m)} = y^{(m)}\left(x_n\right)$, $m = 1, 2, 3$ by the localizing assumption. By taking $k = 2$, (2) becomes

$$y_{n+2} = a_0 + \frac{a_1h\left(1 + a_3h\right)}{1 + a_2h + a_3h}, 1 + a_2h + a_3h \neq 0, \tag{20}$$

which is in the form of (3). We indicate (20) based on (19) as RMM2(2,3) is given as

$$y_{n+2} = y_n + 2hy'_n + \frac{6h^2\left(y''_n\right)^2}{3y''_n - 2hy'''_n}, \tag{21}$$

provided $|y''_n| + |y'''_n| \neq 0$, as to ensure that the denominator of the rational expression in (21) does not equal to zero. From Definition 2 and (18), LTE of RMM2(2,3) becomes

$$\text{LTE}_{\text{RMM2(2,3)}} = h^4\left(-\frac{8\left(y'''_n\right)^2}{9y''_n} + \frac{2}{3}y_n^{(4)}\right) + O\left(h^5\right), \tag{22}$$

where $y_n^{(m)} = y^{(m)}(x_n)$, $m = 2,3,4$ by the localizing assumption. This LTE analysis has confirmed that RMM2(2,3) is a third order method. If we apply RMM2(2,3) to the Dahlquist's test equation $y' = \lambda y$, $Re(\lambda) < 0$, yielding the difference equation

$$y_{n+2} = \frac{3 + 4h\lambda + 2h^2\lambda^2}{3 - 2h\lambda} y_n. \tag{23}$$

Setting $z = h\lambda$, $y_{n+2} = \xi^2$ and $y_n = \xi^0 = 1$ in (23) yield the characteristic equation

$$\xi^2 - \frac{3 + 4z + 2z^2}{3 - 2z} = 0. \tag{24}$$

The roots of (24) are given by

$$\xi_{(24,1)} = -\frac{\sqrt{3 + 4z + 2z^2}}{\sqrt{3 - 2z}} \text{ and } \xi_{(24,2)} = \frac{\sqrt{3 + 4z + 2z^2}}{\sqrt{3 - 2z}}.$$

By taking $z = x + iy$ in the roots of (24), we get the region of absolute stability of RMM2(2,3) as shown in Figure 2.

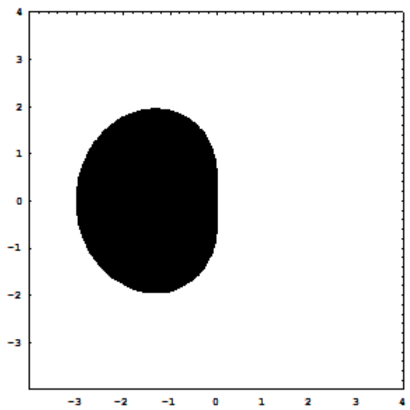

Figure 2: Stability Region of RMM2(2,3)

The shaded region in Figure 2 is the region of absolute stability of RMM2(2,3), where the conditions: $\left|\xi_{(24,1)}\right| \leq 1$ and $\left|\xi_{(24,2)}\right| \leq 1$ are satisfied. From Figure 2, we can see that the region of absolute stability of RMM2(2,3) is a bounded region on the left-hand half plane, which show that RMM2(2,3) is not A-stable.

4 2-step Fourth Order RMM2

In order to derive a fourth order RMM2, we have to take $k = 3$ in (2) and express in the form of (3). Next, from (4), expand $y(x + 2h)$ into series, and the following expression is obtained:

$$\begin{aligned}
&L\left[y\left(x\right);h\right]_{\mathrm{RMM2}}\\
&=-a_0+y\left(x\right)+h\left(-a_1-a_0a_2-a_0a_3-a_0a_4+a_2y\left(x\right)+a_3y\left(x\right)+a_4y\left(x\right)+2y'\left(x\right)\right)\\
&\quad+h^2\left(-a_1a_3-a_1a_4-a_0a_2a_4+a_2a_4y\left(x\right)+2a_2y'\left(x\right)+2a_3y'\left(x\right)+2a_4y'\left(x\right)+2y''\left(x\right)\right)\\
&\quad+h^3\left(2a_2a_4y'\left(x\right)+2a_2y''\left(x\right)+2a_3y''\left(x\right)+2a_4y''\left(x\right)+\tfrac{4}{3}y'''\left(x\right)\right)\\
&\quad+h^4\left(2a_2a_4y''\left(x\right)+\tfrac{4}{3}a_2y'''\left(x\right)+\tfrac{4}{3}a_3y'''\left(x\right)+\tfrac{4}{3}a_4y'''\left(x\right)+\tfrac{2}{3}y^{(4)}\left(x\right)\right)\\
&\quad+h^5\left(\tfrac{4}{3}a_2a_4y'''\left(x\right)+\tfrac{2}{3}a_2y^{(4)}\left(x\right)+\tfrac{2}{3}a_3y^{(4)}\left(x\right)+\tfrac{2}{3}a_4y^{(4)}\left(x\right)+\tfrac{4}{15}y^{(5)}\left(x\right)\right)+O\left(h^6\right).
\end{aligned} \tag{25}$$

Following Definition 1 and (5), it is readily deduced that

$$\begin{aligned}
&\{C_0=-a_0+y\left(x\right),C_1=-a_1-a_0a_2-a_0a_3-a_0a_4+a_2y\left(x\right)+a_3y\left(x\right)+a_4y\left(x\right)+2y'\left(x\right),\\
&\ C_2=-a_1a_3-a_1a_4-a_0a_2a_4+a_2a_4y\left(x\right)+2a_2y'\left(x\right)+2a_3y'\left(x\right)+2a_4y'\left(x\right)+2y''\left(x\right),\\
&\ C_3=2a_2a_4y'\left(x\right)+2a_2y''\left(x\right)+2a_3y''\left(x\right)+2a_4y''\left(x\right)+\tfrac{4}{3}y'''\left(x\right),\\
&\ C_4=2a_2a_4y''\left(x\right)+\tfrac{4}{3}a_2y'''\left(x\right)+\tfrac{4}{3}a_3y'''\left(x\right)+\tfrac{4}{3}a_4y'''\left(x\right)+\tfrac{2}{3}y^{(4)}\left(x\right),\\
&\ C_5=\tfrac{4}{3}a_2a_4y'''\left(x\right)+\tfrac{2}{3}a_2y^{(4)}\left(x\right)+\tfrac{2}{3}a_3y^{(4)}\left(x\right)+\tfrac{2}{3}a_4y^{(4)}\left(x\right)+\tfrac{4}{15}y^{(5)}\left(x\right)\}.
\end{aligned}$$

With $C_0=C_1=C_2=C_3=C_4=0$, we obtain a system of five simultaneous equations which have the following solutions:

$$\left\{\begin{aligned}
&a_0=y\left(x\right),a_1=2y'\left(x\right),a_2=-\tfrac{y''(x)}{y'(x)},a_3=\tfrac{3y''(x)^2-2y'(x)y'''(x)}{3y'(x)y''(x)},\\
&a_4=\tfrac{-4y'(x)y'''(x)^2+3y'(x)y''(x)y^{(4)}(x)}{3y''(x)\left(3y''(x)^2-2y'(x)y'''(x)\right)}
\end{aligned}\right\}. \tag{26}$$

Substituting the parameters in (26) into C_5, we obtain

$$C_5=\frac{16y'''\left(x\right)^3-24y''\left(x\right)y'''\left(x\right)y^{(4)}\left(x\right)+6y'\left(x\right)y^{(4)}\left(x\right)^2}{27y''\left(x\right)^2-18y'\left(x\right)y'''\left(x\right)}+\frac{4}{15}y^{(5)}\left(x\right). \tag{27}$$

When $y\left(x\right)$ is now taken as the theoretical solution of the initial value problem (1) at a point x_n i.e. $y\left(x\right)=y\left(x_n\right)$, (26) may be written as

$$\left\{a_0=y_n,a_1=2y'_n,a_2=-\frac{y''_n}{y'_n},a_3=\frac{3\left(y''_n\right)^2-2y'_ny'''_n}{3y'_ny''_n},a_4=\frac{-4y'_n\left(y'''_n\right)^2+3y'_ny''_ny^{(4)}_n}{3y''_n\left(3\left(y''_n\right)^2-2y'_ny'''_n\right)}\right\}, \tag{28}$$

where $y_n=y\left(x_n\right)$ and $y^{(m)}_n=y^{(m)}\left(x_n\right)$, $m=1,2,3,4$ by the localizing assumption. By taking $k=3$, (2) becomes

$$y_{n+2}=a_0+\frac{a_1h\left(1+a_3h+a_4h\right)}{1+a_2h+a_3h+a_4h+a_2a_4h^2},1+a_2h+a_3h+a_4h+a_2a_4h^2\neq0, \tag{29}$$

which is in the form of (3). We indicate (29) based on (28) as RMM2(2,4) is given as

$$\begin{aligned}
y_{n+2}&=y_n+\frac{2h\left(y'_n\right)^2}{y'_n-hy''_n}\\
&\quad+\frac{2h^3\left(-3\left(y''_n\right)^2+2y'_ny'''_n\right)^2}{\left(-y'_n+hy''_n\right)\left(9\left(y''_n\right)^2-6y'_ny'''_n-6hy''_ny'''_n+4h^2\left(y'''_n\right)^2+3hy'_ny^{(4)}_n-3h^2y''_ny'''_n\right)},
\end{aligned} \tag{30}$$

provided $|y_n'| + |y_n''| \neq 0$ and $|y_n''| + |y_n'''| + \left|y_n^{(4)}\right| \neq 0$, as to ensure that the denominators of the two rational expressions in (30) do not equal to zero. We also note that the conditions $|y_n'| + |y_n''| \neq 0$ and $|y_n''| + |y_n'''| + \left|y_n^{(4)}\right| \neq 0$ are only impose to the formula (30), not on the parameters in (28). From Definition 2 and (27), LTE of RMM2(2,4) becomes

$$\mathrm{LTE}_{\mathrm{RMM2(2,4)}} = h^5 \left(\frac{16\left(y_n'''\right)^3 - 24 y_n'' y_n''' y_n^{(4)} + 6 y_n' \left(y_n^{(4)}\right)^2}{27\left(y_n''\right)^2 - 18 y_n' y_n'''} + \frac{4}{15} y_n^{(5)} \right) + O\left(h^6\right), \quad (31)$$

where $y_n^{(m)} = y^{(m)}(x_n)$, $m = 1,2,3,4,5$ by the localizing assumption. This LTE analysis has confirmed that RMM2(2,4) is a fourth order method. If we apply RMM2(2,4) to the Dahlquist's test equation $y' = \lambda y$, $Re(\lambda) < 0$, yielding the difference equation

$$y_{n+2} = \frac{3 + 3h\lambda + h^2\lambda^2}{3 - 3h\lambda + h^2\lambda^2} y_n. \quad (32)$$

Setting $z = h\lambda$, $y_{n+2} = \xi^2$ and $y_n = \xi^0 = 1$ in (32) yield the characteristic equation

$$\xi^2 - \frac{3 + 3z + z^2}{3 - 3z + z^2} = 0. \quad (33)$$

The roots of (33) are given by

$$\xi_{(33,1)} = -\frac{\sqrt{3 + 3z + z^2}}{\sqrt{3 - 3z + z^2}} \text{ and } \xi_{(33,2)} = \frac{\sqrt{3 + 3z + z^2}}{\sqrt{3 - 3z + z^2}}.$$

By taking $z = x + \mathrm{i}y$ in the roots of (33), we get the region of absolute stability of RMM2(2,4) as shown in Figure 3.

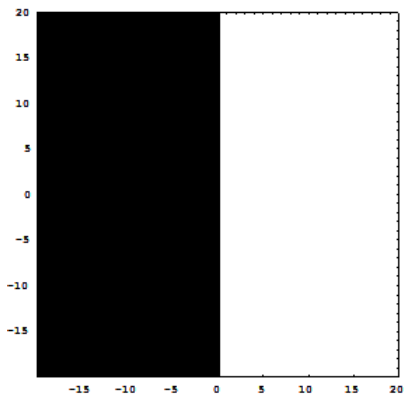

Figure 3: Stability Region of RMM2(2,4)

The shaded region in Figure 3 is the region of absolute stability of RMM2(2,4), where the conditions: $\left|\xi_{(33,1)}\right| \leq 1$ and $\left|\xi_{(33,2)}\right| \leq 1$ are satisfied. From Figure 3, we can see that the region of absolute stability of RMM2(2,4) contains the whole left-hand half plane, which show that RMM2(2,4) is A-stable.

5 2-step Fifth Order RMM2

In order to derive a fifth order RMM2, we have to take $k = 4$ in (2) and express in the form of (3). Next, from (4), expand $y\left(x+2h\right)$ into series, and the following expression is obtained:

$$
\begin{aligned}
&L\left[y\left(x\right);h\right]_{\mathrm{RMM2}} \\
&= -a_0 + y\left(x\right) \\
&\quad + h\left(-a_1 - a_0a_2 - a_0a_3 - a_0a_4 - a_0a_5 + a_2y\left(x\right) + a_3y\left(x\right) + a_4y\left(x\right) + a_5y\left(x\right) + 2y'\left(x\right)\right) \\
&\quad + h^2\left(-a_1a_3 - a_1a_4 - a_0a_2a_4 - a_1a_5 - a_0a_2a_5 - a_0a_3a_5 + a_2a_4y\left(x\right) + a_2a_5y\left(x\right)\right. \\
&\qquad \left. + a_3a_5y\left(x\right) + 2a_2y'\left(x\right) + 2a_3y'\left(x\right) + 2a_4y'\left(x\right) + 2a_5y'\left(x\right) + 2y''\left(x\right)\right) \\
&\quad + h^3\left(-a_1a_3a_5 + 2a_2a_4y'\left(x\right) + 2a_2a_5y'\left(x\right) + 2a_3a_5y'\left(x\right) + 2a_2y''\left(x\right) + 2a_3y''\left(x\right)\right. \\
&\qquad \left. +2a_4y''\left(x\right) + 2a_5y''\left(x\right) + \tfrac{4}{3}y'''\left(x\right)\right) \\
&\quad + h^4\left(2a_2a_4y''\left(x\right) + 2a_2a_5y''\left(x\right) + 2a_3a_5y''\left(x\right) + \tfrac{4}{3}a_2y'''\left(x\right) + \tfrac{4}{3}a_3y'''\left(x\right) + \tfrac{4}{3}a_4y'''\left(x\right)\right. \\
&\qquad \left. +\tfrac{4}{3}a_5y'''\left(x\right) + \tfrac{2}{3}y^{(4)}\left(x\right)\right) \\
&\quad + h^5\left(\tfrac{4}{3}a_2a_4y'''\left(x\right) + \tfrac{4}{3}a_2a_5y'''\left(x\right) + \tfrac{4}{3}a_3a_5y'''\left(x\right) + \tfrac{2}{3}a_2y^{(4)}\left(x\right) + \tfrac{2}{3}a_3y^{(4)}\left(x\right)\right. \\
&\qquad \left. +\tfrac{2}{3}a_4y^{(4)}\left(x\right) + \tfrac{2}{3}a_5y^{(4)}\left(x\right) + \tfrac{4}{15}y^{(5)}\left(x\right)\right) \\
&\quad + h^6\left(\tfrac{2}{3}a_2a_4y^{(4)}\left(x\right) + \tfrac{2}{3}a_2a_5y^{(4)}\left(x\right) + \tfrac{2}{3}a_3a_5y^{(4)}\left(x\right) + \tfrac{4}{15}a_2y^{(5)}\left(x\right) + \tfrac{4}{15}a_3y^{(5)}\left(x\right)\right. \\
&\qquad \left. +\tfrac{4}{15}a_4y^{(5)}\left(x\right) + \tfrac{4}{15}a_5y^{(5)}\left(x\right) + \tfrac{4}{45}y^{(6)}\left(x\right)\right) + O\left(h^7\right).
\end{aligned}
\tag{34}
$$

Following Definition 1 and (5), it is readily deduced that

$$
\begin{aligned}
\{C_0 &= -a_0 + y\left(x\right), \\
C_1 &= -a_1 - a_0a_2 - a_0a_3 - a_0a_4 - a_0a_5 + a_2y\left(x\right) + a_3y\left(x\right) + a_4y\left(x\right) + a_5y\left(x\right) + 2y'\left(x\right), \\
C_2 &= -a_1a_3 - a_1a_4 - a_0a_2a_4 - a_1a_5 - a_0a_2a_5 - a_0a_3a_5 + a_2a_4y\left(x\right) + a_2a_5y\left(x\right) \\
&\quad + a_3a_5y\left(x\right) + 2a_2y'\left(x\right) + 2a_3y'\left(x\right) + 2a_4y'\left(x\right) + 2a_5y'\left(x\right) + 2y''\left(x\right), \\
C_3 &= -a_1a_3a_5 + 2a_2a_4y'\left(x\right) + 2a_2a_5y'\left(x\right) + 2a_3a_5y'\left(x\right) + 2a_2y''\left(x\right) + 2a_3y''\left(x\right) \\
&\quad + 2a_4y''\left(x\right) + 2a_5y''\left(x\right) + \tfrac{4}{3}y'''\left(x\right), \\
C_4 &= 2a_2a_4y''\left(x\right) + 2a_2a_5y''\left(x\right) + 2a_3a_5y''\left(x\right) + \tfrac{4}{3}a_2y'''\left(x\right) + \tfrac{4}{3}a_3y'''\left(x\right) + \tfrac{4}{3}a_4y'''\left(x\right) \\
&\quad + \tfrac{4}{3}a_5y'''\left(x\right) + \tfrac{2}{3}y^{(4)}\left(x\right), \\
C_5 &= \tfrac{4}{3}a_2a_4y'''\left(x\right) + \tfrac{4}{3}a_2a_5y'''\left(x\right) + \tfrac{4}{3}a_3a_5y'''\left(x\right) + \tfrac{2}{3}a_2y^{(4)}\left(x\right) + \tfrac{2}{3}a_3y^{(4)}\left(x\right) \\
&\quad + \tfrac{2}{3}a_4y^{(4)}\left(x\right) + \tfrac{2}{3}a_5y^{(4)}\left(x\right) + \tfrac{4}{15}y^{(5)}\left(x\right), \\
C_6 &= \tfrac{2}{3}a_2a_4y^{(4)}\left(x\right) + \tfrac{2}{3}a_2a_5y^{(4)}\left(x\right) + \tfrac{2}{3}a_3a_5y^{(4)}\left(x\right) + \tfrac{4}{15}a_2y^{(5)}\left(x\right) + \tfrac{4}{15}a_3y^{(5)}\left(x\right) \\
&\quad + \tfrac{4}{15}a_4y^{(5)}\left(x\right) + \tfrac{4}{15}a_5y^{(5)}\left(x\right) + \tfrac{4}{45}y^{(6)}\left(x\right)\}.
\end{aligned}
$$

With $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0$, we obtain a system of six simultaneous equations which have the following solutions

$$
\begin{aligned}
&\Big\{a_0 = y\left(x\right), a_1 = 2y'\left(x\right), a_2 = -\tfrac{y''(x)}{y'(x)}, a_3 = \tfrac{3y''(x)^2 - 2y'(x)y'''(x)}{3y'(x)y''(x)}, \\
&a_4 = \tfrac{-4y'(x)y'''(x)^2 + 3y'(x)y''(x)y^{(4)}(x)}{3y''(x)\left(3y''(x)^2 - 2y'(x)y'''(x)\right)}, \\
&a_5 = -y''\left(x\right)\Big(40y'''\left(x\right)^3 - 60y''\left(x\right)y'''\left(x\right)y^{(4)}\left(x\right) + 15y'\left(x\right)y^{(4)}\left(x\right)^2 \\
&\qquad +18y''\left(x\right)^2y^{(5)}\left(x\right) - 12y'\left(x\right)y'''\left(x\right)y^{(5)}\left(x\right)\Big) \Big/ \\
&\qquad \Big(5\Big(3y''\left(x\right)^2 - 2y'\left(x\right)y'''\left(x\right)\Big)\Big(-4y'''\left(x\right)^2 + 3y''\left(x\right)y^{(4)}\left(x\right)\Big)\Big)\Big\}.
\end{aligned}
\tag{35}
$$

Substituting the parameters in (35) into C_6, we obtain

$$C_6 = \frac{50y^{(4)}(x)^3 - 80y''(x)\,y^{(4)}(x)\,y^{(5)}(x) + 24y''(x)\,y^{(5)}(x)^2}{300y'''(x)^2 - 225y''(x)\,y^{(4)}(x)} + \frac{4}{45}y^{(6)}(x). \tag{36}$$

When $y(x)$ is now taken as the theoretical solution of the initial value problem (1) at a point x_n i.e. $y(x) = y(x_n)$, (35) may be written as

$$\left\{ \begin{array}{l} a_0 = y_n, a_1 = 2y'_n, a_2 = -\frac{y''_n}{y'_n}, a_3 = \frac{3\left(y''_n\right)^2 - 2y'_n y'''_n}{3y'_n y''_n}, a_4 = \frac{-4y'_n\left(y'''_n\right)^2 + 3y'_n y''_n y^{(4)}_n}{3y''_n\left(3(y''_n)^2 - 2y'_n y'''_n\right)}, \\ a_5 = -\frac{y''_n\left(40\left(y'''_n\right)^3 - 60y''_n y'''_n y^{(4)}_n + 15y'_n\left(y^{(4)}_n\right)^2 + 18\left(y''_n\right)^2 y^{(5)}_n - 12y'_n y'''_n y^{(5)}_n\right)}{5\left(3(y''_n)^2 - 2y'_n y'''_n\right)\left(-4(y'''_n)^2 + 3y''_n y^{(4)}_n\right)} \end{array} \right\}, \tag{37}$$

where $y_n = y(x_n)$ and $y^{(m)}_n = y^{(m)}(x_n)$, $m = 1, 2, 3, 4, 5$ by the localizing assumption. By taking $k = 4$, (2) becomes

$$y_{n+2} = a_0 + \frac{a_1 h\left(1 + a_3 h + a_4 h + a_5 h + a_3 a_5 h^2\right)}{1 + a_2 h + a_3 h + a_4 h + a_5 h + a_2 a_4 h^2 + a_2 a_5 h^2 + a_3 a_5 h^2}, \tag{38}$$

with $1 + a_2 h + a_3 h + a_4 h + a_5 h + a_2 a_4 h^2 + a_2 a_5 h^2 + a_3 a_5 h^2 \neq 0$, which is in the form of (3). We indicate (38) based on (37) as RMM2(2,5) is given as

$$\begin{array}{l} y_{n+2} = y_n + 2hy'_n + \frac{6h^2\left(y''_n\right)^2}{3y''_n - 2hy'''_n} \\ \quad - \frac{10h^4\left(-4\left(y'''_n\right)^2 + 3y''_n y^{(4)}_n\right)^2}{3(3y''_n - 2hy'''_n)\left(20(y'''_n)^2 - 15y''_n y^{(4)}_n - 10hy'''_n y^{(4)}_n + 5h^2\left(y^{(4)}_n\right)^2 + 6hy''_n y^{(5)}_n - 4h^2 y'''_n y^{(5)}_n\right)}, \end{array} \tag{39}$$

provided $|y''_n| + |y'''_n| \neq 0$ and $|y'''_n| + \left|y^{(4)}_n\right| + \left|y^{(5)}_n\right| \neq 0$, as to ensure that the denominators of the two rational expressions in (39) do not equal to zero. We also note that the conditions $|y''_n| + |y'''_n| \neq 0$ and $|y'''_n| + \left|y^{(4)}_n\right| + \left|y^{(5)}_n\right| \neq 0$ are only impose to the formula (39), not on the parameters in (37). From Definition 2 and (36), LTE of RMM2(2,5) becomes

$$\text{LTE}_{\text{RMM2(2,5)}} = h^6\left(\frac{50\left(y^{(4)}_n\right)^3 - 80y'''_n y^{(4)}_n y^{(5)}_n + 24y''_n\left(y^{(5)}_n\right)^2}{300\left(y'''_n\right)^2 - 225y''_n y^{(4)}_n} + \frac{4}{45}y^{(6)}_n\right) + O\left(h^7\right), \tag{40}$$

where $y^{(m)}_n = y^{(m)}(x_n)$, $m = 2, 3, 4, 5, 6$ by the localizing assumption. This LTE analysis has confirmed that RMM2(2,5) is a fifth order method. If we apply RMM2(2,5) to the Dahlquist's test equation $y' = \lambda y$, $Re(\lambda) < 0$, yielding the difference equation

$$y_{n+2} = \frac{15 + 18h\lambda + 9h^2\lambda^2 + 2h^3\lambda^3}{15 - 12h\lambda + 3h^2\lambda^2}y_n. \tag{41}$$

Setting $z = h\lambda$, $y_{n+2} = \xi^2$ and $y_n = \xi^0 = 1$ in (41) yield the characteristic equation

$$\xi^2 - \frac{15 + 18z + 9z^2 + 2z^3}{15 - 12z + 3z^2} = 0. \tag{42}$$

The roots of (42) are given by

$$\xi_{(42,1)} = -\frac{\sqrt{15+18z+9z^2+2z^3}}{\sqrt{15-12z+3z^2}} \text{ and } \xi_{(42,2)} = \frac{\sqrt{15+18z+9z^2+2z^3}}{\sqrt{15-12z+3z^2}}.$$

By taking $z = x+iy$ in the roots of (42), we get the region of absolute stability of RMM2(2,5) as shown in Figure 4.

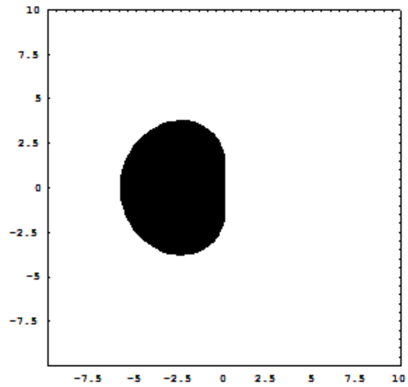

Figure 4: Stability Region of RMM2(2,5)

The shaded region in Figure 4 is the region of absolute stability of RMM2(2,5), where the conditions: $\left|\xi_{(42,1)}\right| \leq 1$ and $\left|\xi_{(42,2)}\right| \leq 1$ are satisfied. From Figure 4, we can see that the region of absolute stability of RMM2(2,5) is a bounded region on the left-hand half plane, which show that RMM2(2,5) is not A-stable.

6 Numerical Experiments and Comparisons

In this section, some test problems are used to check the performance of all newly derived 2-step RMM2 using different number of integration steps. We choose the 6-stage fifth order Kutta-Nyström method shown in page 122 of Lambert [1] as the starting method for 2-step RMM2 of order 2 until order 5. We present the maximum absolute errors over the integration interval given by $\max_{0\leq n\leq N}\{|y(x_n)-y_n|\}$ where N is the number of integration steps; and absolute errors at the end-point given by $|y(x_N)-y_N|$. We note that $y(x_n)$ and y_n represents the exact solution and numerical solution of a test problem at point x_n. The numerical results obtained from our new proposed methods are compared with the numerical results obtained from the RMMs of Okosun and Ademiluyi [5] and Okosun and Ademiluyi [6]. These existing RMMs are 2-step second order method given by

$$y_{n+2} = \frac{y_n^3}{y_n^2 - 2hy_ny_n' + h^2\left(4\left(y_n'\right)^2 - 2y_ny_n''\right)}, \tag{43}$$

3-step third order method given by

$$y_{n+3} = \frac{2\left(y_n\right)^4}{2\left(y_n\right)^3 - 3h\left(y_n\right)^2\left(2y'_n + 3h\left(y''_n + hy'''_n\right)\right) + 18h^2 y_n y'_n\left(y'_n + 3hy''_n\right) - 54h^3\left(y'_n\right)^3}, \tag{44}$$

4-step fourth order method given by

$$\begin{aligned} y_{n+4} = 3\left(y_n\right)^5 \Big/ \Big(& 3\left(y_n\right)^4 - 12h\left(y_n\right)^3 y'_n + h^2\left(48\left(y_n\right)^2\left(y'_n\right)^2 - 24\left(y_n\right)^3 y''_n\right) \\ &+ h^3\left(192\left(y_n\right)^2 y'_n y''_n - 192 y_n\left(y'_n\right)^3 - 32\left(y_n\right)^3 y'''_n\right) \\ &+ h^4\Big(768\left(y'_n\right)^4 - 1152 y_n\left(y'_n\right)^2 y''_n + 192\left(y_n\right)^2\left(y''_n\right)^2 \\ &\quad + 256\left(y_n\right)^2 y'_n y'''_n - 32\left(y_n\right)^3 y_n^{(4)}\Big)\Big), \end{aligned} \tag{45}$$

and 5-step fifth order method given by

$$\begin{aligned} y_{n+5} = 24\left(y_n\right)^6 \Big/ \Big(& 24\left(y_n\right)^5 - 120h\left(y_n\right)^4 y'_n - 300h^2\left(y_n\right)^3\left(y_n y''_n - 2\left(y'_n\right)^2\right) \\ &- 500h^3\left(y_n\right)^2\left(6\left(y'_n\right)^3 - 6y_n y'_n y''_n + \left(y_n\right)^2 y'''_n\right) \\ &- 625h^4 y_n\Big(36 y_n\left(y'_n\right)^2 y''_n - 24\left(y'_n\right)^4 - 8\left(y_n\right)^2 y'_n y'''_n \\ &\quad + \left(y_n\right)^2\left(y_n y_n^{(4)} - 6\left(y''_n\right)^2\right)\Big) \\ &- 625h^5\Big(120\left(y'_n\right)^5 - 240 y_n\left(y'_n\right)^3 y''_n + 60\left(y_n\right)^2\left(y'_n\right)^2 y'''_n \\ &\quad -10\left(y_n\right)^2 y'_n\left(y_n y_n^{(4)} - 9\left(y''_n\right)^2\right) + \left(y_n\right)^3\left(y_n y_n^{(5)} - 20 y''_n y'''_n\right)\Big)\Big). \end{aligned} \tag{46}$$

The starting method for (43) – (46) is the same 6-stage fifth order Kutta-Nyström method mentioned above. It is very clear that all methods in (43) – (46) cannot solve problem with initial value equals to zero.

Problem 1 ([7])

$$y'(x) = -100y(x) + 99e^{2x}, y(0) = 0, x \in [0, 0.5].$$

The exact solution is given by $y(x) = \frac{33}{34}\left(e^{2x} - e^{-100x}\right)$.

Problem 2 ([8])

$$y''(x) + 101y'(x) + 100y(x) = 0, y(0) = 1.01, y'(0) = -2, x \in [0, 10].$$

The exact solutions is given by $y(x) = 0.01e^{-100x} + e^{-x}$. *Problem 2* can also be written as a system, i.e.

$$y'_1(x) = y_2(x), y_1(0) = 1.01, x \in [0, 10];$$

$$y'_2(x) = -100y_1(x) - 101y_2(x), y_2(0) = -2, x \in [0, 10].$$

The exact solutions of this system are given by $y_1(x) = y(x) = 0.01e^{-100x} + e^{-x}$, $y_2(x) = y'(x) = -e^{-100x} - e^{-x}$.

Problem 3 ([7])

$$y'(x) = 1 + y(x)^2, y(0) = 1, x \in [0, 0.8].$$

The exact solution is $y(x) = \tan(x + \pi/4)$. From the exact solution, we notice that the solution becomes unbounded in the neighbourhood of the singularity at $x = \pi/4 \approx 0.785398163367448$.

Table 1: Maximum absolute errors for various second order methods with respect to number of steps (*Problem 1*)

N	Method (43)	RMM2(2,2)
64	-	7.81545(-02)
128	-	1.78169(-02)
256	-	4.14749(-03)
512	-	1.03195(-03)

Table 2: Absolute Errors at the End-point for Various Second Order Methods with Respect to Number of Steps (*Problem 1*)

N	Method (43)	RMM2(2,2)
64	-	2.06636(-05)
128	-	1.74957(-06)
256	-	3.28580(-07)
512	-	7.30578(-08)

Table 3: Maximum Absolute Errors for Various Third Order Methods with Respect to Number of Steps (*Problem 1*))

N	Method (44)	RMM2(2,3)
64	-	2.97028(-02)
128	-	2.95546(-03)
256	-	3.28246(-04)
512	-	3.91259(-05)

7 Discussion and Conclusion

All 2-step RMM2 of order 2 until order 5 proposed above have no problem in solving *Problem 1*, *Problem 2* and *Problem 3*. As expected, all RMMs proposed by Okosun and Ademiluyi [5] and Okosun and Ademiluyi [6] cannot solve *Problem 1* whose initial value equals to zero, while all RMM2 do not face such difficulty. Next, in solving *Problem 2*, we have observed that all RMM2 give smaller absolute errors along the integration interval compare to all existing RMMs of order 2 and order 5 of Okosun and Ademiluyi [5] and

Table 4: Absolute Errors at the End-point for Various Third Order Methods with Respect to Number of Steps (*Problem 1*)

N	Method (44)	RMM2(2,3)
64	-	3.49581(-08)
128	-	3.86154(-09)
256	-	4.13332(-10)
512	-	4.71689(-11)

Table 5: Maximum Absolute Errors for Various Fourth Order Methods with Respect to Number of Steps (*Problem 1*)

N	Method (45)	RMM2(2,4)
64	-	3.39927(-03)
128	-	2.13926(-04)
256	-	2.16793(-05)
512	-	2.35624(-06)

Table 6: Absolute Errors at the End-point for Various Fourth Order Methods with Respect to Number of Steps (*Problem 1*)

N	Method (45)	RMM2(2,4)
64	-	1.49681(-10)
128	-	6.23945(-12)
256	-	3.24629(-13)
512	-	1.82077(-14)

Table 7: Maximum Absolute Errors for Various Fifth Order Methods with Respect to Number of Steps (*Problem 1*)

N	Method (46)	RMM2(2,5)
64	-	5.93616(-04)
128	-	1.64123(-05)

Table 8: Absolute Errors at the End-point for Various Fifth Order Methods with Respect to Number of Steps (*Problem 1*)

N	Method (46)	RMM2(2,5)
64	-	4.16556(-13)
128	-	9.32587(-15)

Table 9: Maximum Absolute Errors for Various Second Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (43)	RMM2(2,2)
2560	1.20431(-03)	8.25702(-04)
5120	2.88964(-04)	2.19023(-04)
10240	7.04627(-05)	5.63484(-05)

Table 10: Absolute Errors at the End-point for Various Second Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (43)	RMM2(2,2)
2560	6.02652(-08)	3.41360(-08)
5120	1.45421(-08)	9.56597(-09)
10240	3.55574(-09)	2.52341(-09)

Table 11: Maximum Absolute Errors for Various Third Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (44)	RMM2(2,3)
2560	2.59963(-03)	1.18777(-03)
5120	6.90425(-04)	3.33834(-04)
10240	1.79972(-04)	8.86875(-05)

Table 12: Absolute Errors at the End-point for Various Third Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (44)	RMM2(2,3)
2560	9.00560(-08)	3.90264(-10)
5120	3.29976(-08)	5.66864(-11)
10240	4.76091(-09)	7.71492(-12)

Table 13: Maximum Absolute Errors for Various Fourth Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (45)	RMM2(2,4)
2560	2.43180(-03)	1.19841(-03)
5120	8.32097(-04)	3.34438(-04)
10240	2.42351(-04)	8.87017(-05)

Table 14: Absolute Errors at the End-point for Various Fourth Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (45)	RMM2(2,4)
2560	1.89700(-08)	9.96633(-10)
5120	9.87620(-10)	5.68070(-11)
10240	5.54945(-11)	3.37347(-12)

Table 15: Maximum Absolute Errors for Various Fifth Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (46)	RMM2(2,5)
2560	3.30441(-03)	1.18719(-03)
5120	1.05235(-03)	3.33575(-04)
10240	3.11077(-04)	8.86222(-05)

Table 16: Absolute Errors at the End-point for Various Fifth Order Methods with Respect to Number of Steps (*Problem 2*)

N	Method (46)	RMM2(2,5)
2560	1.04503(-08)	5.77391(-13)
5120	2.26729(-10)	2.02457(-13)
10240	3.63159(-12)	5.10026(-15)

Table 17: Maximum Absolute Errors for Various Second Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (43)	RMM2(2,2)
64	7.85505(+01)	3.95730(+01)
128	1.78097(+01)	9.46824(+00)
256	1.74431(+01)	9.62127(+00)
512	1.61376(+01)	8.81944(+00)

Table 18: Absolute Errors at the End-point for Various Second Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (43)	RMM2(2,2)
64	1.49174(+00)	7.90470(-01)
128	3.60665(-01)	1.95977(-01)
256	8.90473(-02)	4.88927(-02)
512	2.21457(-02)	1.22169(-02)

Table 19: Maximum Absolute Errors for Various Third Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (44)	RMM2(2,3)
64	4.96315(+00)	1.35285(-01)
128	5.99106(-01)	1.72803(-02)
256	3.01238(-01)	8.96655(-03)
512	1.35801(-01)	4.03688(-03)

Table 20: Absolute Errors at the End-point for Various Third Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (44)	RMM2(2,3)
64	9.82065(-02)	2.90726(-03)
128	1.17654(-02)	3.58019(-04)
256	1.49877(-03)	4.44172(-05)
512	1.85270(-04)	5.53127(-06)

Table 21: Maximum Absolute Errors for Various Fourth Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (45)	RMM2(2,4)
64	5.86819(-01)	1.52254(-03)
128	3.89199(-02)	9.67086(-05)
256	1.00813(-02)	2.53043(-05)
512	2.28366(-03)	5.71485(-06)

Table 22: Absolute Errors at the End-point for Various Fourth Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (45)	RMM2(2,4)
64	1.47062(-02)	3.25678(-05)
128	8.60947(-04)	2.03572(-06)
256	5.20720(-05)	1.27235(-07)
512	3.20141(-06)	7.95028(-09)

Table 23: Maximum Absolute Errors for Various Fifth Order Methods with Respect to Number of Steps (*Problem 3*)

N	Method (46)	RMM2(2,5)
64	9.07345(-02)	3.68169(-05)
128	3.12186(-03)	1.40410(-06)
256	3.97995(-04)	1.95575(-07)
512	4.31393(-05)	3.14233(-08)

Table 24: Absolute Errors at the End-point for Various Fifth Order Methods with Fespect to Number of Steps (*Problem 3*)

N	Method (46)	RMM2(2,5)
64	1.73730(-03)	9.87275(-07)
128	5.97527(-05)	2.28826(-07)
256	1.95537(-06)	5.27152(-10)
512	5.96851(-08)	2.50395(-11)

Okosun and Ademiluyi [6]. Lastly, we also discovered that all RMM2 are more accurate than existing RMMs of Okosun and Ademiluyi [5] and Okosun and Ademiluyi [6] in solving *Problem 3*, as the superiority of RMM2 can be observed from Table 19 to Table 24. In conclusions, all RMM2 of different order can be used to solve initial value problems of different dimension due to less computational effort since they are all explicit methods.

In this paper, we have showed the existence of 2-step variable order RMM2. If 2-step variable order RMM2 do exist, it is reasonable to deduce that r-step variable order RMM2 are possible as well. From (2), we generalize 2-step RMM2 to r-step RMM2 given by

$$y_{n+r} = a_0 + \cfrac{a_1 h}{1 + \cfrac{a_2 h}{1 + \cfrac{a_3 h}{1 + \cdots \cfrac{\vdots}{\cfrac{a_k h}{1 + a_{k+1} h}}}}}. \tag{47}$$

Before establishing the difference operator for (47), we need to simplify the right-hand side of (47). We assume that the simplified version of (47) is given by

$$y_{n+r} = a_0 + \frac{P(a_j, h)}{Q(a_j, h)}, \tag{48}$$

where $P(a_j, h)$ and $Q(a_j, h)$ are functions that contain the parameters

$$a_j \quad \text{for} \quad j = 1, 2, \ldots, k, k+1 \quad \text{and} \quad k \geq 1.$$

With the r-step RMM2 in (48), we associate the difference operator L defined by

$$L[y(x); h]_{\text{RMM2}} = (y(x + rh) - a_0) \times Q(a_j, h) - P(a_j, h), \tag{49}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $x \in [a, b] \subset \mathbb{R}$. Expanding $y(x + rh)$ as Taylor series and collecting terms in (49) gives the following expressions:

$$L[y(x); h]_{\text{RMM2}} = C_0 h^0 + C_1 h^1 + \cdots + C_k h^k + C_{k+1} h^{k+1} + \cdots. \tag{50}$$

We note that the "C" in (50) contain corresponding parameters that need to be determined in the derivation processes. Therefore, the order and local truncation error of r-step RMM2 based on (47) are defined as follows.

Definition 3 *The difference operator* (49) *and the associated rational multistep method* (47) *are said to be of order* $p = k + 1$ *if, in* (50), $C_0 = C_1 = \cdots = C_{k+1} = 0$, $C_{k+2} \neq 0$.

Definition 4 *The local truncation error at* x_{n+r} *of* (47) *is defined to be the expression* $L[y(x_n); h]_{\text{RMM2}}$ *given by* (49), *when* $y(x_n)$ *is the theoretical solution of the initial value problem* (1) *at a point* x_n. *The local truncation error of* (47) *is then*

$$L[y(x_n); h]_{\text{RMM2}} = C_{k+2} h^{k+2} + O\left(h^{k+3}\right). \tag{51}$$

From Definition 3 and Definition 4, we have noticed that the order of accuracy of a r-step RMM2 is not affected by the number of step r. In other words, there exist 4-step RMM2 of order 2 and even 5-step RMM2 of order 2. However, in the sense of cheaper computational cost but higher accuracy, we found that a RMM2 with r greater than the order possessed has less practical use. Below, we show those r-step RMM2 which have more value in computational practice in Table 25.

Table 25: Potential r-step RMM2 of Order p

	p					
r	2	3	4	5	6	$\cdots$
2	✓					
3	✓	✓				
4	✓	✓	✓			
5	✓	✓	✓	✓		
6	✓	✓	✓	✓	✓	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

References

[1] Lambert, J. D. *Computational Methods in Ordinary Differential Equations*. London: John Wiley & Sons. 1973.

[2] Van Niekerk, F. D. Rational one-step methods for initial value problems. *Comput. Math. Appl.* 1988. 16(12): 1035 – 1039.

[3] Ikhile, M. N. O. Coefficients for studying one-step rational schemes for IVPs in ODEs: I. *Comput. Math. Appl.* 2001. 41(5 – 6): 769 – 781.

[4] Lambert, J. D. *Numerical Methods for Ordinary Differential Systems*. Chichester: John Wiley & Sons. 1991.

[5] Okosun, K. O. and Ademiluyi, R. A. A two step second order inverse polynomial methods for integration of differential equations with singularities. *Research Journal of Applied Sciences*. 2007. 2(1): 13 – 16.

[6] Okosun, K. O. and Ademiluyi, R. A. A three step rational methods for integration of differential equations with singularities. *Research Journal of Applied Sciences*. 2007. 2(1): 84 – 88.

[7] Ramos, H. A Non-standard explicit integration scheme for initial-value problems. *Appl. Math. Comput.* 2007. 189(1): 710 – 718.

[8] Yaakub, A. R. and Evans, D. J. New L-stable modified trapezoidal methods for the initial value problems. *Int. J. Comput. Math.* 2003. 80(1): 95 – 104.