MATEMATIKA, 2011, Volume 27, Number 2, 139–148

Approximating the Stieltjes Integral Using the Generalized Midpoint Rule

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Abstract Abstract. Accurate approximations for the Stieltjes integral by the generalized midpoint rule. The generalized midpoint rule is established on the notion of the derivative of function with respect to the strictly increasing function defined in [9].

Keywords Keywords: Stieltjes integral, generalired midpoint rule, the derivative of the function with respect to the strictly increasing function, generalized formula of Taylor.

2010 Mathematics Subject Classification 26D15; 26A42.

1 Infroduction

Our aim is to describe the generalized midpoint rule for the approximation of Stieltjes integral

$$I = \int_a^b f(x)du(x), a < b, a, b \in R, \tag{1}$$

where $f(x)$ is a given continuous function on $[a, b]$ and $u(x)$ is a given function of bounded variation on $[a, b]$.

It is known $[7, 205p.]$, that the function $u(x)$ presented in the form

$$u(x) = \varphi(x) - \psi(x), x \in [a, b] \tag{2}$$

where $\varphi(x)$ and $\psi(x)$ are the known increasing functions on $[a, b]$.
It is suggested different methods of the approximate calculation of the Stieltjes integral in works $[1 - 6]$. Particularly, in 1998 Dragomir and Fedotov [2], in order to approximate the Stieltjes integral (1) with the simpler expression

$$\frac{1}{b-a}[u(b) - u(a)] \int_a^b f(x)dx,$$

introduced the following error functional

$$D(f, u; a, b) := \int_a^b f(x)du(x) - \frac{1}{b-a}[u(b) - u(a)] \int_a^b f(x)dx.$$

In this work for the approximate calculation of the Stieltjes integral (1) it is suggested the generalized midpoint rule which is based on the notion of the derivative of function with respect to strictly increasing function [9]. The generalized midpoint rule summarizes the midpoint rule [8]. Then we need the concept of the derivative defined in the work [9] and the theorems with proofs connected with it. Apparently the first notion of the derivative with respect to the strictly increasing function was introduced in [9].

Definition 1 The derivative of a function $f(x)$ with respect to $\varphi(x)$ is the function $f_{\varphi}^{'}(x)$ whose value at $x \in (a,b)$ is the number:

$$f_{\varphi}^{'}(x) = \lim_{\Delta\to 0} \frac{f(x+\Delta)-f(x)}{\varphi(x+\Delta)-\varphi(x)}, \tag{3}$$

where $\varphi(t)$ is a given the strictly increasing continuous function in (a,b).

If the limit in equation (3) exists, we say that $f(x)$ has a derivative (is differentiable) with respect to $\varphi(x)$. The first derivative $f_{\varphi}^{'}(x)$ may also be differentiable function with respect to $\varphi(x)$ at every point $x \in (a,b)$. If so, its derivative

$$f_{\varphi}^{''}(x) = (f_{\varphi}^{'}(x))_{\varphi}^{'},$$

is called the second derivative of $f(x)$ with respect to $\varphi(x)$. The names continue as you imagine they would, with

$$f_{\varphi}^{(n)}(x) = (f_{\varphi}^{(n-1)}(x))_{\varphi}^{'}$$

denoting the n-th derivative of $f(x)$ with respect to $\varphi(x)$.

Example 1 The function $f(x) = |x|$ is nondifferentiable at the point $x = 0$. If

$$\varphi(x) = \begin{cases} -|x|^{\frac{1}{3}}, & x < 0, \\ |x|^{\frac{1}{3}}, & x \geq 0, \end{cases}$$

then the function $\varphi(x)$ is the strictly increasing continuous function in $(-\infty, \infty)$. We shall show that the function $f(x) = |x|$ has the continuous derivative by means of $\varphi(x)$ at every point $x \in (-\infty, \infty)$.

Let $x < 0$, then from Definition 1,we obtain

$$f_{\varphi}'(x) = \lim_{\Delta x\to 0} \frac{|x+\Delta x|-|x|}{\varphi(x+\Delta x)-\varphi(x)} = \lim_{\Delta x\to 0} \frac{\left(|x+\Delta x|^{\frac{1}{3}}\right)^3 - \left(|x|^{\frac{1}{3}}\right)^3}{-\left(|x+\Delta x|^{\frac{1}{3}} - |x|^{\frac{1}{3}}\right)}$$

$$= -\lim_{\Delta x\to 0} \frac{\left(|x+\Delta x|^{\frac{1}{3}} - |x|^{\frac{1}{3}}\right)\left(|x+\Delta x|^{\frac{2}{3}} + |x+\Delta x|^{\frac{1}{3}}|x|^{\frac{1}{3}} + |x|^{\frac{2}{3}}\right)}{\left(|x+\Delta x|^{\frac{1}{3}} - |x|^{\frac{1}{3}}\right)} = -3|x|^{\frac{2}{3}}.$$

If $x > 0$, then $f_{\varphi}^{'}(x) = 3|x|^{\frac{2}{3}}$.

Let $x = 0$ and $\Delta x > 0$. Then

$$\lim_{\Delta x\to 0} \frac{f(\Delta x)-f(0)}{\varphi(\Delta x)-\varphi(0)} = \lim_{\Delta x\to 0} \frac{|\Delta x|}{|\Delta x|^{\frac{1}{3}}} = 0.$$

If $x = 0$ and $\Delta x < 0$, then

$$\lim_{\Delta x\to 0} \frac{f(\Delta x)-f(0)}{\varphi(\Delta x)-\varphi(0)} = \lim_{\Delta x\to 0} \frac{|\Delta x|}{-|\Delta x|^{\frac{1}{3}}} = 0,$$

i.e.

$$f'_{\varphi}(0) = 0.$$

Then:

$$f'_{\varphi}(x) = \begin{cases} -3|x|^{\frac{2}{3}}, & x < 0, \\ 3|x|^{\frac{2}{3}}, & x \geq 0. \end{cases}$$

It is clear that the function $f'_{\varphi}(x)$ is the continuous function in $(-\infty, \infty)$.

Theorem 1 *(Generalized Fermat's Theorem).*
Let $\varphi(x)$ is strictly increasing continuous function in $[a,b]$, the function $f(x)$ is continuous at every point of the closed interval $[a,b]$ and the function $f(x)$ has a local maximum or a local minimum value in an interior point $x_0 \in (a,b)$ and $\exists f'_{\varphi}(x_0)$. Then $f'_{\varphi}(x_0) = 0$.

Proof The point x_0 can not be the point of increase (or decrease). Because if the point x_0 is the point of increase (or decrease)then in some deleted neighborhood of this point

$$\frac{f(x) - f(x_0)}{\varphi(x) - \varphi(x_0)} > 0, \left(\text{or } \frac{f(x) - f(x_0)}{\varphi(x) - \varphi(x_0)} < 0 \right)$$

That is why inequality $f'_y(x_0) > 0$ or $f'_y(x_0) < 0$ is impossible. It remains to accept, that $f'_{\varphi}(x_0) = 0$. Theorem 1 is proved.

Theorem 1 generalizes the Fermat's Theorem [10, 185p.].

Theorem 2 *(Generalized Rolle's Theorem).*
Let $\varphi(x)$ is strictly increasing continuous function in $[a,b]$, the function $f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable with respect to $\varphi(x)$ at every point of its interior (a,b) and $f(a) = f(b)$. Then there is at least one number ξ between a and b at which $f'_{\varphi}(\xi) = 0$.

Proof On the ground of the theorem of Weierstrass the function $f(x)$, which is continuons in $[a,b]$, gets the maximum M and the minimum m in it. If both of these value are got on the ends of $[a,b]$, then by the data they are equal $(M = m)$. And it means that the function $f(x)$ is identically constant in $[a,b]$. Tnen the derivative $f'_{\varphi}(x)$ is equal to 0 in all points of $[a,b]$. Even if one of these values - maximum or minimum - is got in point $\xi \in (a,b)$ (that is $m < M$), then on the strength of Theorem 1, $f'_{\varphi}(\xi) = 0$. Theorem 2 is proved.

Theorem 2 generalizes Rolle's Theorem [10, 186p.].

Theorem 3 *(Generalized Taylor's Theorem).*
Let $\varphi(x)$ is strictly increasing continuous function in $[a,b]$, $f(x)$ has the $(n+1)$ th derivative with respect to $\varphi(x)$ in (a,b) and $\forall x_0, x_1 \in (a,b)$. Then

$$R_n(x_1) = f(x_1) - f_n(x_0, x_1) = \frac{f_{\varphi}^{(n+1)}(c)}{(n+1)!} (\varphi(x_1) - \varphi(x_0))^{n+1}, \tag{4}$$

where $x_0 < c < x_1$(or $x_1 < c < x_0$)

$$f_n(x_0, x_1) = f(x_0) + \sum_{k=1}^{n} \frac{f_{\varphi}^{(k)}(x_0)}{k!} (\varphi(x_1) - \varphi(x_0))^k.$$

Proof Let $x_1 > x_0$. We define the number H by the equality

$$H = \frac{f(x_1) - f_n(x_0, x_1)}{(\varphi(x_1) - \varphi(x_0))^{n+1}}.$$

Hence we have

$$f(x_1) - f_n(x_0, x_1) - H(\varphi(x_1) - \varphi(x_0))^{n+1} = 0 \tag{5}$$

We consider the function $p(x)$, defined on $[x_0, x1]$ by the correlation

$$p(x) = f(x_1) - f_n(x, x_1) - H(\varphi(x_1) - \varphi(x))^{n+1}. \tag{6}$$

Then taking into account (5) from (6) we have $p(x_0) = 0$. In addition, the function $p(x)$ is differentiable with respect to $\varphi(x)$ at every point on (x_0, x_1) and is continuous on $[x_0, x_1]$. Taking into account $f_n(x_1, x_1) = f(x_1)$ from (6) we obtain $p(x_1) = 0$. Therefore, on the strength of Theorem 2

$$p_\varphi^{'}(c) = 0, c \in (x_0, x_1).$$

But

$$p_\varphi^{'}(x) = f_\varphi^{'}(x) - \sum_{k=1}^{n} \left[\frac{f_\varphi^{(k)}(x)}{k!}(\varphi(x_1) - \varphi(x))^k\right]_\varphi^{'} + (n+1)H(\varphi(x_1) - \varphi(x))^n,$$

that is

$$p_\varphi^{'}(x) = (n+1)H(\varphi(x_1) - \varphi(x))^n - \frac{f_\varphi^{(n+1)}(x)}{n!}(\varphi(x_1) - \varphi(x))^n$$

Here for $x = c$ we obtain

$$H = \frac{f_\varphi^{(n+1)}(c)}{(n+1)!}.$$

Analogously we prove the case $x_1 < x_0$. Theorem 3 is proved.

Theorem 3 generalizes Taylor's Theorem [10, 598p.].

Theorem 4 *Let function $f(x)$ is continuous function in $[a, b]$, $\varphi(x)$ is strictly increasing continuous function in $[a, b]$ and*

$$F(x) = \int_a^x f(t)d\varphi(t), \quad x \in [a, b].$$

Then

$$F_\varphi^{'}(x) = \left(\int_a^x f(t)d\varphi(t)\right)_\varphi^{'} = f(x), \quad x \in [a, b],$$

where

$$F_\varphi^{'}(a) = \lim_{\Delta x \to 0+} \frac{F(a+\Delta x) - F(a)}{\varphi(a+\Delta x) - \varphi(a)}, \quad F_\varphi^{'}(b) = \lim_{\Delta x \to 0-} \frac{F(b+\Delta x) - F(b)}{\varphi(b+\Delta x) - \varphi(b)}$$

Proof From definition of $F'_\varphi(x)$, we have

$$F'_\varphi(x) = \lim_{\Delta x \to 0} \left(f(x) \int_x^{x+\Delta x} d\varphi(t) - \int_x^{x+\Delta x} (f(x) - f(t)) d\varphi(t) \right) / [\varphi(x + \Delta x) - \varphi(x)]$$
$$= f(x) - \lim_{\Delta x \to 0} \psi(x, \Delta x),$$

where

$$\psi(x, \Delta x) = \left(\int_x^{x+\Delta x} (f(x) - f(t)) d\varphi(t) \right) / [\varphi(x + \Delta x) - \varphi(x)].$$

Then

$$|\psi(x, \Delta x)| \le \left[\omega_f(\Delta x) \left(\int_x^{x+\Delta x} d\varphi(t) \right) \right] / [\varphi(x + \Delta x) - \varphi(x)] = \omega_f(\Delta x),$$

where

$$\omega_f(\delta) = \sup_{|t-x| \le \delta} |f(x) - f(t)|,$$

and $\lim_{\delta \to 0} \omega_f(\delta) = 0$. Therefore

$$\lim_{\Delta x \to 0} |\psi(x, \Delta x)| \le \lim_{\Delta x \to 0} \omega_f(|\Delta x|) = 0.$$

Hence, $F'_\varphi(x) = f(x)$. Analogously the other cases are proved. Theorem 4 is proved.

Theorem 4 generalizes the Fundamental Theorem of Calculus [10, 279p.].

Corollary 1 Let $F_0(x) = f(x) \in C[a, b]$, $\varphi(x)$ is strictly increasing continuous function on $[a, b]$ and

$$F_i(x) = \int_a^x F_{i-1}(t) d\varphi(t), x \in [a, b], i = 1, ..., n.$$

Then $F_n(x) \in C_\varphi^{(n)}[a, b]$, where $C_\varphi^{(n)}[a, b]$ is the linear space of all continuous functions $v(x)$ defined in $[a, b]$ such that $v_\varphi^{(n)}(x) \in C[a, b]$.

2 The Generalized Midpoint Rule

Let us divide the closed interval $[a, b]$ into $N = 2n$ partition where

$$x_i = a + ih, i = 0, 1, ..., 2n, h = \frac{b - a}{2n}.$$

Then the approximate value of the integral (1) we define the formula

$$A = \sum_{i=1}^{n} f(x_{2i-1})[u(x_{2i}) - u(x_{2i-2})]. \tag{7}$$

Theorem 5 *Let $f(x)$ is the continuous function in $[a,b], u(x)$ is the function of bounded variation in $[a,b]$. Then, the estimate*

$$|I-A| \leq \omega_f(h)[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)], \tag{8}$$

holds, where $\omega_f(h)$ is the modulus of continuity of function $f(x), \varphi(x)$ and $\psi(x)$ are defined with the help of the formula (2).

Proof Let us introduce the notations:

$$I_i = \int_{x_{2i-2}}^{x_{2i}} f(x)du(x) = \int_{x_{2i-2}}^{x_{2i}} f(x)d\varphi(x) - \int_{x_{2i-2}}^{x_{2i}} f(x)d\psi(x), \tag{9}$$

$$A_i = \int_{x_{2i-2}}^{x_{2i}} f(x_{2i-1})d\varphi(x) - \int_{x_{2i-2}}^{x_{2i}} f(x_{2i-1})d\psi(x), \tag{10}$$

where $i = 1, 2, ..., n$. Then taking into account (9), (10), (2), from (1) and (7) we obtain

$$I = \sum_{i=1}^{n} I_i, A = \sum_{i=1}^{n} A_i. \tag{11}$$

On the strength of (9) and (10) we have

$$\begin{aligned} |I_i - A_i| &\leq \int_{x_{2i-2}}^{x_{2i}} |f(x)-f(x_{2i-1})|d\varphi(x) + \int_{x_{2i-2}}^{x_{2i}} |f(x)-f(x_{2i-1})|d\psi(x) \\ &\leq \omega_f(h)[\varphi(x_{2i})-\varphi(x_{2i-2})] + [\psi(x_{2i})-\psi(x_{2i-2})], \end{aligned} \tag{12}$$

where $i = 1, 2, ..., n$. Taking into account (12) from (11) we obtain

$$\begin{aligned} |I-A| \leq \sum_{i=1}^{n} |I_i - A_i| &\leq \omega_f(h) \sum_{i=1}^{n} \{[\varphi(x_{2i})-\varphi(x_{2i-2})] \\ &+ [\psi(x_{2i})-\psi(x_{2i-2})]\} = \omega_f(h)[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)]. \end{aligned}$$

Theorem 5 is proved.

Theorem 5 generalizes the midpoint rule [8].

Corollary 2 Let $f(x) \in C^{\alpha}[a,b], 0 < \alpha \leq 1, u(x)$ is the function of bounded variation in $[a,b]$, where $C^{\alpha}[a,b]$ is the Holder space i.e. $\forall x_1, x_2 \in [a,b]$ the estimate $|f(x_1)-f(x_2)| \leq c|x_1-x_2|^{\alpha}$ holds, c is a positive constant dependent on $f(x)$ but independent on x_1, x_2. Then, the estimate

$$|I-A| \leq ch^{\alpha}[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)] \tag{13}$$

holds.

Let

$$h = \frac{b-a}{2n}, \ x_j = a + jh, \ j = 0, 1, ..., 2n,$$

$$\begin{cases} y_i = \varphi^{-1}(\frac{1}{2}(\varphi(x_{2i}) + \varphi(x_{2i-2}))), \\ \\ z_i = \psi^{-1}(\frac{1}{2}(\psi(x_{2i}) + \psi(x_{2i-2}))), \quad i = 1, 2, ..., n, \end{cases} \tag{14}$$

where $\varphi^{-1}(t)$ is inverse function to $\varphi(x)$ and $\psi^{-1}(s)$ is the inverse function to $\psi(x)$. From (14), we obtain $y_i, z_i \in (x_{2i-2}, x_{2i}),\ i = 1, 2, ..., n$. Then the approximate value we define by the formula

$$B_n = \sum_{i=1}^{n} f(y_i)[\varphi(x_{2i}) - \varphi(x_{2i-2})] + \sum_{i=1}^{n} f(z_i)[\psi(x_{2i}) - \psi(x_{2i-2})]. \tag{15}$$

Theorem 6 *Let $\varphi(x)$ and $\psi(x)$ are the strictly increasing continuous functions in $[a, b]$ and $f_\varphi^{''}(x)$ and $f_\psi^{''}(x)$ are the continuous functions in $[a, b]$. Then the estimate*

$$|I - B_n| \le \frac{c_0}{24}[\varphi(b) - \varphi(a)](\omega_\varphi(2h))^2 + \frac{c_1}{24}[\psi(b) - \psi(a)](\omega_\psi(2h))^2, \tag{16}$$

holds, where

$$\begin{cases} c_0 = \sup\limits_{x\in[a,b]} |f_\varphi^{''}(x)|, c_1 = \sup\limits_{x\in[a,b]} |f_\psi^{''}(x)|, \\ \\ \omega_\varphi(\delta) = \sup\limits_{|x_1-x_2|\le\delta} |\varphi(x_1) - \varphi(x_2)|, \omega_\psi(\delta) = \sup\limits_{|x_1-x_2|\le\delta} |\psi(x_1) - \psi(x_2)|. \end{cases} \tag{17}$$

Proof Let us introduce the notations:

$$\begin{cases} P_i = \int\limits_{x_{2i-2}}^{x_{2i}} f(x)d\varphi(x),\ Q_i = \int\limits_{x_{2i-2}}^{x_{2i}} f(x)d\psi(x), \\ \\ M_i = f(y_i)[\varphi(x_{2i}) - \varphi(x_{2i-2})] = \int\limits_{x_{2i-2}}^{x_{2i}} f(y_i)d\varphi(x), \\ \\ N_i = f(z_i)[\psi(x_{2i}) - \psi(x_{2i-2})] = \int\limits_{x_{2i-2}}^{x_{2i}} f(z_i)d\psi(x),\ i = 1, 2, ..., n. \end{cases} \tag{18}$$

Using Theorem 3 we can write

$$\begin{aligned} f(x) &= f(y_i) + f_\varphi^{'}(y_i)(\varphi(x) - \varphi(y_i)) \\ &\quad + \frac{1}{2} f_\varphi^{''}(y_i + \theta_1(x - y_i))(\varphi(x) - \varphi(y_i))^2,\ 0 < \theta_1 < 1, \end{aligned} \tag{19}$$

$$\begin{aligned} f(x) &= f(z_i) + f_\psi^{'}(z_i)(\psi(x) - \psi(z_i)) \\ &\quad + \frac{1}{2} f_\psi^{''}(z_i + \theta_2(x - z_i))(\psi(x) - \psi(z_i))^2,\ 0 < \theta_2 < 1. \end{aligned} \tag{20}$$

Taking into account (18), (19) and (20) we obtain

$$
\begin{aligned}
P_i - M_i &= \int_{x_{2i-2}}^{x_{2i}} [f(x) - f(y_i)]d\varphi(x) \\
&= \frac{1}{2} f_{\varphi}^{'}(y_i)(\varphi(x) - \varphi(y_i))^2|_{x_{2i-2}}^{x_{2i}} + \frac{1}{2}\int_{x_{2i-2}}^{x_{2i}} f_{\varphi}^{''}(y_i + \theta_1(x - y_i))(\varphi(x) - \varphi(y_i))^2 d\varphi(x) \\
&= \frac{1}{2}\int_{x_{2i-2}}^{x_{2i}} f_{\varphi}^{''}(y_i + \theta_1(x - y_i))(\varphi(x) - \varphi(y_i))^2 d\varphi(x),\ i = 1, 2, ..., n, \qquad (21) \\
Q_i - N_i &= \int_{x_{2i-2}}^{x_{2i}} [f(x) - f(z_i)]d\varphi(x) \\
&= \frac{1}{2}\int_{x_{2i-2}}^{x_{2i}} f_{\psi}^{''}(z_i + \theta_2(x - z_i))(\psi(x) - \psi(z_i))^2 d\psi(x),\ i = 1, 2, ..., n. \qquad (22)
\end{aligned}
$$

On the strength of (17) from (21) and (22) we have

$$
\begin{aligned}
|P_i - M_i| &\le \frac{c_0}{6}(\varphi(x) - \varphi(y_i))^3|_{x_{2i-2}}^{x_{2i}} = \frac{c_0}{6}\cdot 2[\varphi(x_{2i}) - \frac{\varphi(x_{2i}) + \varphi(x_{2i-2})}{2}]^3 \\
&= \frac{c_0}{24}[\varphi(x_{2i}) - \varphi(x_{2i-2})]^3,\ i = 1, 2, ..., n, \qquad (23) \\
|Q_i - N_i| &\le \frac{c_1}{24}[\psi(x_{2i}) - \psi(x_{2i-2})]^3,\ i = 1, 2, ..., n. \qquad (24)
\end{aligned}
$$

Then, taking into account (9), (15), (18), (23) and (24) we obtain

$$
\begin{aligned}
|I - B_n| &\le \sum_{i=1}^{n}|P_i - M_i| + \sum_{i=1}^{n}|Q_i - N_i| \\
&\le \frac{c_0}{24}(\omega_{\varphi}(2h))^2\sum_{i=1}^{n}[\varphi(x_{2i}) - \varphi(x_{2i-2})] \\
&\quad + \frac{c_1}{24}(\omega_{\psi}(2h))^2\sum_{i=1}^{n}[\psi(x_{2i}) - \psi(x_{2i-2})] \\
&= \frac{c_0}{24}[\varphi(b) - \varphi(a)](\omega_{\varphi}(2h))^2 + \frac{c_1}{24}[\psi(b) - \psi(a)](\omega_{\psi}(2h))^2.
\end{aligned}
$$

Theorem 6 is proved.

The Theorem 6 generalizes the Error Estimate for the midpoint rule [8].

Corollary 3 Let $\varphi(x)$ and $\psi(x)$ are the strictly increasing continuous functions on $[a, b]$, $f_{\varphi}^{''}(x)$ and $f_{\psi}^{''}(x)$ are the continuous functions in $[a, b]$,
$\varphi(x) \in C^{\alpha}[a, b],\ 0 < \alpha \le 1, \psi(x) \in C^{\beta}[a, b],\ 0 < \beta \le 1$, i.e. $\forall x_1,\ x_2 \in [a, b]$ the estimates

$$|\varphi(x_1) - \varphi(x_2)| \le c_2|x_1 - x_2|^{\alpha}, |\psi(x_1) - \psi(x_2)| \le c_3|x_1 - x_2|^{\beta}$$

hold. Then the estimate

$$|I - B_n| \le \frac{1}{6}c_0 c_2^2[\varphi(b) - \varphi(a)]h^{2\alpha} + \frac{1}{6}c_1 c_3^2[\psi(b) - \psi(a)]h^{2\beta}$$

holds.

Corollary 4 Let $\psi(x) = 0$ for all $x \in [a,b], \varphi(x)$ is the strictly increasing continuous function on $[a,b]$ and $f_{\varphi}^{''}$ is the continuous function on $[a,b]$. Then

$$|I_0 - B_{on}| \leq \frac{c_0}{24}[\varphi(b) - \varphi(a)](\omega_{\varphi}(2h))^2,$$

where

$$I_0 = \int_a^b f(x)d\varphi(x),\ B_{on} = \sum_{i=1}^{n} f(y_i)[\varphi(x_{2i}) - \varphi(x_{2i-2})].$$

Corollary 5 Let $\varphi(x)$ is the strictly increasing continuous function on $[a,b]$, and $f_{\varphi}^{''}$ is the continuous function on $[a,b].\psi(x) = 0$ for all $x \in [a,b], f_{\varphi}^{''}$ is the continuous function on $[a,b], \varphi(x) \in C^{\alpha}[a,b], 0 < \alpha \leq 1$, i.e. $\forall x_1, x_2 \in [a,b]$ the estimate

$$|\varphi(x_1) - \varphi(x_2)| \leq c_2|x_1 - x_2|^{\alpha}$$

hold. Then

$$|I_0 - B_{on}| \leq \frac{c_0}{6}[\varphi(b) - \varphi(a)]c_2^2 h^{2\alpha}.$$

Corollary 6 Let $\varphi(x) = x$ and $\psi(x) = 0$ for all $x \in [a,b], f^{''}(x) \in C[a,b]$. Then

$$|I_0 - B_{on}| \leq \frac{c_0}{6}(b-a)h^2.$$

3 Conclusion

In the work [9] the new notion of derivative of function with respect to strictly increasing function is defined. This definition of the derivative generalizes the classical definition of the derivative of function. On the basis of the new definition of derivative the main theorems of Mathematical Analyses are generalized. In particular the Fundamental Theorem of Calculus was generalized for Stieltjes integral. In this article on the basis of the derivative of function with respect to strictly increasing function the generalized midpoint rule is suggested for the for the approximate calculation of the Stieltjes integral. The rule suggested in this article generalyses the midpoint rule.

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