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Strong Convergence Theorems for a Finite Family of Quasi-nonexpansive and Asymptotically Quasi-nonexpansive Mappings in Banach spaces

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Abstract In this paper, we give the sufficient condition for strong convergence of finite step iteration sequences with errors to a common fixed point for a pair of a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in closed convex subset of a Banach spaces. The results presented in this paper generalize, improve and unify the corresponding results in [1, 2, 4, 5, 9, 10].

Keywords Quasi-nonexpansive mapping; asymptotically quasi nonexpansive mapping; common fixed point; multi-step iteration scheme.

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1 Introduction and Preliminaries

Let E be a real Banach space, K be a nonempty subset of E and $S, T \colon K \to K$ be two mappings. We recall the following definitions:

Definition 1 Let $T \colon K \to K$ be a mapping:

(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$.

(2) T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in K$, $p \in F(T)$.

(3) T is said to be asymptotically nonexpansive if there exists a sequence $\{\lambda_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda_n) \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

(4) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{\lambda_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda_n) \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(5) T is said to be uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

(6) T is said to be uniformly quasi-Lipschitzian if there exists $L \in [1, +\infty)$ such that

$$\|T^n x - p\| \leq L \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

From the above definitions, it follows that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian. But the converse does not hold.

Recall also that strong convergence is the type of convergence usually associated with convergence of a sequence. More formally, a sequence $\{x_n\}$ of vectors in a normed space (and, in particular, in an inner product space E) is called convergent to a vector x in E if

$$\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty$$

.

In 1973, Petryshyn and Williamson [9] established a necessary and sufficient condition for a Mann [8] iterative sequence to converge strongly to a fixed point of a quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [2] extended the results of [9] and obtained some necessary and sufficient condition for an Ishikawa-type iterative sequence to converge to a fixed point of a quasi-nonexpansive mapping. In 2001, Liu in [4, 5] extended the results of Ghosh and Debnath [2] to the more general asymptotically quasi-nonexpansive mappings. In 2006, Shahzad and Udomene [10] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space. In 2007, Chidume and Ali [1] gave the necessary and sufficient condition for a strong convergence of common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in real Banach spaces.

Recently, Liu et al. in [6, 7] study the weak and strong convergence of common fixed points for modified two and modified three-step iteration sequence with errors with respect to a pair of mappings S and T.

Motivated and inspired by Liu et al. in [6,7] and others, we study the following iteration scheme for a pair of a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings. Our scheme is as follows:

Definition 2 Let K be a nonempty convex subset of a normed linear space E and

$$S_i, T_i \colon K \to K$$

for all $i = 1, 2, \ldots, N$ be two families of mappings. For an arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n \geq 1}$ defined by:

$$
\begin{array}{rcl}
x_{n+1} = x_n^{(N)} & = & \alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} S_N x_n + \gamma_n^{(N)} u_n^{(N)} \\
x_n^{(N-1)} & = & \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} S_{N-1} x_n + \gamma_n^{(N-1)} u_n^{(N-1)} \\
\dots & = & \dots \\
\dots & = & \dots \\
x_n^{(3)} & = & \alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} S_3 x_n + \gamma_n^{(3)} u_n^{(3)} \\
x_n^{(2)} & = & \alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} S_2 x_n + \gamma_n^{(2)} u_n^{(2)} \\
x_n^{(1)} & = & \alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} S_1 x_n + \gamma_n^{(1)} u_n^{(1)}
\end{array} \tag{1}
$$

is called multi-step iterative sequences with errors, where $\{u_n^{(i)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, for all $i = 1, 2, \dots, N$.

Remark 1 In case $S_1 = S_2 = \cdots = S_N = I$, then the sequence $\{x_n\}_{n \geq 1}$ generated in (1) reduces to the multi-step iteration with errors for N asymptotically quasi-nonexpansive mappings.

The purpose of this paper is to give necessary and sufficient condition for strong convergence of finite step iteration sequences with error terms defined by (1) to a common fixed point for a pair of a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in a real Banach space. The results presented in this paper generalize, improve and unify the corresponding results of [1, 2, 4, 5, 9, 10] and many others.

In the sequel we need the following lemmas to prove our main results:

Lemma 1 ([11]; Lemma 1): Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2 Let K be a nonempty convex subset of a normed linear space E. Let

$$S_1, S_2, \dots, S_N \colon K \to K$$

be a finite family of quasi-nonexpansive mappings and $T_1, T_2, \dots, T_N \colon K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\} \subset [0, \infty)$ satisfying $\lim_{n\to\infty} \lambda_{in} = 0$ such that $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \dots, N$ and

$$F(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset.$$

Let the sequence $\{x_n\}_{n\geq 1}$ defined by (1) with the restrictions $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$. Then

(a) $\lim_{n\to\infty}\|x_n - q\|$ exists for any $q \in F(S,T)$.

(b) There exists a constant $K' > 0$ such that $\|x_{n+m} - q\| \le K' \|x_n - q\| + K' \sum_{k=n}^{n+m-1} d_k^{(N)}$, for all $n, m \ge 1$ and $q \in F(S,T)$, where $K' = e^{\sum_{k=n}^{n+m-1} t_k}$.

Proof (a) Let $q \in F(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i)$. Since $\{u_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ are bounded sequences in K. So we can set

$$D = \max\{\sup_{n\ge1} \left\|u_n^{(i)} - q\right\| : i = 1, 2, \dots, N\}.$$

Since $S_1, S_2, \dots, S_N$ are quasi-nonexpansive and $T_1, T_2, \dots, T_N$ are asymptotically quasi-nonexpansive mappings, it follows from (1), that

$$\begin{aligned}
\left\|x_n^{(1)} - q\right\| &= \left\|\alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} S_1 x_n + \gamma_n^{(1)} u_n^{(1)} - q\right\| \\
&\le \alpha_n^{(1)} \|T_1^n x_n - q\| + \beta_n^{(1)} \|S_1 x_n - q\| + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&\le \alpha_n^{(1)} (1 + \lambda_{1n}) \|x_n - q\| + \beta_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&\le [\alpha_n^{(1)} + \beta_n^{(1)}](1 + \lambda_{1n}) \|x_n - q\| + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&= [1 - \gamma_n^{(1)}](1 + \lambda_{1n}) \|x_n - q\| + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&\le (1 + \lambda_{1n}) \|x_n - q\| + \gamma_n^{(1)} D \\
&\le (1 + \lambda_{1n}) \|x_n - q\| + d_n^{(1)}, \qquad (2)
\end{aligned}$$

where $d_n^{(1)} = \gamma_n^{(1)} D$. Since $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (2) that

$$\begin{aligned}
\left\|x_n^{(2)} - q\right\| &\le \alpha_n^{(2)} \left\|T_2^n x_n^{(1)} - q\right\| + \beta_n^{(2)} \|S_2 x_n - q\| + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&\le \alpha_n^{(2)} (1 + \lambda_{2n}) \left\|x_n^{(1)} - q\right\| + \beta_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&\le \alpha_n^{(2)} (1 + \lambda_{2n}) [(1 + \lambda_{1n}) \|x_n - q\| + d_n^{(1)}] + \beta_n^{(2)} \|x_n - q\| \\
&\quad + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&\le [\alpha_n^{(2)} + \beta_n^{(2)}](1 + \lambda_{1n})(1 + \lambda_{2n}) \|x_n - q\| + \alpha_n^{(2)} (1 + \lambda_{2n}) d_n^{(1)} \\
&\quad + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&= [1 - \gamma_n^{(2)}](1 + \lambda_{1n})(1 + \lambda_{2n}) \|x_n - q\| + \alpha_n^{(2)} (1 + \lambda_{2n}) d_n^{(1)} \\
&\quad + \gamma_n^{(2)} D \\
&\le (1 + \lambda_{1n})(1 + \lambda_{2n}) \|x_n - q\| + (1 + \lambda_{2n}) d_n^{(1)} \\
&\quad + \gamma_n^{(2)} D \\
&\le [1 + \lambda_{1n} + \lambda_{2n}(1 + \lambda_{1n})] \|x_n - q\| + d_n^{(2)} \qquad (3)
\end{aligned}$$

where

$$d_n^{(2)} = (1 + \lambda_{2n}) d_n^{(1)} + \gamma_n^{(2)} D.$$

Since $\sum_{n=1}^{\infty} \gamma_n^{(2)} < \infty$ and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, we can see that

$$\sum_{n=1}^{\infty} d_n^{(2)} < \infty.$$

It follows from (3) that

$$\begin{aligned}
\left\| x_n^{(3)} - q \right\| &\le \alpha_n^{(3)} \left\| T_3^n x_n^{(2)} - q \right\| + \beta_n^{(3)} \| S_3 x_n - q \| + \gamma_n^{(3)} \left\| u_n^{(3)} - q \right\| \\
&\le \alpha_n^{(3)}(1+\lambda_{3n}) \left\| x_n^{(2)} - q \right\| + \beta_n^{(3)} \| x_n - q \| + \gamma_n^{(3)} \left\| u_n^{(3)} - q \right\| \\
&\le \alpha_n^{(3)}(1+\lambda_{3n})[(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})) \| x_n - q \| + d_n^{(2)}] \\
&\quad + \beta_n^{(3)} \| x_n - q \| + \gamma_n^{(3)} \left\| u_n^{(3)} - q \right\| \\
&\le (\alpha_n^{(3)} + \beta_n^{(3)})[(1+\lambda_{3n})(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))] \| x_n - q \| \\
&\quad + \alpha_n^{(3)}(1+\lambda_{3n}) d_n^{(2)} + \gamma_n^{(3)} \left\| u_n^{(3)} - q \right\| \\
&= [1-\gamma_n^{(3)}][(1+\lambda_{3n})(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))] \| x_n - q \| \\
&\quad + \alpha_n^{(3)}(1+\lambda_{3n}) d_n^{(2)} + \gamma_n^{(3)} D \\
&\le [(1+\lambda_{3n})(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))] \| x_n - q \| \\
&\quad + (1+\lambda_{3n}) d_n^{(2)} + \gamma_n^{(3)} D \\
&\le [1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})+\lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n})] \| x_n - q \| + d_n^{(3)}, \quad (4)
\end{aligned}$$

where

$$d_n^{(3)} = (1+\lambda_{3n}) d_n^{(2)} + \gamma_n^{(3)} D.$$

Since $\sum_{n=1}^{\infty} \gamma_n^{(3)} < \infty$ and $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$, we can see that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. Continuing the above process, we get

$$\begin{aligned}
\| x_{n+1} - q \| &= \left\| x_n^{(N)} - q \right\| \\
&\le [1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})+\lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n}) \\
&\quad + \cdots + \lambda_{Nn}(1+\lambda_{1n})(1+\lambda_{2n}) \ldots (1+\lambda_{(N-1)n})] \| x_n - q \| + d_n^{(N)} \\
&\le (1+t_n) \| x_n - q \| + d_n^{(N)} \quad (5)
\end{aligned}$$

where

$$\begin{aligned}
t_n = \lambda_{1n} &+ \lambda_{2n}(1+\lambda_{1n}) + \lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n}) + \ldots \\
&+ \lambda_{Nn}(1+\lambda_{1n})(1+\lambda_{2n}) \ldots (1+\lambda_{(N-1)n}).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \ldots, N$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$ and by assumptions of the theorem $\sum_{n=1}^{\infty} d_n^{(N)} < \infty$. From Lemma 1, we know that $\lim_{n\to\infty} \| x_n - q \|$ exists. This completes the proof of part (a).

(b) Since $1+x \leq e^x$ for all $x>0$. Then from part (a) it can be obtained that

$$\begin{aligned}
\|x_{n+m}-q\| &\leq (1+t_{n+m-1})\,\|x_{n+m-1}-q\| + d_{n+m-1}^{(N)} \\
&\leq e^{t_{n+m-1}}\,\|x_{n+m-1}-q\| + d_{n+m-1}^{(N)} \\
&\leq e^{t_{n+m-1}}[e^{t_{n+m-2}}\,\|x_{n+m-2}-q\| + d_{n+m-2}^{(N)}] + d_{n+m-1}^{(N)} \\
&\leq e^{(t_{n+m-1}+t_{n+m-2})}\,\|x_{n+m-2}-q\| + e^{t_{n+m-1}} d_{n+m-2}^{(N)} + d_{n+m-1}^{(N)} \\
&\leq e^{(t_{n+m-1}+t_{n+m-2})}\,\|x_{n+m-2}-q\| + e^{t_{n+m-1}}[d_{n+m-2}^{(N)} + d_{n+m-1}^{(N)}] \\
&\leq \dots \\
&\leq \dots \\
&\leq e^{\sum_{k=n}^{n+m-1} t_k}.\,\|x_n-q\| + e^{\sum_{k=n}^{n+m-1} t_k}.\sum_{k=n}^{n+m-1} d_k^{(N)} \\
&\leq K'\,\|x_n-q\| + K'\sum_{k=n}^{n+m-1} d_k^{(N)}, \text{ where } K' = e^{\sum_{k=n}^{n+m-1} t_k}.
\end{aligned}$$

This completes the proof of part (b). □

Now, we are in the position to prove the main result of this paper.

2 Main Results

In this section, we prove the strong convergence theorems for a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings by using iteration scheme (1) in the framework of Banach spaces.

Theorem 1 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2, \dots, S_N\colon K \to K$ be a finite family of quasi-nonexpansive mappings and $T_1, T_2, \dots, T_N\colon K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\} \subset [0,\infty)$ satisfying $\lim_{n\to\infty}\lambda_{in}=0$ such that $\sum_{n=1}^{\infty}\lambda_{in}<\infty$ for all $i=1,2,\dots,N$ and $F(S,T)=\bigcap_{i=1}^{N}F(S_i)\cap F(T_i)\neq\emptyset$. Let the sequence $\{x_n\}_{n\geq 1}$ defined by (1) with the restrictions $\sum_{n=1}^{\infty}\gamma_n^{(i)}<\infty$ for all $i=1,2,\dots,N$. Then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$ if and only if $\liminf_{n\to\infty} d(x_n, F(S,T))=0$, where $d(x, F(S,T))$ denotes the distance between x and the set $F(S,T)$.*

Proof The necessity is obvious. Thus we only prove the sufficiency. For all $q\in F(S,T)$, by equation (5) of Lemma 2, we have

$$\|x_{n+1}-q\| \leq (1+t_n)\,\|x_n-q\| + d_n^{(N)},\ \forall n\in N \tag{6}$$

since $\sum_{n=1}^{\infty} t_n<\infty$ and $\sum_{n=1}^{\infty} d_n^{(N)}<\infty$, so from equation (6), we obtain

$$d(x_{n+1},F(S,T)) \leq (1+t_n)d(x_n,F(S,T)) + d_n^{(N)} \tag{7}$$

Since $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$ and from Lemma 1, we have $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$.

Next we will show that $\{x_n\}$ is a Cauchy sequence. For all $\varepsilon_1 > 0$, from Lemma 2(b), it can be known there must exists a constant $K' > 0$ such that

$$\begin{aligned} \|x_{n+m} - q\| &\leq K' \|x_n - q\| \\ &\quad + K' \sum_{k=n}^{n+m-1} d_k^{(N)}, \ \forall n, m \in N, \ \forall q \in F(S,T). \end{aligned} \tag{8}$$

Since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$ and $\sum_{k=n}^{\infty} d_k^{(N)} < \infty$, then there must exists a constant N_1, such that when $n \geq N_1$

$$d(x_n, F(S,T)) < \frac{\varepsilon_1}{4(K'+1)}, \ \textit{and} \ \sum_{k=n}^{\infty} d_k^{(N)} < \frac{\varepsilon_1}{2K'}.$$

So there must exists $q^* \in F(S,T)$, such that

$$d(x_{N_1}, F(S,T)) = \|x_{N_1} - q^*\| < \frac{\varepsilon_1}{2(K'+1)}.$$

From (8), it can be obtained that when $n \geq N_1$

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q^*\| + \|x_n - q^*\| \\ &\leq K' \|x_{N_1} - q^*\| + K' \sum_{k=N_1}^{\infty} d_k^{(N)} + \|x_{N_1} - q^*\| \\ &\leq (K'+1) \|x_{N_1} - p^*\| + K' \sum_{k=N_1}^{\infty} d_k^{(N)} \\ &< (K'+1). \frac{\varepsilon_1}{2(K'+1)} + K'. \frac{\varepsilon_1}{2K'} \\ &< \varepsilon_1 \end{aligned}$$

that is

$$\|x_{n+m} - x_n\| < \varepsilon_1.$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = y^*$. Then $y^* \in K$. It remains to show that $y^* \in F(S,T)$. Let $\varepsilon_2 > 0$ be given. Then there exists a natural number N_2 such that

$$\|x_n - y^*\| < \frac{\varepsilon_2}{2(L+1)}, \ \forall n \geq N_2.$$

Since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$, there must exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$, we have

$$d(x_n, F(S,T)) < \frac{\varepsilon_2}{3(L+1)},$$

and in particular, we have

$$d(x_{N_3}, F(S,T)) < \frac{\varepsilon_2}{3(L+1)}.$$

Therefore, there exists $z^* \in F(S,T)$ such that

$$\|x_{N_3} - z^*\| < \frac{\varepsilon_2}{2(L+1)}.$$

Consequently, we have

$$\begin{aligned}
\|T_i y^* - y^*\| &= \|T_i y^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\
&\le \|T y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le L\|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le L\|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le L[\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le (L+1)\|y^* - x_{N_3}\| + (L+1)\|z^* - x_{N_3}\| \\
&< (L+1).\frac{\varepsilon_2}{2(L+1)} + (L+1).\frac{\varepsilon_2}{2(L+1)} \\
&< \varepsilon_2.
\end{aligned}$$

This implies that $y^* \in F(T_i)$ for all $i = 1, 2, \ldots, N$. Similarly, we can show that $y^* \in F(S_i)$ for all $i = 1, 2, \ldots, N$. Since S_i for all $i = 1, 2, \ldots, N$ is quasi-nonexpansive, so it is uniformly quasi-1 Lipschitzian, so here taking $L = 1$, we have

$$\begin{aligned}
\|S_i y^* - y^*\| &= \|S_i y^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\
&\le \|S y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le \|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le \|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le [\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le 2\|y^* - x_{N_3}\| + 2\|z^* - x_{N_3}\| \\
&< 2.\frac{\varepsilon_2}{4} + 2.\frac{\varepsilon_2}{4} \\
&< \varepsilon_2.
\end{aligned}$$

This shows that $y^* \in F(S_i)$ for all $i = 1, 2, \ldots, N$. Thus $y^* \in \bigcap_{i=1}^N F(S_i) \cap F(T_i)$, that is, y^* is a common fixed point of $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$. This completes the proof. □

Theorem 2 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2, \ldots, S_N \colon K \to K$ be a finite family of quasi-nonexpansive mappings and $T_1, T_2, \ldots, T_N \colon K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\} \subset [0, \infty)$ satisfying $\lim_{n\to\infty} \lambda_{in} = 0$ such that $\sum_{n=1}^\infty \lambda_{in} < \infty$ for all $i = 1, 2, \ldots, N$ and $F(S,T) = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}_{n\ge1}$ defined by (1) with the restrictions $\sum_{n=1}^\infty \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \ldots, N$. Then $\{x_n\}_{n\ge1}$ converges strongly to a common fixed point of $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to q.*

Proof The proof of Theorem 2 follows from Lemma 1 and Theorem 1. This completes the proof. □

Theorem 3 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2, \dots, S_N : K \to K$ be a finite family of quasi-nonexpansive mappings and $T_1, T_2, \dots, T_N : K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\} \subset [0, \infty)$ satisfying $\lim_{n\to\infty} \lambda_{in} = 0$ such that $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \dots, N$ and $F(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}_{n\geq 1}$ defined by (1) with the restrictions $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$. Suppose that the mappings S_i and T_i for all $i = 1, 2, \dots, N$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for all* $i = 1, 2, \dots, N$*;*

(ii) *there exists a constant $A > 0$ such that*

$$\{\|x_n - S_i x_n\| + \|x_n - T_i x_n\|\} \geq A d(x_n, F(S,T))$$

for all $i = 1, 2, \dots, N$ and for all $n \geq 1$.

Then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$.

Proof From condition (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$, it follows as in the proof of Theorem 1, that $\{x_n\}_{n\geq 1}$ must converges strongly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$. This completes the proof. □

Remark 2 Theorem 1 extend, improve and unify the corresponding results of [1,2,4,5,9,10]. Especially Theorem 1 extend, improve and unify Theorem 1 and 2 in [5], Theorem 1 in [4] and Theorem 3.2 in [10] in the following ways:

(1) The identity mapping in [4], [5] and [10] is replaced by the more general quasi-nonexpansive mapping.

(2) The usual Ishikawa iteration scheme [3] in [4], the usual modified Ishikawa iteration scheme with errors in [5] and the usual modified Ishikawa iteration scheme with errors for two mappings in [10] are extended to the multi-step iteration scheme with errors with respect to a pair of a finite family of mappings.

Remark 3 Theorem 2 extend, improve and unify Theorem 3 in [5] and Theorem 3 extend, improve and unify Theorem 3 in [4] in the following aspects:

(1) The identity mapping in [4] and [5] is replaced by the more general quasi-nonexpansive mapping.

(2) The usual Ishikawa iteration scheme [3] in [4] and the usual modified Ishikawa iteration scheme with errors in [5] are extended to the multi-step iteration scheme with errors with respect to a pair of a finite family of mappings.

Remark 4 Theorem 1 also extends, improves and unifies Theorem 3.2 in [1] in the following aspects:

(1) The identity mapping in [1] is replaced by the more general quasi-nonexpansive mapping.

(2) The finite step iteration scheme in [1] is extended to the multi-step iteration scheme with errors with respect to a pair of a finite families of mappings.

3 Conclusion

Our results are good improvement and generalization of the corresponding results of [1, 2, 4, 5, 9, 10].

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References

[1] Chidume abd Bashir Ali, C. E. Convergence theorems for finite families of asymptotically quasi-nonexpansive mappings. *J. Inequal. Appl.* 2007. Article ID 68616: 10.

[2] Ghosh, M. K. and Debnath, L. Convergence of Ishikawa iterates of quasi-nonexpansive mappings. *J. Math. Anal. Appl.* 1997. 207: 96–103.

[3] Ishikawa, S. Fixed points by a new iteration method. *Proc. Amer. Math. Soc.* 1974. 44: 147–150.

[4] Liu, Q. H. Iterative sequences for asymptotically quasi-nonexpansive mappings. *J. Math. Anal. Appl.* 2001. 259: 1–7.

[5] Liu, Q. H. Iterative sequences for asymptotically quasi-nonexpansive mappings with error member. *J. Math. Anal. Appl.*. 2001. 259: 18–24.

[6] Zeqing Liu, Agarwal, Ravi, P., Chi Feng and Shin Min Kang. Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*. 2005. 44: 83–96.

[7] Zeqing Liu, Chi Feng, Jeong Sheok Ume and Shin Min Kang. Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings. *Taiw. J. Math.*. 2007. 11(1): 27–42.

[8] Mann, W. R. Mean value methods in iteration. *Proc. Amer. Math. Soc.* 1953. 4: 506–510.

[9] Petryshyn, W. V. and Williamson, T.E. Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings. *J. Math. Anal. Appl.*. 1973. 43: 459–497.

[10] Shahzad, N. and Udomene, A. Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.* 2006. Article ID 18909:10.

[11] Tan, K. K. and Xu, H. K. Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. *J. Math. Anal. Appl.* 1993. 178: 301–308.