A New Class of Rational Multistep Methods
for the Numerical Solution of First Order Initial Value Problems

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Abstract In this paper, we have developed a new class of 2-step rational multistep methods (RMMs) in second to fifth order of accuracy. We have presented the developments of these RMMs, as well as the local truncation error and stability analysis for each RMM that we have developed. Numerical experiments have shown that all RMMs presented in this paper are suitable to solve initial value problem of various dimensions and also stiff problems.

Keywords Rational functions; rational multistep methods; initial value problems; problems whose solutions posses singularities; stiff problems.

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1 Introduction

Let us consider the initial value problem given by

\[ y' = f(x, y), \quad y(a) = \eta, \]

where \( f \) is assumed to satisfy all the conditions in order that (1) has a unique solution.

Conventional linear multistep method given by

\[ \sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \]

is based on the local representation of a polynomial of the theoretical solution to (1). If a linear multistep method was used to pursue the numerical solutions possess singularities, then it fails woefully near the singular points, [1], [2] and [3]. This is because a linear multistep method is formulated on the basis that (1) satisfies the existence and uniqueness theorem, so that polynomial interpolation can be applied quite successfully in the formulation, [3]. Therefore, a natural step would appear to be the replacement of the polynomial function for a linear multistep method, by a rational function due to its smooth behaviour in the neighbourhood of singularities, [3]. We have addressed multistep methods that based on rational interpolant as rational multistep methods or in brief as RMMs. In this paper, we have developed a new class of RMMs that based on the rational interpolants introduced by Van Niekerk [2]. The discussions of these new RMMs are presented as follows.
Suppose that we have solved (1) numerically up to a point $x_n$ and have obtained a value $y_n$ as an approximation of $y(x_n)$, which is the theoretical solution of (1). Following Lambert [1] and Lambert [4], we assume that no previous truncation errors have been made i.e. $y_n = y(x_n)$, we are interested in obtaining $y_{n+2}$ as the approximation of $y(x_{n+2})$. For that purpose, we suggest an approximation to the theoretical solution $y(x_{n+2})$ of (1) given by

$$y_{n+2} = a_0 + \frac{a_1 h}{1 + \frac{a_2 h}{1 + \cdots + \frac{a_k h}{1 + a_{k+1} h}}}$$

(2)

where $a_i$ for $i = 0, 1, \ldots, k, k + 1$ are parameters that may contain approximations of $y(x_n)$ and higher derivatives of $y(x_n)$.

RMM (2) is defined as 2-step $p$-th order RMM2 or in brief as RMM2(2, $p$) with $p = 2, 3, \ldots$. Before establishing the difference operator for (2), we need to simplify the right-hand side of (2). The simplified version of (2) can be written in the form of

$$y_{n+2} = a_0 + P(a_j, h) Q(a_j, h)$$

(3)

where $P(a_j, h)$ and $Q(a_j, h)$ are functions that contain the parameters $a_j$ for $j = 1, 2, \ldots, k, k + 1$ and $k \geq 1$.

With the RMM2 of the form in (3), we associate the difference operator $L$ defined by

$$L[y(x); h]_{\text{RMM2}} = (y(x + 2h) - a_0) \times Q(a_j, h) - P(a_j, h)$$

(4)

where $y(x)$ is an arbitrary function, continuously differentiable on $x \in [a, b] \subset \mathbb{R}$. Expanding $y(x + 2h)$ as Taylor series and collecting terms in (4) gives the following general expression:

$$L[y(x); h]_{\text{RMM2}} = C_0 h^0 + C_1 h^1 + \cdots + C_k h^k + C_{k+1} h^{k+1} + \cdots$$

(5)

We note that the “C” in (5) contain corresponding parameters that need to be determined in the derivation processes. Therefore, the order and local truncation errors of RMM2 based on (2) are defined as follows.

**Definition 1** The difference operator (4) and the associated rational multistep method (2) are said to be of order $p = k + 1$ if, in (5), $C_0 = C_1 = \cdots = C_{k+1} = 0$, $C_{k+2} \neq 0$.

**Definition 2** The local truncation error at $x_{n+2}$ of (2) is defined to be the expression $L[y(x_n); h]_{\text{RMM2}}$ given by (4), when $y(x_n)$ is the theoretical solution of the initial value problem (1) at a point $x_n$. The local truncation error of (2) is then

$$L[y(x_n); h]_{\text{RMM2}} = C_{k+2} h^{k+2} + O(h^{k+3})$$

(6)
2 2-step Second Order RMM2

In order to derive a second order RMM2, we have to take \( k = 1 \) in (2) and express in the form of (3). Next, from (4), expand \( y(x + 2h) \) into series, and the following expression is obtained:

\[
L[y(x); h]_{\text{RMM2}} = -a_0 + y(x) + h(-a_1 - a_0a_2 + a_2y(x) + 2y'(x)) + h^2(2a_2y'(x) + 2y''(x)) + O(h^3).
\]

(7)

Following Definition 1 and (5), it is readily deduced that

\[
\{C_0 = -a_0 + y(x), C_1 = -a_1 - a_0a_2 + a_2y(x) + 2y'(x), C_2 = 2a_2y'(x) + 2y''(x), C_3 = 2a_2y''(x) + \frac{4}{3}y'''(x)\}.
\]

With \( C_0 = C_1 = C_2 = 0 \), we obtain a system of three simultaneous equations which have the following solutions:

\[
\left\{\begin{array}{l}
a_0 = y(x), a_1 = 2y'(x), a_2 = -\frac{y''(x)}{y'(x)}
\end{array}\right.
\]

(8)

Substituting the parameters in (8) into \( C_3 \), we obtain

\[
C_3 = -\frac{2y''(x)^2}{y'(x)} + \frac{4}{3}y'''(x).
\]

(9)

When \( y(x) \) is now taken as the theoretical solution of the initial value problem (1) at a point \( x_n \) i.e. \( y(x) = y(x_n) \), (8) may be written as

\[
\left\{\begin{array}{l}
a_0 = y_n, a_1 = 2y'_n, a_2 = -\frac{y''_n}{y'_n}
\end{array}\right.
\]

(10)

where \( y_n = y(x_n) \) and \( y_n^{(m)} = y^{(m)}(x_n) \), \( m = 1, 2 \) by the localizing assumption. By taking \( k = 1 \), (2) becomes

\[
y_{n+2} = a_0 + \frac{a_1h}{1 + a_2h}, 1 + a_2h \neq 0.
\]

(11)

We indicate (11) based on (10) as RMM2(2,2) is given as

\[
y_{n+2} = y_n + \frac{2h(y'_n)^2}{y'_n - hy''_n},
\]

(12)

provided \( |y'_n| + |y''_n| \neq 0 \), as to ensure that the denominator of the rational expression in (12) does not equal to zero. From Definition 2 and (9), the local truncation error (in brief as LTE) of RMM2(2,2) becomes

\[
\text{LTE}_{\text{RMM2(2,2)}} = h^3\left(-\frac{2(y''_n)^2}{y'_n} + \frac{4}{3}y'''_n\right) + O(h^4),
\]

(13)
where \( y^{(m)}_n = y^{(m)}(x_n) \), \( m = 1, 2, 3 \) by the localizing assumption. This LTE analysis has confirmed that RMM2(2,2) is a second order method. If we apply RMM2(2,2) to the Dahlquist’s test equation \( y' = \lambda y \), \( \text{Re} (\lambda) < 0 \), yielding the difference equation

\[
y_{n+2} = \frac{1 + h\lambda}{1 - h\lambda} y_n.
\]

(14)

Setting \( z = h\lambda \), \( y_{n+2} = \xi^2 \) and \( y_n = \xi^0 = 1 \) in (14) yield the characteristic equation

\[
\xi^2 - \frac{1 + z}{1 - z} = 0.
\]

(15)

The roots of (15) are given by

\[
\xi_{(15,1)} = \frac{-1 + \sqrt{1 + z}}{\sqrt{1 - z}} \quad \text{and} \quad \xi_{(15,2)} = \frac{1 + \sqrt{1 + z}}{\sqrt{1 - z}}.
\]

By taking \( z = x + iy \) in the roots of (15), we obtain the region of absolute stability of RMM2(2,2) as shown in Figure 1.

The shaded region in Figure 1 is the region of absolute stability of RMM2(2,2), where the conditions: \( |\xi_{(15,1)}| \leq 1 \) and \( |\xi_{(15,2)}| \leq 1 \) are satisfied. From Figure 1, we can see that the region of absolute stability of RMM2(2,2) contains the whole left-hand half plane, which show that RMM2(2,2) is A-stable.

### 3 2-step Third Order RMM2

In order to derive a third order RMM2, we have to take \( k = 2 \) in (2) and express in the form of (3). Next, from (4), expand \( y(x + 2h) \) into series, and the following expression is
obtained:

\[
L [y(x); h]_{RMM2} = -a_0 + y(x) + h (-a_1 - a_0 a_2 - a_0 a_3 + a_2 y(x) + a_3 y(x) + 2 y'(x)) \\
+ h^2 (-a_1 a_3 + 2a_3 y'(x) + 2a_3 y''(x) + 2 y'''(x)) \\
+ h^3 (2 a_2 y'''(x) + 2a_3 y''(x) + \frac{4}{3} y'''(x)) + h^4 \left( \frac{a_4}{2} a_2 y''''(x) + \frac{4}{3} a_3 y''''(x) + \frac{2}{3} y^{(4)}(x) \right) \\
+ O (h^5). 
\]

Following Definition 1 and (5), it is readily deduced that

\[
\{C_0 = - a_0 + y(x), C_1 = -a_1 - a_0 a_2 - a_0 a_3 + a_2 y(x) + a_3 y(x) + 2 y'(x), C_2 = -a_1 a_3 + 2a_3 y'(x) + 2a_3 y''(x), C_3 = 2 a_3 y''(x) + 2a_3 y''(x) + \frac{4}{3} y'''(x), C_4 = \frac{4}{3} a_2 y'''(x) + \frac{4}{3} a_3 y'''(x) + \frac{2}{3} y^{(4)}(x) \}. 
\]

With \( C_0 = C_1 = C_2 = C_3 = 0 \), we obtain a system of four simultaneous equations which have the following solutions:

\[
\begin{cases}
a_0 = y(x), a_1 = 2 y'(x), a_2 = -\frac{y''(x)}{y'(x)}, a_3 = \frac{3 y''(x)^2 - 2 y'(x) y'''(x)}{3 y'(x) y''(x)} \end{cases}
\]

(17)

Substituting the parameters in (17) into \( C_4 \), we obtain

\[
C_4 = -\frac{8 y'''(x)^2}{9 y''(x)} + \frac{2}{3} y^{(4)}(x). 
\]

(18)

When \( y(x) \) is now taken as the theoretical solution of the initial value problem (1) at a point \( x_n \) i.e. \( y(x) = y(x_n) \), (17) may be written as

\[
\begin{cases}
a_0 = y_n, a_1 = 2 y'_n, a_2 = -\frac{y''_n}{y'_n}, a_3 = \frac{3 (y''_n)^2 - 2 y'_n y'''_n}{3 y'_n y''_n} \end{cases}
\]

(19)

where \( y_n = y(x_n) \) and \( y^{(m)}_n = y^{(m)}(x_n) \), \( m = 1, 2, 3 \) by the localizing assumption. By taking \( k = 2 \), (2) becomes

\[
y_{n+2} = a_0 + \frac{a_1 h (1 + a_2 h)}{1 + a_2 h + a_3 h}, 1 + a_2 h + a_3 h \neq 0, 
\]

(20)

which is in the form of (3). We indicate (20) based on (19) as RMM2(2,3) is given as

\[
y_{n+2} = y_n + 2 h y'_n + \frac{6 h^2 (y''_n)^2}{3 y''_n - 2 h y'''_n} 
\]

(21)

provided \( |y''_n| + |y'''_n| \neq 0 \), as to ensure that the denominator of the rational expression in (21) does not equal to zero. From Definition 2 and (18), LTE of RMM2(2,3) becomes

\[
LTE_{RMM2(2,3)} = h^4 \left( -\frac{8 (y''_n)^2}{9 y''_n} + \frac{2}{3} y^{(4)}_n \right) + O (h^5). 
\]

(22)
where \( y^{(m)} = y^{(m)}(x_n), m = 2, 3, 4 \) by the localizing assumption. This LTE analysis has confirmed that RMM2(2,3) is a third order method. If we apply RMM2(2,3) to the Dahlquist’s test equation \( y' = \lambda y, \ Re(\lambda) < 0 \), yielding the difference equation

\[
y_{n+2} = \frac{3 + 4h\lambda + 2h^2\lambda^2}{3 - 2h\lambda} y_n.
\]

Setting \( z = h\lambda \), \( y_{n+2} = \xi^2 \) and \( y_n = \xi^0 = 1 \) in (23) yield the characteristic equation

\[
\xi^2 - \frac{3 + 4z + 2z^2}{3 - 2z} = 0.
\]

The roots of (24) are given by

\[
\xi_{(24,1)} = -\frac{\sqrt{3 + 4z + 2z^2}}{\sqrt{3 - 2z}} \quad \text{and} \quad \xi_{(24,2)} = \frac{\sqrt{3 + 4z + 2z^2}}{\sqrt{3 - 2z}}.
\]

By taking \( z = x + iy \) in the roots of (24), we get the region of absolute stability of RMM2(2,3) as shown in Figure 2.

![Figure 2: Stability Region of RMM2(2,3)](image)

The shaded region in Figure 2 is the region of absolute stability of RMM2(2,3), where the conditions: \( |\xi_{(24,1)}| \leq 1 \) and \( |\xi_{(24,2)}| \leq 1 \) are satisfied. From Figure 2, we can see that the region of absolute stability of RMM2(2,3) is a bounded region on the left-hand half plane, which show that RMM2(2,3) is not \( A \)-stable.

4 2-step Fourth Order RMM2

In order to derive a fourth order RMM2, we have to take \( k = 3 \) in (2) and express in the form of (3). Next, from (4), expand \( y(x + 2h) \) into series, and the following expression is
obtained:

\[ L [y(x); h]_{\text{RMM2}} = -a_0 + y(x) + h (-a_1 - a_0 a_2 - a_0 a_3 - a_0 a_4 + a_2 y(x) + a_3 y(x) + a_4 y(x) + 2 y'(x)) + h^2 (-a_1 a_3 - a_1 a_4 - a_0 a_2 a_4 + a_2 a_4 y(x) + 2 a_2 y'(x) + 2 a_3 y'(x) + 2 a_4 y'(x) + 2 y''(x)) + h^3 (2 a_2 a_4 y'(x) + 2 a_2 y''(x) + 2 a_3 y''(x) + 2 a_4 y''(x) + \frac{1}{2} y'''(x)) + h^4 (2 a_2 a_4 y''(x) + \frac{1}{2} a_2 y'''(x) + \frac{1}{4} a_3 y'''(x) + \frac{1}{4} a_4 y'''(x) + \frac{1}{2} y''''(x)) + h^5 \left( \frac{1}{4} a_2 a_4 y'''(x) + \frac{1}{2} a_2 y''(x) + \frac{1}{2} a_3 y''(x) + \frac{1}{2} a_4 y''(x) + \frac{1}{15} y''''(x) \right) + O(h^6). \]

(25)

Following Definition 1 and (5), it is readily deduced that

\[
\begin{align*}
\{C_0 &= -a_0 + y(x), C_1 = -a_1 - a_0 a_2 - a_0 a_3 - a_0 a_4 + a_2 y(x) + a_3 y(x) + a_4 y(x) + 2 y'(x), \\
C_2 &= -a_1 a_3 - a_1 a_4 - a_0 a_2 a_4 + a_2 a_4 y(x) + 2 a_2 y'(x) + 2 a_3 y'(x) + 2 a_4 y'(x) + 2 y''(x), \\
C_3 &= 2 a_2 a_4 y''(x) + 2 a_2 y''(x) + 2 a_3 y''(x) + 2 a_4 y''(x) + \frac{1}{2} y'''(x), \\
C_4 &= 4 a_2 a_4 y'''(x) + \frac{1}{2} a_2 y'''(x) + \frac{1}{4} a_3 y'''(x) + \frac{1}{4} a_4 y'''(x) + \frac{1}{2} y''''(x), \\
C_5 &= \frac{1}{2} a_2 a_4 y''''(x) + \frac{1}{2} a_2 y''(x) + \frac{1}{2} a_3 y''(x) + \frac{1}{2} a_4 y''(x) + \frac{1}{15} y''''(x) \}
\end{align*}
\]

With \( C_0 = C_1 = C_2 = C_3 = C_4 = 0 \), we obtain a system of five simultaneous equations which have the following solutions:

\[
\begin{align*}
a_0 &= y(x), a_1 = 2 y'(x), a_2 = -\frac{y''(x)}{y'(x)}, a_3 = \frac{3 y'''(x)^2 - 2 y''(x) y''''(x)}{3 y''(x) y''(x)}, \\
a_4 &= -\frac{4 y'(x) y'''(x)^2 + 2 y''(x) y''''(x) y'''(x)}{3 y''(x) (y''(x)^2 - 2 y''(x) y''''(x))}.
\end{align*}
\]

(26)

Substituting the parameters in (26) into \( C_5 \), we obtain

\[
C_5 = \frac{16 y''''(x)^2 - 24 y''(x) y''''(x) y'''(x) + 6 y''(x) y'''(x)^2 + 4}{27 y''(x)^2 - 18 y'(x) y''(x)}.
\]

(27)

When \( y(x) \) is now taken as the theoretical solution of the initial value problem (1) at a point \( x_n \) i.e. \( y(x) = y(x_n) \), (26) may be written as

\[
\begin{align*}
a_0 &= y_n, a_1 = 2 y_n', a_2 = -\frac{y_n''}{y_n'}, a_3 = \frac{3 (y_n')^2 - 2 y_n' y_n'''}{3 y_n' y_n'''}, a_4 = -\frac{4 y_n' (y_n''')^2 + 2 y_n' y_n'''(y_n')(4)}{3 y_n'''' (3 (y_n')^2 - 2 y_n' y_n''')} \}
\end{align*}
\]

(28)

where \( y_n = y(x_n) \) and \( y_n^{(m)} = y^{(m)}(x_n) \), \( m = 1, 2, 3, 4 \) by the localizing assumption. By taking \( k = 3, 2 \) becomes

\[
y_{n+2} = a_0 + \frac{a_1 h (1 + a_3 h + a_4 h)}{1 + a_2 h + a_3 h + a_4 h + a_2 a_4 h^2}, 1 + a_2 h + a_3 h + a_4 h + a_2 a_4 h^2 \neq 0,
\]

(29)

which is in the form of (3). We indicate (29) based on (28) as RMM2(2,4) is given as

\[
y_{n+2} = y_n + \frac{2h (y_n')^2}{y_n' - hy_n''} + \frac{2h^3 (3 (y_n')^2 - 2 y_n' y_n''')}{(-y_n' + hy_n'') \left( 9 (y_n')^2 - 6 y_n' y_n'' - 6 h y_n' y_n'' + 4 h^2 (y_n'')^2 \right) + 3 h y_n' y_n^{(4)} - 3 h^2 y_n' y_n''}.
\]

(30)
provided $|y'_n| + |y''_n| \neq 0$ and $|y''_n| + |y'''_n| + |y^{(4)}_n| \neq 0$, as to ensure that the denominators of the two rational expressions in (30) do not equal to zero. We also note that the conditions $|y'_n| + |y''_n| \neq 0$ and $|y''_n| + |y'''_n| + |y^{(4)}_n| \neq 0$ are only impose to the formula (30), not on the parameters in (28). From Definition 2 and (27), LTE of RMM2(2,4) becomes

$$\text{LTE}_{\text{RMM2}(2,4)} = h^5 \left( \frac{16 (y''_n)^3 - 24 y'''_n y^{(4)}_n + 6 y^{(4)}_n (y^{(4)}_n)^2 + 4 \frac{y^{(5)}_n}{15}}{27 (y''_n)^2 - 18 y''_n y'''_n + 15 y^{(5)}_n} \right) + O(h^6),$$

(31)

where $y^{(m)}_n = y^{(m)}(x_n)$, $m = 1, 2, 3, 4, 5$ by the localizing assumption. This LTE analysis has confirmed that RMM2(2,4) is a fourth order method. If we apply RMM2(2,4) to the Dahlquist’s test equation $y' = y, \ Re(\lambda) < 0$, yielding the difference equation

$$y_{n+2} = 3 + 3h\lambda + 3h^2\lambda^2 - 3h\lambda + h^2\lambda^2 y_n.$$  

(32)

Setting $z = h\lambda$, $y_{n+2} = \xi^2$ and $y_n = \xi^0 = 1$ in (32) yield the characteristic equation

$$\xi^2 - 3 + 3z + z^2 = 0.$$  

(33)

The roots of (33) are given by

$$\xi_{(33,1)} = -\frac{\sqrt{3 + 3z + z^2}}{\sqrt{3 - 3z + z^2}} \text{ and } \xi_{(33,2)} = \frac{\sqrt{3 + 3z + z^2}}{\sqrt{3 - 3z + z^2}}.$$  

By taking $z = x + iy$ in the roots of (33), we get the region of absolute stability of RMM2(2,4) as shown in Figure 3.

![Figure 3: Stability Region of RMM2(2,4)](image)

The shaded region in Figure 3 is the region of absolute stability of RMM2(2,4), where the conditions: $|\xi_{(33,1)}| \leq 1$ and $|\xi_{(33,2)}| \leq 1$ are satisfied. From Figure 3, we can see that the region of absolute stability of RMM2(2,4) contains the whole left-hand half plane, which show that RMM2(2,4) is A-stable.
5 2-step Fifth Order RMM2

In order to derive a fifth order RMM2, we have to take \( k = 4 \) in (2) and express in the form of (3). Next, from (4), expand \( y(x + 2h) \) into series, and the following expression is obtained:

\[
L[y(x); h]_{\text{RMM2}} = -a_0 + y(x) + h(-a_1 + a_0a_2 - a_0a_3 - a_0a_4 + a_0a_5 + a_2y(x) + a_3y(x) + a_4y(x) + a_5y(x) + 2y'(x)) + h^2(-a_1a_3 - a_1a_4 - a_0a_2a_3 - a_0a_2a_4 - a_0a_3a_5 + a_2a_3y(x) + a_2a_5y(x) + a_3a_4y(x) + 2a_2y'(x) + 2a_3y'(x) + 2a_4y'(x) + 2a_5y'(x) + 2y''(x)) + h^3(-a_1a_2a_3 + 2a_1a_2y''(x) + 2a_2a_3y'(x) + 2a_3a_5y'(x) + 2a_2y''(x) + 2a_3y''(x) + 2a_5y''(x)) + h^4(2a_2a_4y'''(x) + 2a_2a_5y'''(x) + 2a_3a_5y'''(x) + 2a_2y'''(x) + 2a_3y'''(x) + 2a_4y'''(x) + 2a_5y'''(x)) + h^5(2a_2a_4y''''(x) + 2a_2a_5y''''(x) + 2a_3a_5y''''(x) + 2a_2y''''(x) + 2a_3y''''(x) + 2a_4y''''(x) + 2a_5y''''(x)) + h^6(2a_2a_4y'''''(x) + 2a_2a_5y'''''(x) + 2a_3a_5y'''''(x) + 2a_2y'''''(x) + 2a_3y'''''(x) + 2a_4y'''''(x) + 2a_5y'''''(x)) + O(h^7).
\]

Following Definition 1 and (5), it is readily deduced that

\[
\begin{align*}
C_0 &= -a_0 + y(x), \\
C_1 &= -a_1 - a_0a_2 - a_0a_3 - a_0a_4 - a_0a_5 + a_2y(x) + a_3y(x) + a_4y(x) + a_5y(x) + 2y'(x), \\
C_2 &= -a_1a_3 - a_1a_4 - a_0a_2a_3 - a_0a_2a_4 - a_0a_3a_5 + a_2a_3y(x) + a_2a_5y(x) + a_3a_4y(x) + 2a_2y'(x) + 2a_3y'(x) + 2a_4y'(x) + 2y''(x), \\
C_3 &= -a_1a_2a_3 + 2a_1a_2y''(x) + 2a_2a_3y'(x) + 2a_3a_5y'(x) + 2a_2y''(x) + 2a_3y''(x) + 2a_4y''(x) + 2a_5y''(x), \\
C_4 &= 2a_2a_4y'''(x) + 2a_2a_5y'''(x) + 2a_3a_5y'''(x) + 2a_2y'''(x) + 2a_3y'''(x) + 2a_4y'''(x) + 2a_5y'''(x) + 2y'''(x), \\
C_5 &= 2a_2a_4y''''(x) + 2a_2a_5y''''(x) + 2a_3a_5y''''(x) + 2a_2y''''(x) + 2a_3y''''(x) + 2a_4y''''(x) + 2a_5y''''(x) + 2y''''(x), \\
C_6 &= \frac{4}{15}a_2a_4y'''(x) + \frac{4}{15}a_2a_5y'''(x) + \frac{4}{15}a_3a_5y'''(x) + \frac{4}{15}a_2y'''(x) + \frac{4}{15}a_3y'''(x) + \frac{8}{15}ay(x) + \frac{4}{15}y''(x) + \frac{4}{15}y'(x).
\end{align*}
\]

With \( C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \), we obtain a system of six simultaneous equations which have the following solutions

\[
\begin{align*}
a_0 &= y(x), & a_1 &= 2y'(x), & a_2 &= -y''(x) y'(x), & a_3 &= \frac{3y''(x)^2 - 2y''(x)y'''(x)}{3y'(x)y''(x)}, \\
a_4 &= \frac{-4y''(x)y''(x)^2 + 3y(x)y''(x)y'''(x)}{3y'(x)y''(x)}, \\
a_5 &= -y''(x) \left( 40y''(x)^3 - 60y''(x)y'''(x)y''(x) + 15y'''(x)(x)^2 \right) \\
&\quad + \left( 5\left(3y''(x)^2 - 2y''(x)y'''(x) - 4y'''(x)^2 + 3y(x)y''(x) \right) \right) \left( 5\left(3y''(x)^2 - 2y''(x)y'''(x) - 4y'''(x)^2 + 3y(x)y''(x) \right) \right).
\end{align*}
\]
Substituting the parameters in (35) into $C_6$, we obtain

$$C_6 = \frac{50y^{(4)}(x)^3 - 80y''(x)y^{(4)}(x)y^{(5)}(x) + 24y''(x)y^{(5)}(x)^2 + 4}{45y^{(6)}(x)}.$$ \hspace{1cm} (36)

When $y(x)$ is now taken as the theoretical solution of the initial value problem (1) at a point $x_n$ i.e. $y(x) = y(x_n)$, (35) may be written as

$$\begin{align*}
  a_0 &= y_n, \quad a_1 = 2y'_n, \quad a_2 = -\frac{y''_n}{y'_n}, \quad a_3 = \frac{3(y''_n)^2 - 2y''_n y'''}{3y'_n y''_n}, \quad a_4 = -\frac{4y''_n(y''''_n)^2 + 3y''_n y''''_n}{3y'_n y''_n}, \\
  a_5 &= -\frac{y''_n(40y''_n)^3 - 60y''_n y''''_n y^{(4)}(x) + 15y''_n(y''_n)^2 - 18y''_n y''''_n}{5(3y''_n)^2 - 2y''_n y'''}.
\end{align*}$$ \hspace{1cm} (37)

where $y_n = y(x_n)$ and $y_n^{(m)} = y^{(m)}(x_n)$, $m = 1, 2, 3, 4, 5$ by the localizing assumption. By taking $k = 4$, (2) becomes

$$y_{n+2} = a_0 + \frac{a_1 h (1 + a_2 h + a_3 h + a_4 h + a_5 h + a_3 a_5 h^2)}{1 + a_2 h + a_3 h + a_4 h + a_5 h + a_2 a_4 h^2 + a_2 a_5 h^2 + a_3 a_5 h^2},$$ \hspace{1cm} (38)

with $1 + a_2 h + a_3 h + a_4 h + a_5 h + a_2 a_4 h^2 + a_2 a_5 h^2 + a_3 a_5 h^2 \neq 0$, which is in the form of (3). We indicate (38) based on (37) as RMM2(2,5) is given as

$$y_{n+2} = y_n + 2hy'_n + \frac{6h^2(y''_n)^2}{6y''_n - 2y''''_n} - \frac{10h^4(-4(y''_n)^2 + 3y''_n y''''_n)^2}{3(y''_n)^2 - 2y''_n y'''}$$

$$- \frac{20(y''_n)^2 - 15y''_n y^{(4)}(x) - 10h y''_n y^{(4)} + 5h^2(y^{(4)}(x)^2 + 6hy''_n y^{(5)} - 4h^2 y''''_n y^{(5)} + O(h^7)),}$$ \hspace{1cm} (39)

provided $|y''_n| + |y'''_n| \neq 0$ and $|y''''_n| + |y^{(4)}(x)| + |y^{(5)}(x)| \neq 0$, as to ensure that the denominators of the two rational expressions in (39) do not equal to zero. We also note that the conditions $|y''_n| + |y'''_n| \neq 0$ and $|y''''_n| + |y^{(4)}(x)| + |y^{(5)}(x)| \neq 0$ are only impose to the formula (39), not on the parameters in (37). From Definition 2 and (36), LTE of RMM2(2,5) becomes

$$\text{LTE}_{\text{RMM2}(2,5)} = h^6 \left( \frac{50(y^{(4)}(x))^3 - 80y''_n y^{(4)}y^{(5)} + 24y''_n(y^{(5)}(x))^2 + 4}{300(y''_n)^2 - 225y''_n y^{(4)}} \right) + O(h^7),$$ \hspace{1cm} (40)

where $y_n^{(m)} = y^{(m)}(x_n)$, $m = 2, 3, 4, 5, 6$ by the localizing assumption. This LTE analysis has confirmed that RMM2(2,5) is a fifth order method. If we apply RMM2(2,5) to the Dahlquist’s test question $y' = \lambda y$, $Re(\lambda) < 0$, yielding the difference equation

$$y_{n+2} = \frac{15 + 18h\lambda + 9h^2\lambda^2 + 2h^3\lambda^3}{15 - 12h\lambda + 3h^2\lambda^2}y_n.$$ \hspace{1cm} (41)

Setting $z = h\lambda$, $y_{n+2} = \xi^2$ and $y_n = \xi^0 = 1$ in (41) yield the characteristic equation

$$\xi^2 - \frac{15 + 18z + 9z^2 + 2z^3}{15 - 12z + 3z^2} = 0.$$ \hspace{1cm} (42)
The roots of (42) are given by
\[ \xi_{(42,1)} = -\frac{\sqrt{15 + 18z + 9z^2 + 2z^3}}{\sqrt{15 - 12z + 3z^2}} \] and
\[ \xi_{(42,2)} = \frac{\sqrt{15 + 18z + 9z^2 + 2z^3}}{\sqrt{15 - 12z + 3z^2}}. \]

By taking \( z = x + iy \) in the roots of (42), we get the region of absolute stability of RMM2(2,5) as shown in Figure 4.

![Figure 4: Stability Region of RMM2(2,5)](image)

The shaded region in Figure 4 is the region of absolute stability of RMM2(2,5), where the conditions: \( |\xi_{(42,1)}| \leq 1 \) and \( |\xi_{(42,2)}| \leq 1 \) are satisfied. From Figure 4, we can see that the region of absolute stability of RMM2(2,5) is a bounded region on the left-hand half plane, which show that RMM2(2,5) is not \( A \)-stable.

6 Numerical Experiments and Comparisons

In this section, some test problems are used to check the performance of all newly derived 2-step RMM2 using different number of integration steps. We choose the 6-stage fifth order Kutta-Nyström method shown in page 122 of Lambert [1] as the starting method for 2-step RMM2 of order 2 until order 5. We present the maximum absolute errors over the integration interval given by \( \max_{0 \leq n \leq N} \{|y(x_n) - y_n|\} \) where \( N \) is the number of integration steps; and absolute errors at the end-point given by \( |y(x_N) - y_N| \). We note that \( y(x_n) \) and \( y_n \) represents the exact solution and numerical solution of a test problem at point \( x_n \).

The numerical results obtained from our new proposed methods are compared with the numerical results obtained from the RMMs of Okosun and Ademiluyi [5] and Okosun and Ademiluyi [6]. These existing RMMs are 2-step second order method given by

\[ y_{n+2} = \frac{y_n^3}{y_n^2 - 2hy_ny_n' + h^2 \left( 4 (y_n')^2 - 2y_ny''_n \right)}, \] (43)
3-step third order method given by
\[ y_{n+3} = \frac{2 (y_n)^4}{2 (y_n)^3 - 3h (y_n)^2 (2y_n' + 3h (y_n'' + hy_n''')) + 18h^2 y_n y_n' (y_n' + 3hy_n'') - 54h^3 (y_n')^3} \]
(44)

4-step fourth order method given by
\[
y_{n+4} = 3 (y_n)^5 \left/ \left(3 (y_n)^4 - 12h (y_n)^3 y_n' + h^2 \left(48 (y_n)^2 (y_n')^2 - 24 (y_n)^3 y_n'' \right) + h^3 \left(192 (y_n)^2 y_n'''y_n'' - 192y_n (y_n')^3 - 32 (y_n)^3 y_n''''\right) + h^4 \left(768 (y_n')^4 - 1152y_n (y_n')^2 y_n'' + 192 (y_n)^2 (y_n'')^2 \right. \right. \\
\left. \left. + 256 (y_n)^2 y_n'''y_n'' - 32 (y_n)^3 y_n'''' \right) \right) \right) ,
\]
(45)

and 5-step fifth order method given by
\[
y_{n+5} = 24 (y_n)^6 \left/ \left(24 (y_n)^5 - 120h (y_n)^4 y_n' - 300h^2 (y_n)^3 \left(y_n y_n'' - 2 (y_n')^2 \right) \\
- 500h^3 (y_n)^2 \left(6 (y_n)^3 - 6y_n y_n y_n'' + (y_n')^2 y_n''\right) \\
- 625h^4 y_n \left(36y_n (y_n')^2 y_n'' - 24 (y_n)^4 - 8 (y_n)^3 y_n' y_n'' + (y_n)^2 y_n''' \right) \\
+ (y_n)^2 \left(y_n y_n^{(4)} - 6 (y_n'')^2 \right) \right) \right) ,
\]
(46)

The starting method for (43) − (46) is the same 6-stage fifth order Kutta-Nyström method mentioned above. It is very clear that all methods in (43) − (46) cannot solve problem with initial value equals to zero.

**Problem 1** ([7])
\[ y'(x) = -100y(x) + 99e^{2x}, \quad y(0) = 0, x \in [0, 0.5]. \]
The exact solution is given by \( y(x) = \frac{99}{101} (e^{2x} - e^{-100x}) \).

**Problem 2** ([8])
\[ y''(x) + 101y'(x) + 100y(x) = 0, y(0) = 1.01, y'(0) = -2, x \in [0, 10]. \]
The exact solutions is given by \( y(x) = 0.01e^{-100x} + e^{-x} \). **Problem 2** can also be written as a system, i.e.
\[
y_1'(x) = y_2(x), \quad y_1(0) = 1.01, x \in [0, 10]; \\
y_2'(x) = -100y_1(x) - 101y_2(x), \quad y_2(0) = -2, x \in [0, 10].
\]
The exact solutions of this system are given by \( y_1(x) = y(x) = 0.01e^{-100x} + e^{-x}, \quad y_2(x) = y'(x) = -e^{-100x} - e^{-x}. \)

**Problem 3** ([7])
\[ y'(x) = 1 + y(x)^2, \quad y(0) = 1, x \in [0, 0.8]. \]
The exact solution is $y(x) = \tan(x + \pi/4)$. From the exact solution, we notice that the solution becomes unbounded in the neighbourhood of the singularity at $x = \pi/4 \approx 0.785398163367448$.

Table 1: Maximum absolute errors for various second order methods with respect to number of steps (Problem 1)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (43)</th>
<th>RMM2(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>7.81545(-02)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>1.78169(-02)</td>
</tr>
<tr>
<td>256</td>
<td>-</td>
<td>4.14749(-03)</td>
</tr>
<tr>
<td>512</td>
<td>-</td>
<td>1.03195(-03)</td>
</tr>
</tbody>
</table>

Table 2: Absolute Errors at the End-point for Various Second Order Methods with Respect to Number of Steps (Problem 1)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (43)</th>
<th>RMM2(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>2.06636(-05)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>1.74957(-06)</td>
</tr>
<tr>
<td>256</td>
<td>-</td>
<td>3.28580(-07)</td>
</tr>
<tr>
<td>512</td>
<td>-</td>
<td>7.30578(-08)</td>
</tr>
</tbody>
</table>

Table 3: Maximum Absolute Errors for Various Third Order Methods with Respect to Number of Steps (Problem 1)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (44)</th>
<th>RMM2(2,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>2.97028(-02)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>2.95546(-03)</td>
</tr>
<tr>
<td>256</td>
<td>-</td>
<td>3.28246(-04)</td>
</tr>
<tr>
<td>512</td>
<td>-</td>
<td>3.91259(-05)</td>
</tr>
</tbody>
</table>

7 Discussion and Conclusion

All 2-step RMM2 of order 2 until order 5 proposed above have no problem in solving Problem 1, Problem 2 and Problem 3. As expected, all RMMs proposed by Okosun and Ademiluyi [5] and Okosun and Ademiluyi [6] cannot solve Problem 1 whose initial value equals to zero, while all RMM2 do not face such difficulty. Next, in solving Problem 2, we have observed that all RMM2 give smaller absolute errors along the integration interval compare to all existing RMMs of order 2 and order 5 of Okosun and Ademiluyi [5] and
Table 4: Absolute Errors at the End-point for Various Third Order Methods with Respect to Number of Steps (*Problem 1*)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (44)</th>
<th>RMM2(2,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>3.49581(-08)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>3.86154(-09)</td>
</tr>
<tr>
<td>256</td>
<td>-</td>
<td>4.13332(-10)</td>
</tr>
<tr>
<td>512</td>
<td>-</td>
<td>4.71689(-11)</td>
</tr>
</tbody>
</table>

Table 5: Maximum Absolute Errors for Various Fourth Order Methods with Respect to Number of Steps (*Problem 1*)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (45)</th>
<th>RMM2(2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>3.39927(-03)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>2.13926(-04)</td>
</tr>
<tr>
<td>256</td>
<td>-</td>
<td>2.16793(-05)</td>
</tr>
<tr>
<td>512</td>
<td>-</td>
<td>2.35624(-06)</td>
</tr>
</tbody>
</table>

Table 6: Absolute Errors at the End-point for Various Fourth Order Methods with Respect to Number of Steps (*Problem 1*)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (45)</th>
<th>RMM2(2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>1.49681(-10)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>6.23945(-12)</td>
</tr>
<tr>
<td>256</td>
<td>-</td>
<td>3.24629(-13)</td>
</tr>
<tr>
<td>512</td>
<td>-</td>
<td>1.82077(-14)</td>
</tr>
</tbody>
</table>

Table 7: Maximum Absolute Errors for Various Fifth Order Methods with Respect to Number of Steps (*Problem 1*)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (46)</th>
<th>RMM2(2,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>5.93616(-04)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>1.64123(-05)</td>
</tr>
</tbody>
</table>

Table 8: Absolute Errors at the End-point for Various Fifth Order Methods with Respect to Number of Steps (*Problem 1*)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (46)</th>
<th>RMM2(2,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-</td>
<td>4.16556(-13)</td>
</tr>
<tr>
<td>128</td>
<td>-</td>
<td>9.32587(-15)</td>
</tr>
</tbody>
</table>
Table 9: Maximum Absolute Errors for Various Second Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (43)</th>
<th>RMM2(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>1.20431(-03)</td>
<td>8.25702(-04)</td>
</tr>
<tr>
<td>5120</td>
<td>2.88964(-04)</td>
<td>2.19023(-04)</td>
</tr>
<tr>
<td>10240</td>
<td>7.04627(-05)</td>
<td>5.63484(-05)</td>
</tr>
</tbody>
</table>

Table 10: Absolute Errors at the End-point for Various Second Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (43)</th>
<th>RMM2(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>6.02652(-08)</td>
<td>3.41360(-08)</td>
</tr>
<tr>
<td>5120</td>
<td>1.45421(-08)</td>
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</tr>
<tr>
<td>10240</td>
<td>3.55574(-09)</td>
<td>2.52341(-09)</td>
</tr>
</tbody>
</table>

Table 11: Maximum Absolute Errors for Various Third Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (44)</th>
<th>RMM2(2,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>2.59963(-03)</td>
<td>1.18777(-03)</td>
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<tr>
<td>5120</td>
<td>6.90425(-04)</td>
<td>3.33834(-04)</td>
</tr>
<tr>
<td>10240</td>
<td>1.79972(-04)</td>
<td>8.86875(-05)</td>
</tr>
</tbody>
</table>

Table 12: Absolute Errors at the End-point for Various Third Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (44)</th>
<th>RMM2(2,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>9.00560(-03)</td>
<td>3.90264(-10)</td>
</tr>
<tr>
<td>5120</td>
<td>3.29976(-03)</td>
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</tr>
<tr>
<td>10240</td>
<td>4.76091(-03)</td>
<td>7.71492(-12)</td>
</tr>
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</table>

Table 13: Maximum Absolute Errors for Various Fourth Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method (45)</th>
<th>RMM2(2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>2.43180(-03)</td>
<td>1.19841(-03)</td>
</tr>
<tr>
<td>5120</td>
<td>8.32997(-04)</td>
<td>3.34438(-04)</td>
</tr>
<tr>
<td>10240</td>
<td>2.42351(-04)</td>
<td>8.87017(-05)</td>
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</table>
Table 14: Absolute Errors at the End-point for Various Fourth Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (45)</th>
<th>RMM2(2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>1.89700(-08)</td>
<td>9.96633(-10)</td>
</tr>
<tr>
<td>5120</td>
<td>9.87620(-10)</td>
<td>5.68070(-11)</td>
</tr>
<tr>
<td>10240</td>
<td>5.54945(-11)</td>
<td>3.37347(-12)</td>
</tr>
</tbody>
</table>

Table 15: Maximum Absolute Errors for Various Fifth Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (46)</th>
<th>RMM2(2,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>3.30441(-03)</td>
<td>1.18719(-03)</td>
</tr>
<tr>
<td>5120</td>
<td>1.05235(-03)</td>
<td>3.33575(-04)</td>
</tr>
<tr>
<td>10240</td>
<td>3.11077(-04)</td>
<td>8.86222(-05)</td>
</tr>
</tbody>
</table>

Table 16: Absolute Errors at the End-point for Various Fifth Order Methods with Respect to Number of Steps (Problem 2)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (46)</th>
<th>RMM2(2,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>1.04503(-08)</td>
<td>5.77391(-13)</td>
</tr>
<tr>
<td>5120</td>
<td>2.26729(-10)</td>
<td>2.02457(-13)</td>
</tr>
<tr>
<td>10240</td>
<td>3.63159(-12)</td>
<td>5.10026(-15)</td>
</tr>
</tbody>
</table>

Table 17: Maximum Absolute Errors for Various Second Order Methods with Respect to Number of Steps (Problem 3)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (43)</th>
<th>RMM2(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>7.85505(+01)</td>
<td>3.95730(+01)</td>
</tr>
<tr>
<td>128</td>
<td>1.78097(+01)</td>
<td>9.46824(+00)</td>
</tr>
<tr>
<td>256</td>
<td>1.74431(+01)</td>
<td>9.62127(+00)</td>
</tr>
<tr>
<td>512</td>
<td>1.61376(+01)</td>
<td>8.81944(+00)</td>
</tr>
</tbody>
</table>
Table 18: Absolute Errors at the End-point for Various Second Order Methods with Respect to Number of Steps (Problem 3)

<table>
<thead>
<tr>
<th>N</th>
<th>Method (43)</th>
<th>RMM2(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.49174(+00)</td>
<td>7.90470(-01)</td>
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<tr>
<td>128</td>
<td>3.60665(-01)</td>
<td>1.95977(-01)</td>
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<tr>
<td>256</td>
<td>8.90473(-02)</td>
<td>4.88927(-02)</td>
</tr>
<tr>
<td>512</td>
<td>2.21457(-02)</td>
<td>1.22169(-02)</td>
</tr>
</tbody>
</table>

Table 19: Maximum Absolute Errors for Various Third Order Methods with Respect to Number of Steps (Problem 3)

<table>
<thead>
<tr>
<th>N</th>
<th>Method (44)</th>
<th>RMM2(2,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>4.96315(+00)</td>
<td>1.35285(-01)</td>
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<tr>
<td>128</td>
<td>5.99106(-01)</td>
<td>1.72803(-02)</td>
</tr>
<tr>
<td>256</td>
<td>3.01238(-01)</td>
<td>8.96655(-03)</td>
</tr>
<tr>
<td>512</td>
<td>1.35801(-01)</td>
<td>4.03688(-03)</td>
</tr>
</tbody>
</table>

Table 20: Absolute Errors at the End-point for Various Third Order Methods with Respect to Number of Steps (Problem 3)

<table>
<thead>
<tr>
<th>N</th>
<th>Method (44)</th>
<th>RMM2(2,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>9.82065(-02)</td>
<td>2.90726(-03)</td>
</tr>
<tr>
<td>128</td>
<td>1.17654(-02)</td>
<td>3.58019(-04)</td>
</tr>
<tr>
<td>256</td>
<td>1.49877(-03)</td>
<td>4.44172(-05)</td>
</tr>
<tr>
<td>512</td>
<td>1.85270(-04)</td>
<td>5.53127(-06)</td>
</tr>
</tbody>
</table>

Table 21: Maximum Absolute Errors for Various Fourth Order Methods with Respect to Number of Steps (Problem 3)

<table>
<thead>
<tr>
<th>N</th>
<th>Method (45)</th>
<th>RMM2(2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>5.86819(-01)</td>
<td>1.52254(-03)</td>
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<tr>
<td>128</td>
<td>3.89199(-02)</td>
<td>9.67086(-05)</td>
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<tr>
<td>256</td>
<td>1.00813(-02)</td>
<td>2.53043(-05)</td>
</tr>
<tr>
<td>512</td>
<td>2.28366(-03)</td>
<td>5.71485(-06)</td>
</tr>
</tbody>
</table>
Table 22: Absolute Errors at the End-point for Various Fourth Order Methods with Respect to Number of Steps (*Problem 3*)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (45)</th>
<th>RMM2(2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.47062(-02)</td>
<td>3.25678(-05)</td>
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<tr>
<td>128</td>
<td>8.60947(-04)</td>
<td>2.03572(-06)</td>
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<tr>
<td>256</td>
<td>5.20720(-05)</td>
<td>1.27235(-07)</td>
</tr>
<tr>
<td>512</td>
<td>3.20141(-06)</td>
<td>7.95028(-09)</td>
</tr>
</tbody>
</table>

Table 23: Maximum Absolute Errors for Various Fifth Order Methods with Respect to Number of Steps (*Problem 3*)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (46)</th>
<th>RMM2(2,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>9.07345(-02)</td>
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<tr>
<td>128</td>
<td>3.12186(-03)</td>
<td>1.40410(-06)</td>
</tr>
<tr>
<td>256</td>
<td>3.97995(-04)</td>
<td>1.95575(-07)</td>
</tr>
<tr>
<td>512</td>
<td>4.31393(-05)</td>
<td>3.14233(-08)</td>
</tr>
</tbody>
</table>

Table 24: Absolute Errors at the End-point for Various Fifth Order Methods with Respect to Number of Steps (*Problem 3*)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Method (46)</th>
<th>RMM2(2,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.73730(-03)</td>
<td>9.87275(-07)</td>
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<tr>
<td>128</td>
<td>5.97527(-05)</td>
<td>2.28826(-07)</td>
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<tr>
<td>256</td>
<td>1.95537(-06)</td>
<td>5.27152(-10)</td>
</tr>
<tr>
<td>512</td>
<td>5.96851(-08)</td>
<td>2.50395(-11)</td>
</tr>
</tbody>
</table>
Okosun and Ademiluyi [6]. Lastly, we also discovered that all RMM2 are more accurate than existing RMMs of Okosun and Ademiluyi [5] and Okosun and Ademiluyi [6] in solving Problem 3, as the superiority of RMM2 can be observed from Table 19 to Table 24. In conclusions, all RMM2 of different order can be used to solve initial value problems of different dimension due to less computational effort since they are all explicit methods.

In this paper, we have showed the existence of 2-step variable order RMM2. If 2-step variable order RMM2 do exist, it is reasonable to deduce that \( r \)-step variable order RMM2 are possible as well. From (2), we generalize 2-step RMM2 to \( r \)-step RMM2 given by

\[
y_{n+r} = a_0 + \frac{a_1 h}{1 + \frac{a_2 h}{1 + \frac{\cdots}{1 + \frac{a_{k+1} h}{1 + \frac{\cdots}{1 + \frac{a_{k+2} h}{1 + \cdots}}}}}}
\]

(47)

Before establishing the difference operator for (47), we need to simplify the right-hand side of (47). We assume that the simplified version of (47) is given by

\[
y_{n+r} = a_0 + \frac{P(a_j, h)}{Q(a_j, h)}
\]

(48)

where \( P(a_j, h) \) and \( Q(a_j, h) \) are functions that contain the parameters \( a_j \) for \( j = 1, 2, \ldots, k, k+1 \) and \( k \geq 1 \).

With the \( r \)-step RMM2 in (48), we associate the difference operator \( L \) defined by

\[
L[y(x); h]_{\text{RMM2}} = (y(x + rh) - a_0) \times Q(a_j, h) - P(a_j, h),
\]

(49)

where \( y(x) \) is an arbitrary function, continuously differentiable on \( x \in [a, b] \subset \mathbb{R} \). Expanding \( y(x + rh) \) as Taylor series and collecting terms in (49) gives the following expressions:

\[
L[y(x); h]_{\text{RMM2}} = C_0 h^0 + C_1 h^1 + \cdots + C_k h^k + C_{k+1} h^{k+1} + \cdots.
\]

(50)

We note that the “\( C \)” in (50) contain corresponding parameters that need to be determined in the derivation processes. Therefore, the order and local truncation error of \( r \)-step RMM2 based on (47) are defined as follows.

**Definition 3** The difference operator (49) and the associated rational multistep method (47) are said to be of order \( p = k + 1 \) if, in (50), \( C_0 = C_1 = \cdots = C_{k+1} = 0 \), \( C_{k+2} \neq 0 \).

**Definition 4** The local truncation error at \( x_{n+r} \) of (47) is defined to be the expression \( L[y(x_n); h]_{\text{RMM2}} \) given by (49), when \( y(x_n) \) is the theoretical solution of the initial value problem (1) at a point \( x_n \). The local truncation error of (47) is then

\[
L[y(x_n); h]_{\text{RMM2}} = C_{k+2} h^{k+2} + O(h^{k+3}).
\]

(51)

From Definition 3 and Definition 4, we have noticed that the order of accuracy of a \( r \)-step RMM2 is not affected by the number of step \( r \). In other words, there exist 4-step RMM2 of order 2 and even 5-step RMM2 of order 2. However, in the sense of cheaper computational cost but higher accuracy, we found that a RMM2 with \( r \) greater than the order possessed has less practical use. Below, we show those \( r \)-step RMM2 which have more value in computational practice in Table 25.
Table 25: Potential r-step RMM2 of Order p

<table>
<thead>
<tr>
<th>r</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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References


