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Quarter-Sweep Gauss-Seidel Method for Solving First Order Linear Fredholm Integro-differential Equations

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Abstract The purpose of this paper is to scrutinize the application of the Quarter-Sweep Gauss-Seidel (QSGS) method by using the quarter-sweep approximation equation based on backward difference (BD) and repeated trapezoidal (RT) formulas to solve first order linear Fredholm integro-differential equations. The formulation and implementation of the Full-Sweep Gauss-Seidel (FSGS) and Half-Sweep Gauss-Seidel (HSGS) iterative methods are also presented. Then some numerical tests are illustrated to show the effectiveness of QSGS method as compared to the FSGS and HSGS methods.

Keywords Integro-differential equations; Linear Fredholm equations; Finite difference; Quadrature formulas; Quarter-Sweep iteration.

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1 Introduction

Consider the first order linear Fredholm integro-differential equation (FOLFIDE) as given in the general form of

$$y\,'(x) = p(x)y(x) + f(x) + \int_a^b K(x,t)y(t)\,dt\,, \quad a \le x \le b\,,$$
$$y\,(a) = y_a, \tag{1}$$

where the functions $f(x), p(x)$ and the kernel $K(x,t)$ are known and $y(x)$ is the solution to be determined. In the engineering field, numerical methods for solution of linear Fredholm integro-differential equations (LFIDE) have been studied by authors such as compact finite difference method [1], Wavelet-Galerkin method [2], Lagrange interpolation method [3], Tau method [4], quadrature-difference method [5], variational method [6], collocation method [7], homotopy perturbation method [8], Euler-Chebyshev method [9] and GMRES method [10]. LFIDE are usually difficult to solve analytically so numerical approaches are applied to obtain an approximation solution for problem (1). To solve a LFIDE equation numerically, discretization of differential part and integral part is used to construct a system of linear algebraic equations. By considering numerical techniques, actually, there are many schemes that can be used to discretize problem (1) independently for linear differential and integral terms. Many researchers have implemented discretization schemes for linear differential term such as finite difference scheme [11, 12], Taylor polynomial scheme [13], Chebyshev polynomial method [14], Runge-Kutta scheme [15] and Euler implicit schemes [16]. Again in discretizing linear integral term numerically, many discretization schemes can be used for approximation such as quadrature [17, 18], projection method [19, 20], least squares [21] and Cleanshaw-Curtis quadrature formula [22].

The concept of half-sweep iterative method was introduced by Abdullah [23] by employing the Explicit Decoupled Group (EDG) iterative method to solve two-dimensional Poisson equations. In addition, further explorations for application of this concept have been widely used to solve multi-dimensional partial differential equations [24-28]. Othman and Abdullah [29] then extended the concept of quarter-sweep iteration via the Modified Explicit Group (MEG) iterative methods to solve two-dimensional Poisson equations. Again, applications of this concept have been intensively discussed in [30-33]. These two concepts are essential to reduce the computational complexities during the iterative process, whereas the implementation of the half- and quarter-sweep iterations will only consider nearly half and quarter of all node points in a solution domain respectively. In this paper, we investigate the application of the FSGS, HSGS, and QSGS iterative methods by using approximation equation based on finite difference and quadrature schemes for solving first order linear Fredholm integro-differential equations. In point of fact, the standard GS iterative method also called as the Full-Sweep Gauss-Seidel iterative method is implemented to be a control method in order to examine the performance of HSGS and QSGS iterative methods.

The organization of this paper is as follows: In Section 2, the formulation of the full-, half-, and quarter-sweep finite difference and quadrature approximation equations will be elaborated. In Section 3, formulation of the FSGS, HSGS and QSGS methods will be demonstrated. Some numerical results will be illustrated to emphasize the effectiveness of proposed methods in Section 4 and in Section 5 conclusion is given.

2 Formulation of the Quarter-Sweep Approximation Equation

Based on Figure 1, the full-, half- and quarter-sweep iterative methods will compute approximate values at solid node points only until the convergence criterion is reached. It seems that the implementation of the quarter-sweep iterative method just involves by nearly one-quarter of whole inner points as shown in Figure 1(c) compared with the full-sweep iterative method. Then other approximation solutions for the remaining points are calculated by using direct methods [23-33]. Actually there are many direct methods that can be considered. In this paper, however, we propose the linear interpolation to compute approximate values of remaining points for both half and quarter-sweep cases.

2.1 Derivation of Quarter-Sweep Backward Difference Scheme

As mentioned in Section 1 in discretizing differential term, the first order Backward Difference (BD) scheme based on finite difference method is used to form an approximation equation for differential term. In solving of FOLFIDE, the first order BD scheme is used to approximate any differential term in problem (1). In general, first order BD scheme can be derived from the Taylor series expansion as given by

$$y'\left(x_i\right) = \frac{y\left(x_i\right) - y\left(x_{i-1}\right)}{h} + O\left(h\right). \tag{2}$$

Eq. (2) is called as the first BD scheme because it involves the values at x_i and x_{i-1}, where h is the size interval between nodes. The notation $O\left(h\right)$ is known as truncations error which will not be considered in this paper.

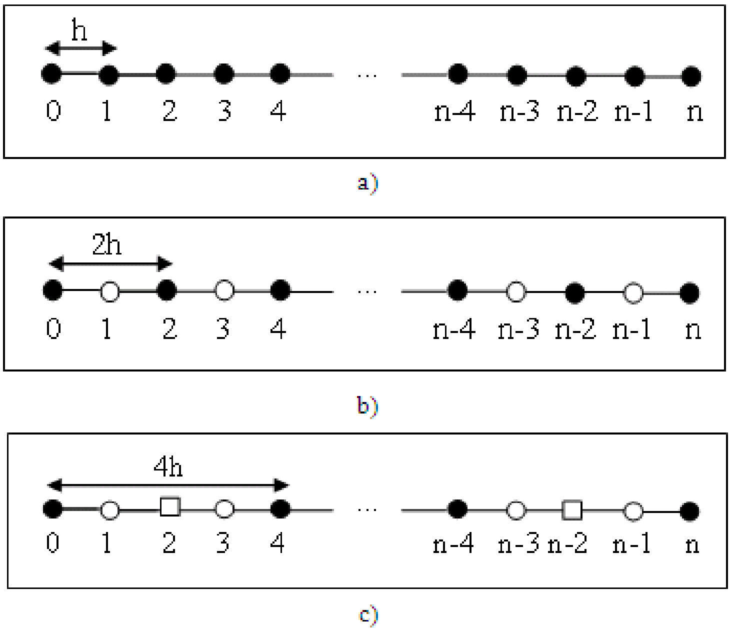

Figure 1: Diagrams a), b) and c) Show Distribution of Uniform Node Points for the Full-, Half- and Quarter-Sweep Cases, Respectively

In order to construct the finite grid networks for formulation of the full-, half- and quarter-sweep finite difference approximation equations over problem (1), further discussions will be restricted onto first-order BD scheme which is as follows

$$y'(x_i) \approx \frac{y(x_i) - y(x_{i-p})}{ph}, \tag{3}$$

where the value of p, which corresponds to 1, 2 and 4, represents the full-, half- and quarter-sweep respectively.

2.2 Derivation of Quarter-Sweep Repeated Trapezoidal Rule

For the integral term, the Repeated Trapezoidal (RT) discretization scheme based on quadrature method is used to construct an approximation equation. In general quadrature formula can be defined as follows

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \ , \tag{4}$$

where t_j $(j = 0, 1, \ldots, n)$ are the abscissas of the partition points of the integration interval $[a,b]$ or quadrature (interpolation) nodes, $A_j (j = 0, 1, \ldots, n)$ are numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of Eq. (2). Based on

RT scheme, numerical coefficients A_j satisfy the following relation

$$A_j = \begin{cases} \frac{1}{2}ph, & j = 0, n \\ ph, & \text{otherwise} \end{cases} \tag{5}$$

where the constant step size, h is defined by

$$h = \frac{b-a}{n} \tag{6}$$

and n is the number of subintervals in the interval $[a,b]$. Meanwhile, the value of p, which corresponds to 1, 2 and 4, represents the full-, half- and quarter-sweep respectively.

By substituting Eqs. (3) and (4) into Eq. (1), a general linear approximation equation for full-, half-, and quarter-sweep cases can be constructed as

$$\frac{y_i - y_{i-p}}{ph} = P_i y_i + f_i + \sum_{j=0,\ p,\ 2p}^{n} A_j K_{i,j} y_j. \tag{7}$$

This approximation equation will be used to create the corresponding linear system, which can be easily shown as

$$M \underset{\sim}{y} = \underset{\sim}{f} \tag{8}$$

where

$$M = \begin{bmatrix} 1 - hP_pA_pK_{p,p} & -hA_{2p}K_{p,2p} & \cdots & -hA_nK_{p,n} \\ -hA_pK_{2p,p} & 1 - hP_pA_{2p}K_{2p,2p} & \cdots & -hA_pK_{2p,n} \\ \vdots & \vdots & \ddots & \vdots \\ -hA_pK_{n,p} & -hA_{2p}K_{n,2p} & \cdots & 1 - hP_nA_nK_{n,n} \end{bmatrix}_{\left(\frac{n}{p}\times\frac{n}{p}\right)}$$

$$\underset{\sim}{y} = \begin{bmatrix} y_p \\ y_{2p} \\ \vdots \\ y_{n-p} \\ y_n \end{bmatrix} \quad \text{and} \quad \underset{\sim}{f} = \begin{bmatrix} (1 + hA_0K_{p,0}) + hf \\ hA_0K_{2p,0} + hf_p \\ \vdots \\ hA_0K_{n-p,0} + hf_{n-p} \\ hA_0K_{n,0} + hf_n \end{bmatrix}.$$

3 Formulation of Family of Gauss-Seidel Iterative Methods

In this paper, FSGS, HSGS and QSGS iterative methods will be applied to solve linear systems (8) generated from the approximation equation (7) through the discretization of problem (1). Let the matrix M be articulated into

$$M = D - L - U, \tag{9}$$

where D, L and U are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Thus, the general scheme for FSGS, HSGS and QSGS iterative methods can be written as

$$\underset{\sim}{y}^{(k+1)} = (D-L)^{-1}\left(U\underset{\sim}{y}^{(k)} + \underset{\sim}{f}\right). \tag{10}$$

Actually, the iterative methods attempt to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the initial vector, $\underset{\sim}{y}^{(0)}$. Basically FSGS, HSGS and QSGS iterative methods will be performed until the solution is within a predetermined acceptable bound on the error. Based on [1] and [31], the general algorithm for FSGS, HSGS and QSGS iterative methods to solve problem (1) would be generally described in Algorithm 1.

Algorithm 1: FSGS, HSGS and QSGS Methods

(i) Initializing all the parameters. Set $k = 0$.

(ii) For $i = \ p,\ 2p,\ \cdots, n-2p,\ n-p,\ n$ and $j = 0,\ p,\ 2p,\ \cdots, n-2p,\ n-p,\ n$

(iii) Calculate

$$y_i^{(k+1)} = \frac{1}{1 - hP_i - hA_iK_{i,i}} \left(hf_i + y_{i-1} - h\sum_{j=0}^{i-1} A_jK_{i,j}y_j^{(k+1)} - h\sum_{j=i+1}^{n} A_jK_{i,j}y_j^{(k)} \right)$$

(iv) Convergence test

(a) If the error of tolerance $\left| y_i^{(k+1)} - y_i^{(k)} \right| < \varepsilon = 10^{-10}$ is satisfied, iteration will be terminated.

(v) Else, set $k = k{+}1$ and go to step (ii).

4 Illustrative examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the FSGS, HSGS, and QSGS iterative methods. These three methods were executed on the computer using a program written in C language. For comparison purpose, three criteria will be considered such as number of iterations, execution time and maximum absolute error.

Example 1 (Darania [34])

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)\,dt \quad 0 \le x \le 1$$

with the condition

$$y(0) = 0.$$

The exact solution of the problem is $y(x) = x$.

Example 2 (Darania [34])

$$y'(x) = xe^x + e^x - x + \int_0^1 y(t)\,dt, \quad 0 \le x \le 1$$

with the condition

$$y(0) = 0.$$

The exact solution of the problem is $y(x) = xe^x$.

Throughout the numerical experiments, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$. The experiments were carried out on several different mesh sizes such as 960, 1920, 3840, 7680, 15360 and 30720. Results of numerical simulations, which were obtained from implementations of the FSGS, HSGS and QSGS iterative methods for Example 1 and Example 2, have been recorded in Table 1 and Table 2, respectively.

Table 1: Comparison of a Number of Iterations, Execution Time (Seconds) and Maximum Absolute Error for the Iterative Methods (Example 1)

Methods	Number of iterations					
	Mesh size					
	960	1920	3840	7680	15360	30720
FSGS+BD+RT	12	12	12	12	12	12
HSGS+BD+RT	12	12	12	12	12	12
QSGS+BD+RT	12	12	12	12	12	12
Methods	Execution time (seconds)					
	Mesh size					
	960	1920	3840	7680	15360	30720
FSGS+BD+RT	0.38	1.52	5.99	23.73	136.66	563.21
HSGS+BD+RT	0.07	0.27	1.09	4.26	17.61	69.81
QSGS+BD+RT	0.02	0.08	0.28	1.05	4.58	18.02
Methods	Maximum absolute error					
	Mesh size					
	960	1920	3840	7680	15360	30720
FSGS+BD+RT	1.0347 E-7	2.5849 E-8	6.4591 E-9	1.6133 E-9	1.6671 E-12	1.6561 E-12
HSGS+BD+RT	4.1438 E-7	1.0347 E-7	2.5849 E-8	6.4591 E-9	1.6132 E-9	4.0203 E-10
QSGS+BD+RT	1.6617 E-6	4.1438 E-7	1.0347 E-7	2.5849 E-8	6.4591 E-9	1.6133 E-9

5 Conclusion

In this paper, the effectiveness for the family of Gauss-Seidel (GS) iterative methods namely FSGS, HSGS and QSGS was investigated to solve LFIDE for first-order integro-differential equations. As mentioned in Section 2, the formulation and implementation of these three methods have been constructed based on combination of first-order BD and first quadrature schemes. Based on these discretization schemes, numerical results tabulated in Tables 1 and 2 show that the application of the half- and quarter-sweep iterative concepts reduces tremendously computational time (refer Table 3) with the acceptable accuracy. For the three iterative methods tested, QSGS iterative method is faster for the computational works compared to HSGS and FSGS iterative methods. This is due to the computational complexity of the QSGS is approximately 75% less than FSGS method. In order to measure the accuracy, these three methods were run with increasing mesh sizes $n = 960$, 1920, 3840, 15360 and 30720. It can be seen clearly from Example 1 and Example 2 in Tables

Table 2: Comparison of a Number of Iterations, Execution Time (Seconds) and Maximum Absolute Error for the Iterative Methods (Example 2)

Methods	Number of Iterations					
	Mesh Size					
	960	1920	3840	7680	15360	30720
FSGS+BD+RT	13	13	13	13	13	13
HSGS+BD+RT	13	13	13	13	13	13
QSGS+BD+RT	13	13	13	13	13	13
Methods	Execution Time (Seconds)					
	Mesh Size					
	960	1920	3840	7680	15360	30720
FSGS+BD+RT	0.39	1.50	6.01	23.83	78.82	326.97
HSGS+BD+RT	0.09	0.30	0.95	4.65	19.49	76.60
QSGS+BD+RT	0.03	0.08	0.29	1.16	5.02	19.78
Methods	Maximum Absolute Error					
	Mesh Size					
	960	1920	3840	7680	15360	30720
FSGS+BD+RT	2.8493 E-3	1.4242 E-3	7.1202 E-4	3.5598 E-4	1.7799 E-4	8.8891 E-5
HSGS+BD+RT	5.7019 E-3	2.8493 E-3	1.4242 E-3	7.1202 E-4	3.5598 E-4	1.7798 E-4
QSGS+BD+RT	1.1417 E-3	5.7020 E-3	2.8493 E-3	1.4242 E-3	7.1202 E-4	3.5598 E-4

Table 3: Percentages of Reduction of the Execution Time for HSGS and QSGS Iterative Methods Compared with FSGS Method

Methods	Execution Time	
	Example 1	Example 2
HSGS+BD+RT	81.57%-87.61%	75.27%-84.19%
QSGS+BD+RT	94.73%-96.80%	93.11%-95.17%

1 and 2 that the use of the largest mesh size $n = 30720$ has shown improvements slightly in the accuracy of the three methods. For future studies, quarter-sweep concept will be applied to other iterative methods for LFIDE problem in order to examine the effectiveness of their combination. Apart from that, the investigation of the high-order quadrature schemes should be carried out to derive high-order approximation equations in order to get high accuracy of numerical solutions.

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