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Orthogonal Functions Based on Chebyshev Polynomials

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Abstract It is known that Chebyshev polynomials are an orthogonal set associated with a certain weight function. In this paper, we present an approach for the construction of a special wavelet function as well as a special scaling function. Main tool of the special wavelet is a first kind Chebyshev polynomial. Based on Chebyshev polynomials and their zero, we define our scaling function and wavelets, and by using Christoffel-Darboux formula for Chebyshev polynomials, we prove that these functions are orthogonal. Finally, we provide several examples of scaling function and wavelets for illustration.

Keywords Chebyshev polynomial, Christoffel-Darboux formula, Wavelets, and Scaling function.

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1 Introduction

Consider the polynomial

$$T_n(t) = \cos(n\theta) \tag{1}$$

where n is a nonnegative integer, $\theta = \arccos(t)$, and $0 \leq \theta \leq \pi$. As θ increases from 0 to π, t decreases from 1 to -1. The polynomial $T_n(t)$ defined by (1) on the interval $[-1, 1]$, is called the first kind Chebyshev polynomial of degree n. For more detail, the properties of Chebyshev polynomial can be found in [1], [2], and [3].

Chui and Mhaskar [4] have introduced wavelet analysis by trigonometric polynomial. Then Kilgore and Prestin [5] used algebraic polynomial to obtain orthogonality with respect to Chebyshev weight between the wavelets and corresponding scaling functions. These underlying functions fullfilled interpolatory condition on extremes and zeros of Chebyshev polynomials [5]. Wavelet techniques for polynomials wavelet have been developed in [6], [7], and [8]. Orthognal polynomial wavelet has been constructed by kernel function of orthogonal polynomials [4], and [5]. For detail discussing kernel function of orthogonal polynomials can be found in [9]. Capobiancho and Themistoclakis [6] combined Lagrange interpolation and the De la Vallée Poussin interpolation process to construct polynomial wavelet based on four kinds of Chebyshev polynomials. A general theory of orthogonal polynomial wavelet has been developed in [7], and [8].

In this paper, in order to introduce our own version of wavelet analysis which is slightly different from what had been constructed in the several articles mentioned above, we only use the zeroes of Chebyshev polynomials which have orthogonality property with respect to Chebyshev weight among the scaling and wavelet functions. The idea for constructing our own different scaling and wavelet, is according to the Christofel-Darboux rule. In this case, we show the orthogonality of scaling and wavelet functions using the Christofel-Darboux formula for Chebyshev polynomials.

The rest of this paper is organized as follows. Some notes about the Chebyshev polynomial are given in Section 2 and Section 3. Section 4 contains our contribution of scaling and wavelet functions including their properties. Some remark and conclusion which are given in Section 5 and Section 6 respectively, will end this paper.

2 Notations

For obtaining the special scaling and wavelet functions as mentioned in Section 1, we need to establish some notes about the Chebyshev polynomials as given in this section.

Throughout this paper, P and $T_n(t)$ denote the space of all real polynomials and the first kind of Chebyshev polynomial of degree n, respectively.

The set of all zeroes of Chebyshev polynomial $T_{n+1}(t)$ which are denoted by

$$\{\eta_0, \eta_1, \eta_2, \cdots, \eta_n\}$$

are located in $[-1, 1]$ ([10]).

The notation

$$\text{span } \{a_0, a_1, \cdots, a_n\}$$

denotes the space of all linear combination of $\{a_0, a_1, \cdots, a_n\}$. In particular, if $\{p_0, p_1, \cdots, p_n\}$ is a set of polynomials, then

$$P_n = \text{span } \{p_0, p_1, \cdots, p_n\}$$

is the space of all real polynomials whose degree does not exceed n.

The inner product space of P is defined by

$$\langle f, g \rangle_{ch} = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}} dt \tag{2}$$

for all $f, g \in P$.

Throughout the inner product space (2), the set $\{p_0, p_1, \cdots, p_n\}$ is orthogonal and becomes the basis for P_n. The best basis polynomials have the valuable extra property that the polynomials are orthogonal to each other.

3 Chebyshev Polynomial

As mentioned in [1], [2], and [3], the Chebyshev polynomials $T_n(t)$ of degree have precisely zeros denoted by $\eta_k \quad (k = 1, ..., n)$ or $\eta_k \quad (k = 0, 1, ..., n-1)$ in the interval $[-1, 1]$ where

$$\eta_k = \cos\left(\frac{\pi\left(k - \frac{1}{2}\right)}{n}\right) = \cos\left(\frac{\pi\,(2k-1)}{2n}\right) \quad (k = 1, 2, \cdots, n).$$

The Chebyshev polynomials are orthogonal in the interval $[-1,1]$ over weight function $\omega(t) = \left(1-t^2\right)^{-1/2}$. In particular,

$$\int_{-1}^{1} \frac{T_m(t)T_n(t)}{\sqrt{1-t^2}}\,dt = \begin{cases} 0 & (m \neq n) \\ \dfrac{\pi}{2} & (m = n = 1,2,3,\cdots) \\ \pi & (m = n = 0) \end{cases} . \tag{3}$$

From trigonometry identity (see [1] and [2]), we obtain

$$\cos\left((n+1)\,\theta\right) + \cos\left((n-1)\,\theta\right) = 2\cos\left(\theta\right)\cos\left(n\theta\right)$$

from which, by using (1), we have the relation

$$T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t), \quad n \geq 2 \tag{4}$$

where $T_0(t) = 1$, $T_1(t) = t$, and also we can produce the result written in the following Theorem 3.1 which has already been proved in [3].

Theorem 3.1 ([3])

Let $T_n(t)$ be the first kind Chebyshev polynomial of degree $n > 0$. Then

$$T_{n+m}\left(t\right) + T_{n-m}\left(t\right) = 2T_n\left(t\right)T_m\left(t\right)$$

for all $t \in [-1,1]$ and $m = 1,2,\cdots,n$.♦

If we observe Theorem 3.1 more carefully, we will obtain the following theorem.

Theorem 3.2

If a and b be the zeros of Chebyshev polynomial $T_n(t)$ then

$$\sum_{m=n+1}^{2n-1} T_m(a)T_m(b) = \sum_{m=1}^{n-1} T_m(a)T_m(b)$$

Proof

Let a and b be the two zeros of Chebyshev polynomial $T_n(t)$. Using Theorem 3.1, we have

$$T_{n+m}(a) + T_{n-m}(a) = 2T_n(a)T_m(a) = 0$$

and

$$T_{n+m}(b) + T_{n-m}(b) = 2T_n(b)T_m(b) = 0$$

for $m = 1,2,3,\cdots,n-1$.Therefore,

$$T_{n+m}(a)T_{n+m}(b) = \left(-T_{n-m}(a)\right)\left(-T_{n-m}(b)\right) = T_{n-m}(a)T_{n-m}(b)$$

for $m = 1,2,3,\cdots,n-1$. By summation, we obtain

$$T_{n+1}\left(a\right)T_{n+1}\left(b\right)+...+T_{n+n-1}\left(a\right)T_{n+n-1}\left(b\right) = T_{n-1}\left(a\right)T_{n-1}\left(b\right)+...+T_{n-n+1}\left(a\right)T_{n-n+1}\left(b\right)$$

$$T_{n+1}(a)\,T_{n+1}(b)+...+T_{2n-1}(a)\,T_{2n-1}(b)=T_{n-1}(a)\,T_{n-1}(b)+...+T_1(a)\,T_1(b)$$

Finally, we have

$$\sum_{m=n+1}^{2n-1} T_m(a)\,T_m(b)=\sum_{m=1}^{n-1} T_m(a)\,T_m(b).\blacklozenge$$

The following theorem is called the **Christofel-Darboux formula** for Chebyshev polynomial. The proof of the theorem can be found in [3].

Theorem 3.3 ([3])

If $T_m(t)$ is the first kind Chebyshev polynomial with $m>0$ order then

$$\sum_{m=0}^{n}{}^{\oplus} T_m(x)T_m(y)=\frac{1}{2}\left[\frac{T_{n+1}(x)T_n(y)-T_n(x)T_{n+1}(y)}{x-y}\right] \tag{5}$$

where

$$\sum_{m=0}^{n}{}^{\oplus} a_m=\tfrac{1}{2}a_0+\sum_{m=1}^{n} a_m \tag{6}$$

for $x,y\in[-1,1]$ and $x\neq y$.$\blacklozenge$

By using Theorem 3.2 and Theorem 3.3, we can show the orthogonality property of Chebyshev polynomials as given in the following corollary.

Corollary 3.1

Let a and b be the zeros of Chebyshev polynomial $T_{n+1}(t)$ with $a\neq b$ then

$$\sum_{m=0}^{n}{}^{\oplus} T_m(a)T_m(b)=0$$

Proof

Let a and b be the zeros of Chebyshev polynomial $T_{n+1}(t)$. Then $T_{n+1}(a)=T_{n+1}(b)=0$. As a consequence of Theorem 3.3, we get

$$\sum_{m=0}^{n}{}^{\oplus} T_m(a)T_m(b)=\frac{1}{2}\left[\frac{T_{n+1}(a)T_n(b)-T_n(a)T_{n+1}(b)}{a-b}\right]=0.\blacklozenge$$

By using the first kind Chebyshev polynomial and its properties as shown in this section, we can construct scaling and wavelet functions as explained in the next section.

4 Scaling Function and Wavelets

Since the scaling and wavelet functions to be built based on the zeroes of Chebyshev polynomial with orthogonality property, we need to establish the scaling and wavelet functions which satisfy the criteria described in Section 2, and we will do these separately.

Let P_n be the space of all polynomials of degree does not exceed n where its basis is an orthogonal basis. We define a new **Scaling function** $\phi_{n,k}$ $(k=0,1,2,\cdots,n)$ which satisfies the orthogonality property of Chebyshev polynomials as follows.

Definition 4.1 (Scaling Function)

If $\eta_0, \eta_1, \cdots, \eta_n$ *are the zeros of Chebyshev polynomial* $T_{n+1}(t)$*. Then we defined*

$$V_n = \text{span}\,\{\phi_{n,k}\,|\,k = 0, 1, 2, \cdots, n\} \tag{7}$$

as a new space of real polynomials where

$$\phi_{n,k}(t) = \sum_{m=0}^{n}{}^{\oplus} T_m(t)\, T_m(\eta_k) \tag{8}$$

for $k = 0, 1, 2, \cdots, n$ *are called the Chebyshev scaling functions in* V_n.♦

From Definition 4.1, we note that $P_n = V_n$ whenever the set $\{\phi_{n,k}\,|\,k = 0, 1, 2, \cdots, n\}$ is orthogonal as shown in the following theorem.

Theorem 4.1

If $\phi_{n,k},\ \phi_{n,s} \in V_n$ *then*

$$\left\langle \phi_{\text{n,k}}, \phi_{\text{n,s}} \right\rangle_{Ch} = d_{k,s}\, \lambda_n$$

where

$$d_{k,s} = \begin{cases} 1 & (k = s) \\ 0 & (k \neq s) \end{cases}$$

and

$$\lambda_n = \frac{\pi}{2} \sum_{m=0}^{n}{}^{\oplus} T_m^2(\eta_k).$$

Proof

Since $\{T_0, T_1, T_2, \cdots, T_n\}$ is an orthogonal set shown by (3), we have

$$\begin{aligned}
\left\langle \phi_{\text{n,k}}, \phi_{\text{n,s}} \right\rangle_{Ch} &= \left\langle \sum_{m=0}^{n}{}^{\oplus} T_m(t)\, T_m(\eta_k), \sum_{m=0}^{n}{}^{\oplus} T_m(t)\, T_m(\eta_s) \right\rangle_{Ch} \\
&= \frac{1}{4} T_0(\eta_s)\, T_0(\eta_k) \int_{-1}^{1} \frac{T_0^2(t)}{\sqrt{1-t^2}} dt + \sum_{m=1}^{n} T_m(\eta_s)\, T_m(\eta_k) \int_{-1}^{1} \frac{T_m^2(t)}{\sqrt{1-t^2}} dt \\
&= \frac{1}{4} T_0(\eta_s) T_0(\eta_k)\,(\pi) + \sum_{m=1}^{n} T_m(\eta_s) T_m(\eta_k) \left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{4} T_0(\eta_s) T_0(\eta_k) + \frac{\pi}{2} \sum_{m=1}^{n} T_m(\eta_s) T_m(\eta_k) \\
&= \frac{\pi}{2} \left[\frac{1}{2} T_0(\eta_s) T_0(\eta_k) + \sum_{m=1}^{n} T_m(\eta_s) T_m(\eta_k) \right] \\
&= \frac{\pi}{2} \left[\sum_{m=0}^{n}{}^{\oplus} T_m(\eta_s) T_m(\eta_k) \right].
\end{aligned} \tag{9}$$

For $k = s$, we obtain

$$\left\langle \phi_{\text{n,k}}, \phi_{\text{n,s}} \right\rangle_{Ch} = \lambda_n$$

where

$$\lambda_n = \frac{\pi}{2} \sum_{m=0}^{\overset{\oplus}{n}} T_m^2(\eta_k).$$

Next, we will show that

$$\left\langle \phi_{\mathrm{n,k}}, \phi_{\mathrm{n,s}} \right\rangle_{Ch} = 0 \quad \text{for} \quad k \neq s.$$

By using Corollary 3.1 and Definition 4.1, the equation (9) becomes

$$\left\langle \phi_{\mathrm{n,k}}, \phi_{\mathrm{n,s}} \right\rangle_{Ch} = \frac{\pi}{2} \left[\sum_{m=0}^{\overset{\oplus}{n}} T_m(\eta_s) T_m(\eta_k) \right] = 0$$

for $k, s = 0, 1, 2, \cdots, n$ and $k \neq s$. The proof is complete.⧫

From Definition 4.1 and properties of Chebyshev polynomial, if $f \in V_n$ then $f \in \text{span}\{T_0, T_1, T_2, \cdots, T_n\}$ and vice versa. It shows that

$$V_n = \text{span}\{T_0, T_1, T_2, \cdots, T_n\}.$$

We shall define wavelets and discuss their proofs for wavelets to be orthogonal to each other. The proof will be done by using Christofel-Darboux formula for Chebyshev polynomial.

We will define a new wavelet function $\psi_{n,k}(t)\,(k = 0, 1, 2, \cdots, n-1)$ which satisfies the orthogonality property of Chebyshev polynomials as follows.

Definition 4.2

Let $\eta_0, \eta_1, \eta_2, \cdots, \eta_{n-1}$ zeroes of Chebyshev polynomial $T_n(t)$. For each n nonnegative integer, we define

$$W_n = \text{span}\ \{\psi_{n,k} \mid k = 0, 1, 2, \cdots, n-1\} \tag{10}$$

as a new space of real polynomials where

$$\psi_{n,k}(t) = \sum_{m=n+1}^{2n-1} T_m(t)\, T_m(\eta_k) + \frac{1}{\sqrt{2}} T_{2n}(t)\, T_{2n}(\eta_k) \tag{11}$$

for $k = 0, 1, 2, \cdots, n-1$.⧫

The function $\psi_{n,k}(t)$ for $k = 0, 1, 2, \cdots, n-1$ defined by (11) are called **Chebyshev wavelet function**.

By using this new definition and Corollary 3.1, we will obtain a new theorem which proves the orthogonality of the set of Chebyshev wavelet functions as follows.

Theorem 4.2

If $\psi_{n,k},\ \psi_{n,s} \in W_n$ with $k \neq s$, then

$$\left\langle \psi_{\mathrm{n,k}}, \psi_{\mathrm{n,s}} \right\rangle_{Ch} = 0.$$

Proof

We denote $\eta_0, \eta_1, \eta_2, \cdots, \eta_{n-1}$ as zeroes of Chebyshev polynomial $T_n(t)$. By using Corollary 3.1, we have

$$\sum_{m=0}^{n-1 \oplus} T_m(\eta_s)T_m(\eta_k) = \left[\frac{T_n(\eta_k)T_{n-1}(\eta_s) - T_n(\eta_s)T_{n-1}(\eta_k)}{\eta_k - \eta_s}\right] = 0 \tag{12}$$

for $k, s = 0, 1, 2, 3, \cdots, n-1$ and $k \neq s$.

Since the polynomial Chebyshev $\{T_{n+1}, T_{n+2}, T_{n+3}, \cdots, T_{2n}\}$ is orthogonal. Therefore, we obtain

$$\begin{aligned}
&\left\langle \psi_{\mathrm{n,k}}, \psi_{\mathrm{n,s}} \right\rangle_{Ch} \\
&= \left\langle \sum_{m=n+1}^{2n-1} T_m(t)\, T_m(\eta_k) + \frac{1}{\sqrt{2}} T_{2n}(t)\, T_{2n}(\eta_k),\ \sum_{m=n+1}^{2n-1} T_m(t)\, T_m(\eta_s) + \frac{1}{\sqrt{2}} T_{2n}(t)\, T_{2n}(\eta_s) \right\rangle \\
&= \frac{1}{2} T_{2n}(\eta_s)\, T_{2n}(\eta_k) \int_{-1}^{1} \frac{T_{2n}^2(t)}{\sqrt{1-t^2}} dt + \sum_{m=n+1}^{2n-1} T_m(\eta_s)\, T_m(\eta_k) \int_{-1}^{1} \frac{T_m^2(x)}{\sqrt{1-t^2}} dt \\
&= \frac{\pi}{4} T_{2n}(\eta_s) T_{2n}(\eta_k) + \frac{\pi}{2} \sum_{m=n+1}^{2n-1} T_m(\eta_s) T_m(\eta_k).
\end{aligned}$$

By using Theorem 3.1, Theorem 3.2, and equation (12), the above equation can be written as

$$\begin{aligned}
\left\langle \psi_{\mathrm{n,k}}, \psi_{\mathrm{n,s}} \right\rangle_{Ch} &= \frac{\pi}{4} T_0(\eta_s) T_0(\eta_k) + \frac{\pi}{2} \sum_{m=1}^{n-1} T_m(\eta_s) T_m(\eta_k) \\
&= \frac{\pi}{2} \left[\frac{1}{2} T_0(\eta_s) T_0(\eta_k) + \sum_{m=1}^{n-1} T_m(\eta_s) T_m(\eta_k) \right] \\
&= \frac{\pi}{2} \left[\sum_{m=0}^{\overset{\oplus}{n-1}} T_m(\eta_s) T_m(\eta_k) \right] \\
&= 0
\end{aligned}$$

for $k \neq s$. The proof is complete.⧫

By the above explanation, the scaling and wavelet functions are constructed as follows. Firstly, choose a Chebyshev polynomial $T_n(t)$ of degree n and identifying its zeroes. Secondly, define the scaling Chebyshev and wavelet Chebyshev functions by using Definition 4.1 and Definition 4.2 respectively. By using the Christofel-Darboux rule and Chebyshev polynomial, both functions can be shown to possess the orthogonality property.

5 Discussion

As mentioned in Section 1, in this paper, we have introduced wavelet function based on Chebyshev polynomial. The wavelet concept to be suggested is really built using the zeroes of Chebyshev polynomials with certain degree as shown in Section 4.

In Definition 4.1, we have created Chebyshev scaling functions in V_n where V_n is defined by (7). These scaling functions have orthogonality property as have been proved by using Christofell-Darboux formula for Chebyshev polynomials in Section 4.

In Definition 4.2, we have defined our own Chebyshev wavelet functions in W_n where W_n is defined by (10). These wavelet functions have orthogonality property as been proven by using Christofell-Darboux formula for Chebyshev polynomials in Section 4.

Theorem 4.1 and Theorem 4.2 show that the set of scaling functions

$$\{\phi_{n,k} \mid k = 0, 1, 2, \cdots, n\}$$

and the set of wavelets

$$\{\psi_{n,k} \mid k = 0, 1, 2, \cdots, n-1\}$$

are the orthogonal sets respectively.

In Figure 1 and Figure 2, we display some scaling and wavelet functions in their certain subspaces V_n and W_n respectively which are obtained by using Matlab. The Matlab codes and further information can be obtained from the authors.

By definitions, $\phi_{16,k}(t)$ $(k = 2, 8, 13)$ and $\psi_{16,k}(t)\,(k = 2, 8, 13)$ are the polynomials of degree 16 respectively. In Figure 1, the position of maximum of $\phi_{16,k}(t)$ $(k = 2, 8, 13)$ will move from right to left as $k$increases from 2 to 13. The same case is applied to $\psi_{16,k}(t)\,(k = 2, 8, 13)$. Therefore, we can conclude that the scaling and wavelet functions have the same pattern of graphs for n = 16 and $(k = 2, 8, 13)$.

6 Conclusion

In this paper, in order to obtain the special scaling and wavelet functions different from what have been created in ([1],[7]), we have used the zeroes of Chebyshev polynomial for creating the scaling and wavelet functions which have orthogonality property. The capability of the said functions for satisfying the property of scaling and wavelet functions is proven by the Christoffel-Darboux formula' These findings are further illustrated in Figure 1 and Figure 2. These scaling and wavelet functions can be used as a choice wavelets in wavelet analysis.

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References

[1] Fox, L. and Parker I. B. *Chebyshev Polynomial in Numerical Analysis*. Glasgow: Oxford University Press. 1968.

[2] Mason, J. C. and Handscomb, D. C. *Chebyshev Polynomial*. Florida: Chapman & Hall/CRC. 2002.

[3] Rivlin, T. *The Chebyshev Polynomials*. New York: John Wiley and Sons. 1974.

[4] Chui, C. K., and Mhaskar, H. N. *On Trigonometric Wavelets*, Constructive Approximation, 1993. 9: 167–190.

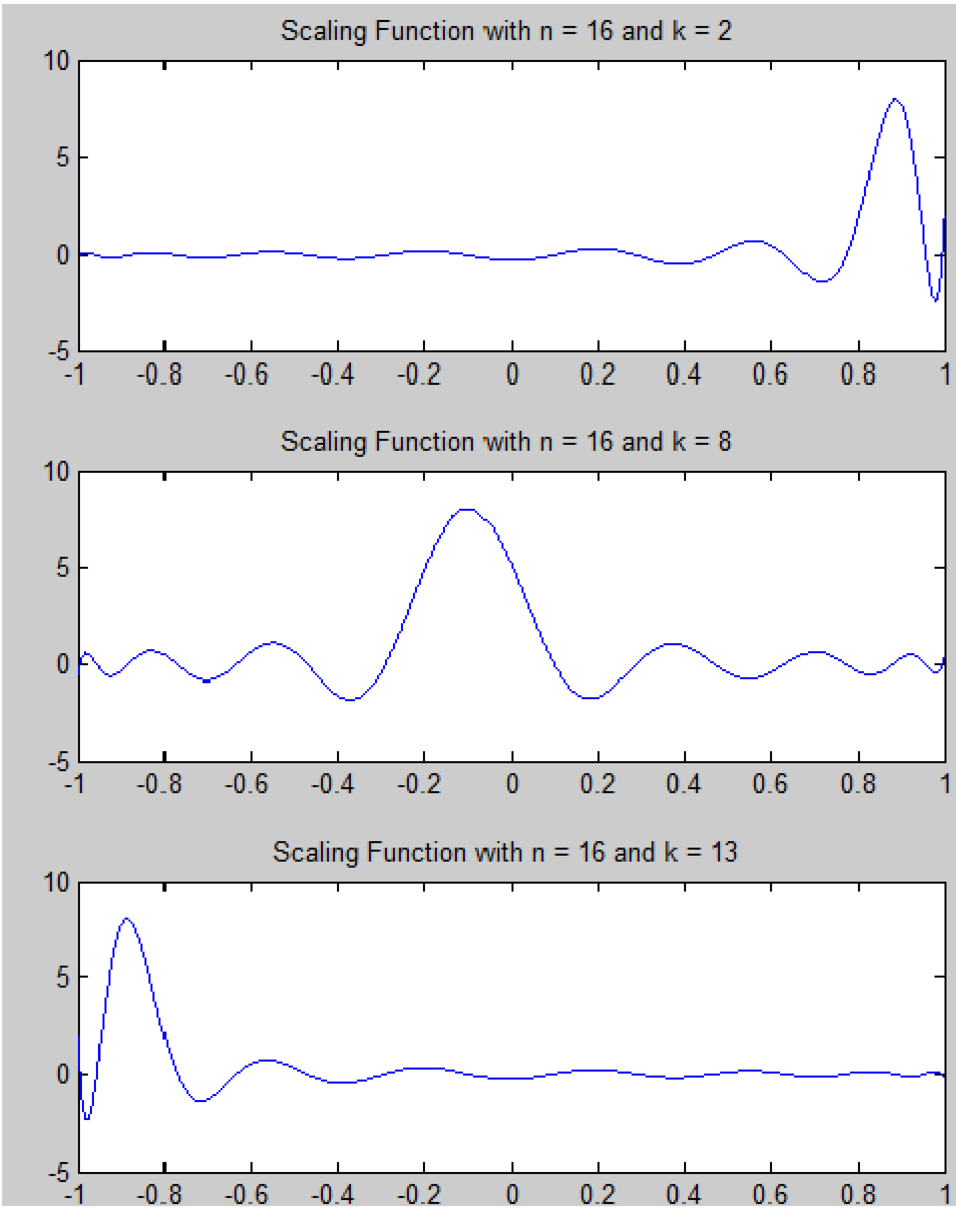

Figure 1: Scaling Function with $n = 16$ for $k = 2, 8, 13$

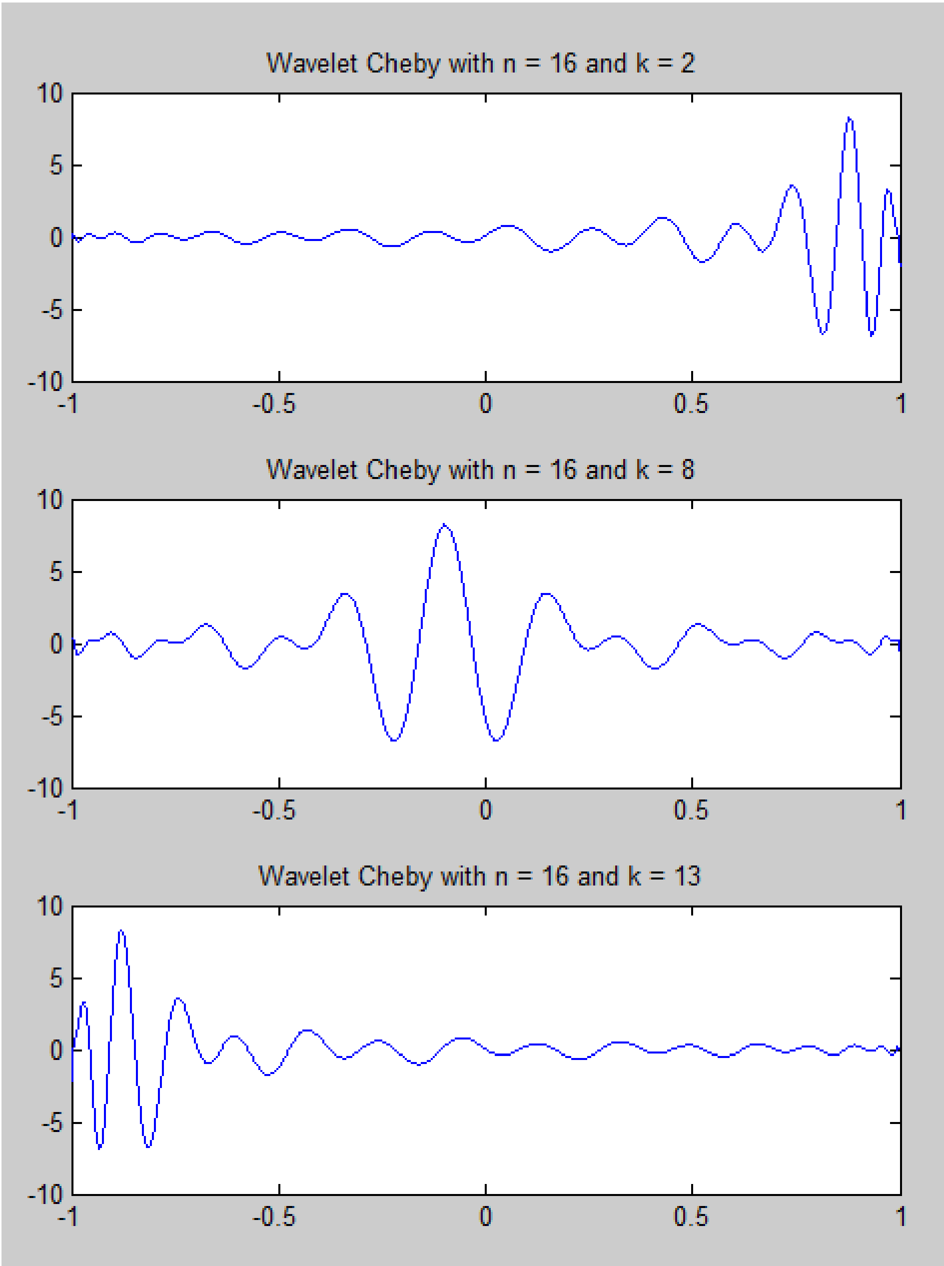

Figure 2: Chebyshev Wavelet with $n = 16$ for $k = 2, 8, 13$

[5] Kilgore, T., and Prestin, J. Polynomial wavelets on an interval. *Constructive Approximation*. 1996. 12(1): 1–18.

[6] Capobiancho, M.R., and Themistoclakis, W. Interpolating polynomial wavelet on $[-1, 1]$. *Advanced in Computational Mathematics*. 2005. 23: 353–374.

[7] Fischer, B. and Themistoclakis, W. (2002). Orthogonal polynomial wavelets. *Numerical Algorithms*. 30: 37–58.

[8] Fischer, B. and Prestin, J. Wavelet based on orthogonal polynomials. *Math. Comp.* 1997. 66: 1593–1618.

[9] Davis, P. J. *Interpolation and Approximation.* New York : Dover Publishing Inc. 1973.