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NLS Equation with a Variable Coefficient in a Stenosed Elastic Tube Filled with an Averaged Inviscid Fluid

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Abstract Wave modulation in an elastic tube has been studied by researchers since year 1979. In the present work, by considering the artery as a prestressed thin elastic tube with a symmetrical stenosis and the blood as an incompressible averaged inviscid fluid. We have studied the amplitude modulation of non-linear waves in such a composite medium by using the reductive perturbation method. The governing equations can be reduced to the non-linear Schrodinger (NLS) equation with variable coefficient. The progressive wave solution to the above non-linear evolution equation is then sought. The non-linear Schrodinger (NLS) equation with variable coefficient shows increasing shock wave propagates to the right as real time passed.

Keywords Elastic tube; stenosis; inviscid fluid; nonlinear Schrodinger equation.

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1 Introduction

Due to its applications in arterial mechanics, the propagation of pressure pulses in fluid-filled distensible tubes has been studied by several researchers [1,2]. As far as the biological applications are concerned, most of the works on wave propagation in compliant tubes have considered small amplitude waves ignoring the nonlinear effects and focused on the dispersive character of waves. However, when the nonlinear terms arising from the constitutive equations and kinematical relations are introduced, one has to consider either finite amplitude, or small-but-finite amplitude waves, depending on the order of nonlinearity.

Recently, Tay [3], Tay et al. [4], and Tay and Demiray [5] studied the non-linear wave propagation in a prestressed thin elastic tube with a symmetrical stenonis filled with inviscid, viscous and Newtonian fluid with variable viscosity, they showed that the governing equations can be reduced to forced Kortewed-de Vries, forced perturbed Kortewed-de Vries and forced Kortewed-de Vries-Burgers equations, respectively.

The modulation of small-but-finite amplitude pressure waves in a fluid-filled distensible, linear elastic tube has been examined by Ravindran and Prasad [6]. They obtained the non-linear Schrodinger (NLS) equation. The work of non-linear waves modulation in a prestressed thin elastic tube filled with inviscid or viscous fluid has been carried out by Demiray and co-worker [7–9]. They showed that the governing equations can be reduced to NLS and dissipative NLS equations, respectively. The NLS equation is the simplest representative equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media. It has a balance between the nonlinearity and dispersion.

As far as we know, none of the literature works dealt with wave modulation in a stenosed elastic tube yet. Most of the researches studied the wave modulation in an elastic tube without present of stenosis. The governing equation is NLS equation. In our approach, considering the artery as an incompressible, prestressed, thin-walled elastic tube with a symmetrical stenosis and the blood as an averaged inviscid fluid, we have studied the amplitude modulation of non-linear waves in such a composite medium by using the reductive perturbation method referring the work of Tay [3]. We obtained the NLS equation with a variable coefficient. We then sought the progressive wave solution to the non-linear evolution equation obtained. The result of this work will be discussed in order to determine the blood flow patterns in the stenosed artery.

2 Equations of Tube

In this section, we shall derive the governing equations of an elastic tube filled with an average inviscid fluid. Such a combination of a solid and a fluid is considered to be a model for blood flow in arteries. The equation of motion of the tube in the radial direction takes the following form [3]

$$\mu R_0 \frac{\partial}{\partial z^*}\left\{\frac{\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)}{\Lambda}\frac{\partial \Sigma}{\partial \lambda_1}\right\} - \frac{\mu}{\lambda_z}\frac{\partial \Sigma}{\partial \lambda_2} + \frac{P^*}{H}(r_0 - f^* + u^*) - \rho_0 \frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}} = 0. \tag{1}$$

where the function Λ is defined by

$$\Lambda = \left[1+\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2}.$$

The constant μ is the shear modulus of the tube material, R_0 is the radius of a circularly cylindrical tube, z^* is the axial coordinate at the intermediate configuration, t^* is the time parameter, r_0 is the radius of the origin after finite static deformation, u^* is the finite dynamical radial displacement, $f^*(z^*)$ is the stenosis functions after the deformation, ρ_0 is the mass density of the membrane material, H is the thickness in the undeformed configuration, P^* is the inner pressure applied by the fluid, λ_1 and λ_2 are stretch ratios along the deformed meridional and circumferential curves, respectively, λ_z is the constant stretch ratio along the tube axis and Σ is the strain energy density function.

3 Equations of Fluid

In general, blood is known to be an incompressible non-Newtonian fluid.However, in the course of flow in large arteries, the red blood cells in the vicinity of arterial wall move to the central region of the artery so that hematocrit ratio becomes quite low near the arterial wall, where the shear rate is quite high, as can be seen from Poiseuille flow. Experimental studies indicate when the hematocrit ratio is low and the shear rate is high, blood behaves like a Newtonian fluid (see [2, 10]). For this case,the variation of the field quantities with the radial coordinate may be neglected. Thus, the averaged equations of motion of an incompressible fluid may be given by

$$\frac{\partial A^*}{\partial t^*} + \frac{\partial}{\partial z^*}(A^* w^*) \quad = \quad 0, \tag{2}$$

$$\frac{\partial w^*}{\partial t^*} + w^* \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial P^*}{\partial z^*} = 0, \tag{3}$$

where A^* is the cross-sectional area of the tube, w^* is the averaged axial fluid velocity and P^* is the averaged fluid pressure.

At this stage it is convenient to introduce the following non-dimensional quantities:

$$\begin{aligned} &t^* = \left(\frac{R_0}{c_0}\right) t, \quad && z^* = R_0 z, \quad && u^* = R_0 u, \\ &f^* = R_0 f, && w^* = c_0 w, && P^* = \rho_f c_0^2 p, \\ &r_0 = R_0 \lambda_\theta, && c_0^2 = \frac{\mu H}{\rho_f R_0}, && m = \frac{\rho_0 H}{\rho_f R_0}. \end{aligned} \tag{4}$$

Introducing (4) into the equations (1), (2) and (3), the following non-dimensional equations are obtained

$$\begin{aligned} p \quad = \quad & \frac{m}{\lambda_z(\lambda_\theta - f(z) + u)}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda_z(\lambda_\theta - f(z) + u)}\frac{\partial \Sigma}{\partial \lambda_2} \\ & - \frac{1}{(\lambda_\theta - f(z) + u)}\frac{\partial}{\partial z}\left\{\frac{(-f' + \partial u/\partial z)}{[1 + (-f' + \partial u/\partial z)^2]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\} \end{aligned} \tag{5}$$

$$2\frac{\partial u}{\partial t} + 2w\left[-f' + \frac{\partial u}{\partial z}\right] + [\lambda_\theta - f(z) + u]\frac{\partial w}{\partial z} \quad = \quad 0, \tag{6}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} \quad = \quad 0. \tag{7}$$

The equations (5)-(7) give sufficient relations to determine the field quantities $u,\ w$, and p completely.

4 Non-linear Wave Modulation

In this section, we will examine the amplitude modulation of weakly non-linear waves in a fluid-filled thin elastic with a stenosis whose non-dimensional governing equations are given in equations (5)-(7). Considering the dispersion relation of the linearized field equations and the nature of the problem of concern, which is a boundary-value problem, the following stretched coordinates are introduced:

$$\xi = \epsilon(z - \lambda t), \quad \tau = \epsilon^2 z, \tag{8}$$

where ϵ is a small parameter measuring the weakness of nonlinearity and λ is a constant to be determined from the solution. Solving z in terms of τ, we get

$$z = \epsilon^{-2}\tau. \tag{9}$$

Introducing (9) into the expression of $f(z)$, we obtain

$$f(\epsilon^{-2}\tau) = \hat{h}(\tau). \tag{10}$$

In order to take the effect of stenosis into account, $f(z)$ must be of order ϵ^4. For the present work, we shall assume that $\hat{h}(\tau)$ have the form,

$$\hat{h}(\tau) = \epsilon^2 h(\tau). \tag{11}$$

Assuming that the field variables u, w, and p are functions of the slow variables (ξ, τ) as well as the fast variables (z, t), the following relations hold:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \epsilon\lambda\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \epsilon\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \tau}. \tag{12}$$

We will further assume that the field variables may be expanded into asymptotic series of ϵ as

$$\begin{aligned} u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ..., \\ w &= \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + ..., \\ p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + ..., \\ h(\tau) &= \epsilon^2 h_1(\tau) + \epsilon^3 h_2(\tau) + ... \end{aligned} \tag{13}$$

Introducing the expansions (12) and (13) into the equations (5)-(7), the following sets of differential equations are obtained:

$O(\epsilon)$ equations

$$\begin{aligned} p_1 &= \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^2 u_1}{\partial t^2} - \alpha_0\frac{\partial^2 u_1}{\partial z^2} + \beta_1 u_1, \\ 2\frac{\partial u_1}{\partial t} &+ \lambda_\theta\frac{\partial w_1}{\partial z} = 0, \quad \frac{\partial w_1}{\partial t} + \frac{\partial p_1}{\partial z} = 0. \end{aligned} \tag{14}$$

$O(\epsilon^2)$ equations

$$\begin{aligned} p_2 = {} & \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^2 u_2}{\partial t^2} - \alpha_0\frac{\partial^2 u_2}{\partial z^2} + \beta_1(u_2 - h_1) \\ & - \frac{2m\lambda}{\lambda_\theta\lambda_z}\frac{\partial^2 u_1}{\partial \xi \partial t} - 2\alpha_0\frac{\partial^2 u_1}{\partial \xi \partial z} - \frac{m}{\lambda_\theta^2 \lambda_z} u_1 \frac{\partial^2 u_1}{\partial t^2} \\ & - \alpha_1\left(\frac{\partial u_1}{\partial z}\right)^2 - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right) u_1 \frac{\partial^2 u_1}{\partial z^2} + \beta_2 u_1^2, \\ 2\frac{\partial u_2}{\partial t} & + \lambda_\theta\frac{\partial w_2}{\partial z} - 2\lambda\frac{\partial u_1}{\partial \xi} + \lambda_\theta\frac{\partial w_1}{\partial \xi} + u_1\frac{\partial w_1}{\partial z} + 2w_1\frac{\partial u_1}{\partial z} = 0, \\ \frac{\partial w_2}{\partial t} & + \frac{\partial p_2}{\partial z} - \lambda\frac{\partial w_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} + w_1\frac{\partial w_1}{\partial z} = 0. \end{aligned} \tag{15}$$

$O(\epsilon^3)$ equations

$$\begin{aligned}
p_3 = {} & \frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_3}{\partial t^2} - \alpha_0\frac{\partial^2 u_3}{\partial z^2} - \frac{2m\lambda}{\lambda_\theta\lambda_z}\frac{\partial^2 u_2}{\partial\xi\partial t} - 2\alpha_0\frac{\partial^2 u_2}{\partial\xi\partial z} \\
& -\alpha_0\left(\frac{\partial^2 u_1}{\partial\xi^2} + 2\frac{\partial^2 u_1}{\partial z\partial\tau}\right) + \frac{m\lambda^2}{\lambda_\theta\lambda_z}\frac{\partial^2 u_1}{\partial\xi^2} + \beta_1(u_3 - h_2) \\
& -\frac{m}{\lambda_\theta^2\lambda_z}u_1\left(\frac{\partial^2 u_2}{\partial t^2} - 2\lambda\frac{\partial^2 u_1}{\partial\xi\partial t}\right) - \frac{m}{\lambda_\theta^2\lambda_z}(u_2 - h_1)\frac{\partial^2 u_1}{\partial t^2} \\
& -2\alpha_1\frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial\xi}\right) - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)u_1\left(\frac{\partial^2 u_2}{\partial z^2} + 2\frac{\partial^2 u_1}{\partial z\partial\xi}\right) \\
& -\left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)(u_2 - h_1)\frac{\partial^2 u_1}{\partial z^2} + 2\beta_2 u_1(u_2 - h_1) \\
& +\frac{m}{\lambda_\theta^3\lambda_z}u_1^2\frac{\partial^2 u_1}{\partial t^2} - \left(\alpha_2 - \frac{\alpha_1}{\lambda_\theta}\right)u_1\left(\frac{\partial u_1}{\partial z}\right)^2 \\
& -\left(\alpha_2 - \frac{2\alpha_1}{\lambda_\theta} + \frac{\alpha_0}{\lambda_\theta^2}\right)u_1^2\frac{\partial^2 u_1}{\partial z^2} - 3\left(\gamma_1 - \frac{\alpha_0}{2}\right)\left(\frac{\partial u_1}{\partial z}\right)^2\frac{\partial^2 u_1}{\partial z^2} + \beta_3 u_1^3, \\
2\frac{\partial u_3}{\partial t} & + \lambda_\theta\frac{\partial w_3}{\partial z} - 2\lambda\frac{\partial u_2}{\partial\xi} + \lambda_\theta\frac{\partial w_2}{\partial\xi} + 2w_1\left(\frac{\partial u_1}{\partial\xi} + \frac{\partial u_2}{\partial z}\right) \\
& +2w_2\frac{\partial u_1}{\partial z} + \lambda_\theta\frac{\partial w_1}{\partial\tau} + u_1\left(\frac{\partial w_2}{\partial z} + \frac{\partial w_1}{\partial\xi}\right) + (u_2 - h_1)\frac{\partial w_1}{\partial z} = 0, \qquad (16) \\
\frac{\partial w_3}{\partial t} & + \frac{\partial p_3}{\partial z} - \lambda\frac{\partial w_2}{\partial\xi} + \frac{\partial p_2}{\partial\xi} + \frac{\partial p_1}{\partial\tau} + w_1\frac{\partial w_2}{\partial z} + w_2\frac{\partial w_1}{\partial z} + w_1\frac{\partial w_1}{\partial\xi} = 0. \qquad (17)
\end{aligned}$$

Here the coefficients of α_0, α_1, α_2, β_0, β_1, β_2, β_3 and $\gamma_1$are defined by

$$\begin{aligned}
&\alpha_0 = \frac{1}{\lambda_\theta}\frac{\partial\Sigma}{\partial\lambda_z}, \qquad \alpha_1 = \frac{1}{2\lambda_\theta}\frac{\partial^2\Sigma}{\partial\lambda_\theta\lambda_z}, \qquad \alpha_2 = \frac{1}{2\lambda_\theta}\frac{\partial^3\Sigma}{\partial\lambda_\theta^2\lambda_z}, \\
&\beta_0 = \frac{1}{\lambda_\theta\lambda_z}\frac{\partial\Sigma}{\partial\lambda_\theta}, \qquad \beta_1 = \frac{1}{\lambda_\theta\lambda_z}\frac{\partial^2\Sigma}{\partial\lambda_\theta^2} - \frac{\beta_0}{\lambda_\theta}, \quad \beta_2 = \frac{1}{2\lambda_\theta\lambda_z}\frac{\partial^3\Sigma}{\partial\lambda_\theta^3} - \frac{\beta_1}{\lambda_\theta}, \qquad (18) \\
&\beta_3 = \frac{1}{6\lambda_\theta\lambda_z}\frac{\partial^4\Sigma}{\partial\lambda^4} - \frac{\beta_2}{\lambda_\theta}, \quad \gamma_1 = \frac{\lambda_z}{2\lambda_\theta}\frac{\partial^2\Sigma}{\partial\lambda_z^2}.
\end{aligned}$$

Equations in (18) are defined through series expansion of the stretch ratios λ_1 and λ_2, which read

$$\begin{aligned}
\lambda_1 &= \lambda_z\left[1 + \left(\epsilon\frac{\partial u_1}{\partial z} + \epsilon^2\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial\xi}\right] + \epsilon^3\left[\frac{\partial u_3}{\partial z} + \frac{\partial u_2}{\partial\xi} + \frac{\partial u_1}{\partial\tau}\right]\right)^2\right]^{1/2}, \\
\lambda_2 &= \lambda_\theta + \epsilon u_1 + \epsilon^2[u_2 - h_1(\tau)] + \epsilon^3[u_3 - h_2(\tau)]. \qquad (19)
\end{aligned}$$

5 Solution of the Field Equations

5.1 The Solution for $O(\epsilon)$ *equations*

Seeking the following type of solution to the differential equations (14):

$$\begin{aligned} u_1 &= (U_1 e^{i\theta} + c.c), \\ w_1 &= (W_1 e^{i\theta} + c.c), \\ p_1 &= \left(-\frac{m\omega^2}{\lambda_\theta \lambda_z} + \alpha_0 k^2 + \beta_1\right) U_1 e^{i\theta} + c.c, \end{aligned} \tag{20}$$

where U_1 and W_1 are unknown functions of the slow variables (ξ, τ), $\theta = \omega t - kz$ is the phasor and $c.c$ is the complex conjugate of the corresponding expressions, ω is the angular frequency, k is the wave number. Introducing (20) into (14), we obtain

$$U_1 = U(\xi, \tau), \quad W_1 = \frac{2\omega}{\lambda_\theta k} U, \tag{21}$$

provided that the following dispersion relation holds true:

$$\omega^2 = \frac{\lambda_\theta \lambda_z k^2 (\alpha_0 k^2 + \beta_1)}{2\lambda_z + mk^2}. \tag{22}$$

Here $U(\xi, \tau)$ is an unknown function whose governing equation will be obtained later.

5.2 The Solution for $O(\epsilon^2)$ *equations*

Introducing the solutions (20)-(21) into (15) gives

$$\begin{aligned} p_2 = {} & \frac{m}{\lambda_\theta \lambda_z} \frac{\partial^2 u_2}{\partial t^2} - \alpha_0 \frac{\partial^2 u_2}{\partial z^2} + \beta_1 (u_2 - h_1) + 2\left(\frac{m\omega^2}{\lambda_\theta^2 \lambda_z} + \alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + \beta_2\right) |U|^2 \\ & + 2i\left(\alpha_0 k - \frac{m\omega\lambda}{\lambda_\theta \lambda_z}\right) \frac{\partial U}{\partial \xi} e^{i\theta} + \left(\frac{m\omega^2}{\lambda_\theta^2 \lambda_z} + 3\alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + \beta_2\right) U^2 e^{2i\theta} + c.c, \\ & 2\frac{\partial u_2}{\partial t} + \lambda_\theta \frac{\partial w_2}{\partial z} + 2\left(\frac{\omega}{k} - \lambda\right) \frac{\partial U}{\partial \xi} e^{i\theta} - 6i\frac{\omega}{\lambda_\theta} U^2 e^{2i\theta} + c.c = 0, \\ & \frac{\partial w_2}{\partial t} + \frac{\partial p_2}{\partial z} + \left(-2\frac{\lambda\omega}{\lambda_\theta k} - \frac{m\omega^2}{\lambda_\theta \lambda_z} + \alpha_0 k^2 + \beta_1\right) \frac{\partial U}{\partial \xi} e^{i\theta} \\ & \qquad - 4i\frac{\omega^2}{\lambda_\theta^2 k} U^2 e^{2i\theta} + c.c = 0, \end{aligned} \tag{23}$$

where $|U|^2 = UU^*, U^*$ is the complex conjugate of U. Seeking the following type of solutions

$$\begin{aligned} u_2 &= U_2^{(0)} + \left(\sum_{l=1}^{2} U_2^{(l)} e^{il\theta} + c.c\right), \\ w_2 &= W_2^{(0)} + \left(\sum_{l=1}^{2} W_2^{(l)} e^{il\theta} + c.c\right), \\ p_2 &= P_2^{(0)} + \left(\sum_{l=1}^{2} P_2^{(l)} e^{il\theta} + c.c\right), \end{aligned} \tag{24}$$

to (23) yields:

$$P_2^{(0)} = \beta_1(U_2^{(0)} - h_1) + 2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + \alpha_1 k^2 - \frac{\alpha_0}{\lambda_\theta}k^2 + \beta_2\right)|U|^2, \tag{25}$$

$$2\omega U_2^{(1)} - \lambda_\theta k W_2^{(1)} = 2i\left(\frac{\omega}{k} - \lambda\right)\frac{\partial U}{\partial \xi}, \tag{26}$$

$$\omega W_2^{(1)} - k\left(-\frac{m\omega^2}{\lambda_\theta\lambda_z} + \alpha_0 k^2 + \beta_1\right)U_2^{(1)}$$
$$= i\left(-\frac{2\lambda\omega}{\lambda_\theta k} - \frac{2m\omega\lambda k}{\lambda_\theta\lambda_z} - \frac{m\omega^2}{\lambda_\theta\lambda_z} + 3\alpha_0 k^2 + \beta_1\right)\frac{\partial U}{\partial \xi}, \tag{27}$$

$$P_2^{(1)} = -\frac{m\omega^2}{\lambda_\theta\lambda_z}U_2^{(1)} + \alpha_0 k^2 U_2^{(1)} + \beta_1 U_2^{(1)} + 2i\left(\alpha_0 k - \frac{m\omega\lambda}{\lambda_\theta\lambda_z}\right)\frac{\partial U}{\partial \xi}, \tag{28}$$

$$2\omega U_2^{(2)} - \lambda_\theta k W_2^{(2)} = 3\frac{\omega}{\lambda_\theta}U^2, \tag{29}$$

$$\omega W_2^{(2)} - k\left(-\frac{4m\omega^2}{\lambda_\theta\lambda_z} + 4\alpha_0 k^2 + \beta_1\right)U_2^{(2)}$$
$$= \left(\frac{2\omega^2}{\lambda_\theta^2 k} + \frac{m\omega^2 k}{\lambda_\theta^2\lambda_z} + 3\alpha_1 k^3 - \frac{\alpha_0}{\lambda_\theta}k^3 + \beta_2 k\right)U^2. \tag{30}$$

Taking $U_2^{(1)} = 0$ and solving equation (25), we get

$$W_2^{(1)} = i\frac{2}{\lambda_\theta k}\left(\lambda - \frac{\omega}{k}\right)\frac{\partial U}{\partial \xi}. \tag{31}$$

Introducing equation (31) into equation (27) we have

$$\lambda = \frac{\lambda_z(2\omega^2 + \lambda_\theta\alpha_0 k^4)}{\omega k(2\lambda_z + mk^2)}(\text{group velocity}). \tag{32}$$

Solving equations (29)- (30) leads to

$$U_2^{(2)} = \Phi_0 U^2, \quad W_2^{(2)} = \frac{2\omega}{\lambda_\theta k}U_2^{(2)} - \frac{3\omega}{\lambda_\theta^2 k}U^2,$$
$$\Phi_0 = \frac{\frac{3\omega^2}{\lambda_\theta} + k^2\beta_1 + 3\alpha_1\lambda_\theta k^4 + \lambda_\theta\beta_2 k^2}{3(\beta_1\lambda_\theta k^2 - 2\omega^2)}. \tag{33}$$

5.3 The Solution for $O(\epsilon^3)$ *equations*

Introducing the following type of solutions

$$u_3 = U_3^{(0)} + \left(\sum_{l=1}^{3} U_3^{(l)} e^{il\theta} + c.c\right),$$
$$w_3 = W_3^{(0)} + \left(\sum_{l=1}^{3} W_3^{(l)} e^{il\theta} + c.c\right), \tag{34}$$
$$p_3 = P_3^{(0)} + \left(\sum_{l=1}^{3} P_3^{(l)} e^{il\theta} + c.c\right),$$

into O(ϵ^3) equations (16)-(17), we obtain the zeroth- and first-order equations below:

$$
\begin{aligned}
&-2\lambda\frac{\partial U_2^{(0)}}{\partial\xi}+\lambda_\theta\frac{\partial W_2^{(0)}}{\partial\xi}+\frac{6\omega}{\lambda_\theta k}\frac{\partial}{\partial\xi}|U|^2=0,\\
&-\lambda\frac{\partial W_2^{(0)}}{\partial\xi}+\frac{\partial P_2^{(0)}}{\partial\xi}+\frac{4\omega^2}{\lambda_\theta^2k^2}\frac{\partial}{\partial\xi}|U|^2=0,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
P_3^{(1)}=&\left(\alpha_0k^2-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\beta_1\right)U_3^{(1)}+\left(\frac{m\lambda^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2U}{\partial\xi^2}+2i\alpha_0k\frac{\partial U}{\partial\tau}\\
&+\left(\frac{5m\omega^2}{\lambda_\theta^2\lambda_z}+6\alpha_1k^2-\frac{5\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)U_2^{(2)}U*\\
&\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+2\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)(U_2^{(0)}-h_1)U\\
&+\left[-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z}+2\alpha_2k^2-\frac{5\alpha_1k^2}{\lambda_\theta}+\frac{3\alpha_0k^2}{\lambda_\theta^2}+3\left(\gamma_1-\frac{\alpha_0}{2}\right)k^4+3\beta_3\right]|U|^2U,\\
2i\omega U_3^{(1)}&-ik\lambda_\theta W_3^{(1)}+\lambda_\theta\frac{\partial W_2^{(1)}}{\partial\xi}+\frac{2\omega}{k}\frac{\partial U}{\partial\tau}-\frac{6i\omega}{\lambda_\theta}U_2^{(2)}U^*\\
&-2i\left(kW_2^{(0)}+\frac{\omega}{\lambda_\theta}U_2^{(0)}-\frac{\omega}{\lambda_\theta}h_1\right)U=0,\\
i\omega W_3^{(1)}&-ikP_3^{(1)}-\lambda\frac{\partial W_2^{(1)}}{\partial\xi}+\frac{\partial P_2^{(1)}}{\partial\xi}+\left(-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right)\frac{\partial U}{\partial\tau}\\
&-\frac{2i\omega}{\lambda_\theta}W_2^{(2)}U^*-\frac{2i\omega}{\lambda_\theta}W_2^{(0)}U=0.
\end{aligned}
\tag{36}
$$

From the solution of the equations (35) and (25), results in

$$
\begin{aligned}
&U_2^{(0)}=\Phi_1|U|^2-\Phi_2h_1,\quad W_2^{(0)}=\frac{2\lambda}{\lambda_\theta}U_2^{(0)}-\frac{6\omega}{\lambda_\theta^2k}|U|^2,\\
&\Phi_1=\frac{\frac{3\lambda\omega}{\lambda_\theta k}+\frac{2\omega^2}{\lambda_\theta k^2}+\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_1\lambda_\theta k^2-\alpha_0k^2+\lambda_\theta\beta_2}{\lambda^2-\frac{\lambda_\theta\beta_1}{2}},\\
&\Phi_2=\frac{\lambda_\theta\beta_1}{2\lambda^2-\lambda_\theta\beta1}.
\end{aligned}
\tag{37}
$$

Finally, eliminating $U_3^{(1)}$,$W_3^{(1)}$ and $P_3^{(1)}$ between equation (36) through the use of dispersion relation (22), equations (28), (31), (33) and (37), we obtain the following NLS equation with variable coefficient:

$$
i\frac{\partial U}{\partial\tau}+\mu_1\frac{\partial^2U}{\partial\xi^2}+\mu_2|U|^2U-\mu_3h_1(\tau)U=0,
\tag{38}
$$

where the coefficients μ, μ_1, μ_2 and μ_3 are defined by

$$
\begin{aligned}
\mu &= \frac{2\omega^2}{k} + 3\alpha_0\lambda_\theta k^3 - \frac{mk\omega^2}{\lambda_z} + \lambda_\theta\beta_1 k, \\
\mu_1 &= \mu^{(-1)}\left[-\frac{4\lambda\omega}{k} + \frac{2\omega^2}{k^2} + 2\lambda^2 + \frac{m\lambda^2k^2}{\lambda_z} - 3\alpha_0\lambda_\theta k^2 + \frac{2m\omega\lambda k}{\lambda_z}\right], \\
\mu_2 &= \mu^{(-1)}\Bigg\{\left[\frac{10\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{5m\omega^2}{\lambda_\theta^2\lambda_z} + 6\alpha_1k^2 - \frac{5\alpha_0k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_0 \\
&\quad + \left[\frac{8\omega\lambda k}{\lambda_\theta} + \frac{2\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\alpha_1k^2 - \frac{\alpha_0k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_1 \\
&\quad - \frac{30\omega^2}{\lambda_\theta^2} + \lambda_\theta k^2\left[-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z} + 2\alpha_2k^2 - \frac{5\alpha_1k^2}{\lambda_\theta} + \frac{3\alpha_0k^2}{\lambda_\theta^2} + 3\left(\gamma_1 - \frac{\alpha_0}{2}\right)k^4 + 3\beta_3\right]\Bigg\} \\
\mu_3 &= \mu^{(-1)}\Bigg\{\left[\frac{2\omega^2}{\lambda_\theta} + \frac{8\omega\lambda k}{\lambda_\theta} + k^2\lambda_\theta\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\alpha_1k^2 - \frac{\alpha_0k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_2 \\
&\quad \left[\frac{2\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\alpha_1k^2 - \frac{\alpha_0k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Bigg\}.
\end{aligned} \tag{39}
$$

Introducing the following change of variable:

$$U = V(\xi, \tau)\exp\left[-i\mu_3\int_0^\tau h_1(s)ds\right], \tag{40}$$

equation (38) reduces to the following conventional NLS equations:

$$i\frac{\partial V}{\partial \tau} + \mu_1\frac{\partial^2 V}{\partial \xi^2} + \mu_2|V|^2V = 0. \tag{41}$$

6 Progressive Wave Solution

In this section, we will present the progressive wave solution to the evolution equation given in (41) of the following form :

$$V(\xi,\ \tau) = F(\zeta)\exp\left[i(K\xi - \Omega\tau)\right], \quad \zeta = \xi - c\tau, \tag{42}$$

where Ω, K and c are some constants and $F(\zeta)$ is a real-valued unknown function to be determined from the solution. Introducing (42) into (41), we have

$$\mu_1\frac{\partial^2 F}{\partial \zeta^2} + i(2\mu_1K - c)\frac{\partial F}{\partial \zeta} + (\Omega - \mu_1K^2)F + \mu_2F^3 = 0. \tag{43}$$

Now, letting $c = 2\mu_1K$, the $\frac{\partial F}{\partial \zeta}$ term can be eliminated and choosing $\Omega = \mu_1K^2 - \mu_2a^2$, where a is the amplitude of the wave, we obtain

$$\mu_1\frac{\partial^2 F}{\partial \zeta^2} - \frac{\mu_2a^2}{2}F + \mu_2F^3 = 0. \tag{44}$$

Solving equation (44), the soliton modulated wave solution to NLS equation (41) is given by

$$V(\xi,\tau) = a\tanh\left[\sqrt{\frac{-\mu_2}{2\mu_1}}a(\xi - c\tau)\right]\exp[i(K\xi - \Omega\tau)], \tag{45}$$

where the modulus of $V(\xi,\tau)$ will be given by

$$|V(\xi,\tau)| = a\tanh\left[\sqrt{\frac{-\mu_2}{2\mu_1}}a(\xi - c\tau)\right]. \tag{46}$$

Substituting the solution of standard NLS equation (45) into equation (40), we obtain the solution of the NLS equation with variable coefficient (38) as

$$U(\xi,\tau) = a\tanh\left[\sqrt{\frac{-\mu_2}{2\mu_1}}a(\xi - c\tau)\right]\exp\left[i(K\xi - \Omega\tau - \mu_3\int_0^\tau h_1(s)ds)\right], \tag{47}$$

where the modulus of $U(\xi,\tau)$ is given by

$$|U(\xi,\tau)| = a\,tanh\left[\sqrt{\frac{-\mu_2}{2\mu_1}}a(\xi - c\tau)\right]. \tag{48}$$

The speed of the enveloping wave is constant and equal to $2\mu_1 K$. On the other hand, the speed of the harmonic wave is given by

$$v_p = \frac{K}{\Omega + \mu_3 h_1(\tau)}. \tag{49}$$

7 Numerical Results and Discussions

For numerical calculation, we need to assign values for the coefficients α_0, α_1, α_2, β_0, β_1, β_2, β_3, γ_1,μ_1, μ_2, μ_3, μ_4, In order to do that, we must know the constitutive relation of the tube material. In this work, we will utilize the constitutive relation proposed by Demiray [8] for soft biological tissues. Following Demiray [8], the strain energy density function may be expressed as

$$\Sigma = \frac{1}{2\alpha}\{\exp\left[\alpha(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3)\right] - 1\}, \tag{50}$$

where α is a material constant and I_1 is the first invariant of Finger deformation tensor defined by $I_1 = \lambda_\theta^2 + \lambda_z^2 + 1/(\lambda_\theta^2\lambda_z^2)$. Introducing (50) into equation (18), the explicit expressions of the coefficients α_0, α_1, α_2, β_0, β_1, β_2, β_3 and γ_1 are obtained as :

$$
\begin{aligned}
\alpha_0 &= \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right) G(\lambda_\theta,\lambda_z),\\
\alpha_1 &= \left[\frac{1}{\lambda_\theta^4\lambda_z^3} + \alpha\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)\left(1 - \frac{1}{\lambda_\theta^4\lambda_z^2}\right)\right] G(\lambda_\theta,\lambda_z),\\
\alpha_2 &= \left[\frac{2\alpha^2}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2 + \frac{3\alpha}{\lambda_\theta^3\lambda_z^3}\left(1 - \frac{7}{3\lambda_\theta^4\lambda_z^2}\right)\right.\\
&\quad \left. + \frac{\alpha\lambda_z}{\lambda_\theta} + \frac{3}{\lambda_\theta^5\lambda_z}\left(\alpha - \frac{1}{\lambda_z^2}\right)\right] G(\lambda_\theta,\lambda_z),\\
\beta_0 &= \left[\frac{1}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\right] G(\lambda_\theta,\lambda_z),\\
\beta_1 &= \left[\frac{2\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2 + \frac{4}{\lambda_\theta^5\lambda_z^3}\right] G(\lambda_\theta,\lambda_z),\\
\beta_2 &= \left[\frac{2\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3 - \frac{10}{\lambda_\theta^6\lambda_z^3}\right.\\
&\quad \left. + \frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1 + \frac{11}{\lambda_\theta^4\lambda_z^2}\right)\right] G(\lambda_\theta,\lambda_z),\\
\beta_3 &= \left[\frac{20}{\lambda_\theta^7\lambda_z^3} + \frac{4\alpha^3}{3\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^4 + \frac{4\alpha^2}{\lambda_\theta\lambda_z}\left(1 + \frac{3}{\lambda_\theta^4\lambda_z^2}\right)\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right.\\
&\quad - \frac{16\alpha}{\lambda_\theta^6\lambda_z^3}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right) + \frac{\alpha}{\lambda_\theta\lambda_z}\left(1 + \frac{3}{\lambda_\theta^4\lambda_z^2}\right)^2 - \frac{2\alpha^2}{\lambda_\theta^2\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\\
&\quad \left. - \frac{\alpha}{\lambda_\theta^2\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)(1 + \frac{11}{\lambda_\theta^4\lambda_z^2})\right] G(\lambda_\theta,\lambda_z),\\
\gamma_1 &= \left[\frac{\alpha\lambda_z}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)^2 + \frac{\lambda_z}{2\lambda_\theta} + \frac{3}{2\lambda_\theta^3\lambda_z^3}\right] G(\lambda_\theta,\lambda_z), \qquad (51)
\end{aligned}
$$

where the function G is defined by

$$
G(\lambda_\theta,\lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right]. \qquad (52)
$$

Right now, we need the value of the material constant α. For the static case, the present model was compared by Demiray [9] with the experimental measurement by Simon *et al.* [11] on canine abdominal artery with the characteristics $R_i = 0.31cm$, $R_0 = 0.38cm$ and $\lambda_z = 1.53$ and the value of the material constant α was found to be $\alpha = 1.948$. Using this numerical value of the coefficient α, and for the initial deformation $\lambda_\theta = \lambda_z = 1.6$, we obtain $\alpha_0 = 78.6924$, $\alpha_1 = 233.7666$, $\alpha_2 = 1563.4837$, $\beta_0 = 49.1827$, $\beta_1 = 296.1049$, $\beta_2 = 991.4958$, β_3=2394.7, $\gamma_1 = 418.3605$, $\omega = 41.6845$, $\lambda = 29.2660$, $\Phi_0 = -6.0631$, $\Phi_1 = 7.2986$, $\Phi_2 = 0.3823$, $\mu_1 = -0.1548$, $\mu_2 = 26.4295$, $\mu_3 = 7.3572$, provided $m = 0.1$, $k = 2$ and $K = 2$.

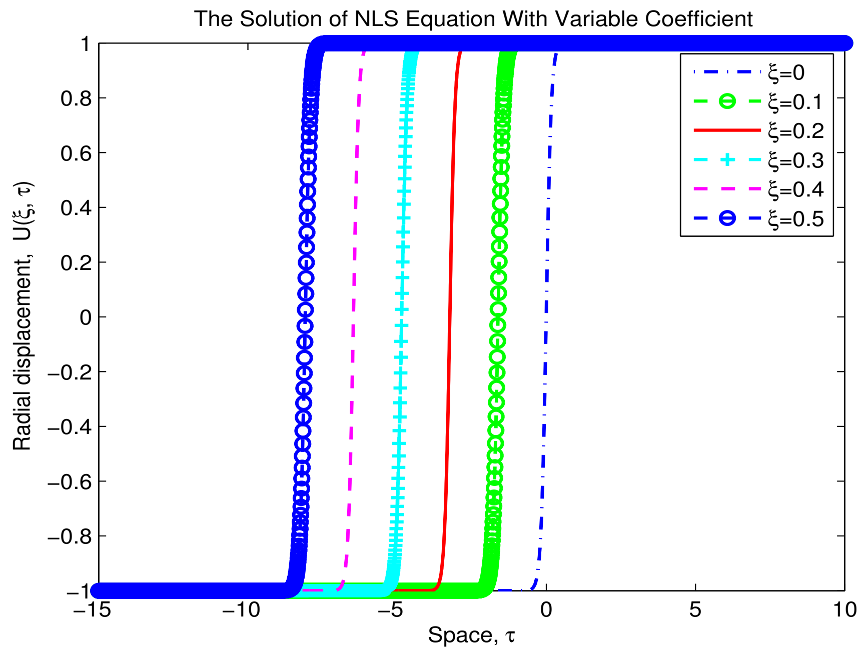

Figure 1: The Solution of NLS Equation with Variable Coefficient Versus Space τ

The solution of the NLS equation with variable coefficient (38) versus space τ at different time ξ is shown in Figure 1. The non-linear Schrödinger equation with variable coefficient shows increasing shock wave propagates to the right as real time, t passed.

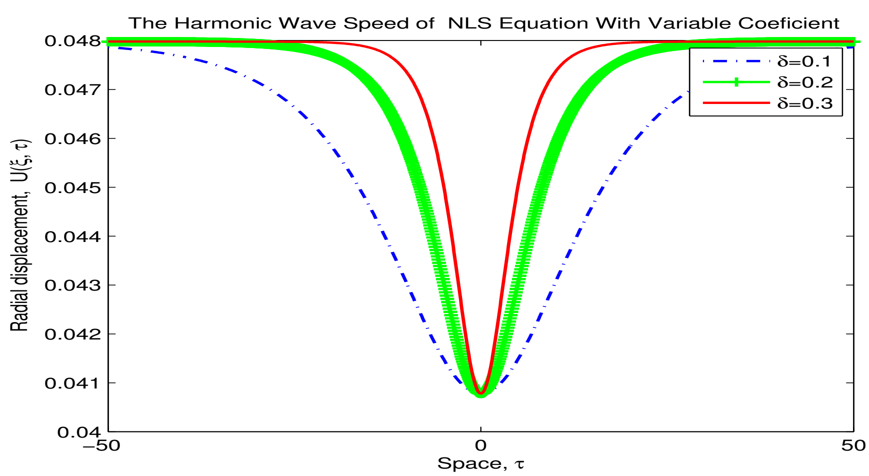

Figure 2: The Speed of Harmonic Wave

Figure 2 illustrates the speed of harmonic wave of NLS equation with variable coefficient versus space τ at different δ, where δ specify the sharpness of stenosis function $f(\tau) = sech(\delta\tau)$. The graph shows the speed is minimum at the center of stenosis and increases to a constant value of 0.048 as it goes away from center of stenosis. If the shape of the stenosis is sharp, the wave speed increase rapidly.

8 Conclusion

It has been presented the wave modulation in a prestressed thin-wall stenosed elastic tube filled with an averaged inviscid fluid. The governing equation is obtained as the nonlinear Schrödinger (NLS) equation with a variable coefficient. The solutions of NLS equation with a variable coefficient admits a increasing shock wave profile and the wave propagates to the left with a constant amplitude as time ξ increases. Besides, the shape of the stenosis is proportional to the speed of solitary wave. It means if the shape of stenosis is sharp (bigger δ), the wave speed increase rapidly.

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