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Certain Matrices Associated with Balancing and Lucas-balancing Numbers

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Abstract Balancing numbers n and balancers r are originally defined as the solution of the Diophantine equation $1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$. These numbers can be generated by the linear recurrence $B_{n+1} = 6B_n - B_{n-1}$ or by the nonlinear recurrence $B_n^2 = 1+B_{n-1}B_{n+1}$. There is another way to generated balancing numbers using powers of a matrix $Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$. The matrix representation, indeed gives many known and new formulas for balancing numbers. In this paper, using matrix algebra we obtain several interesting results on balancing and related numbers.

Keywords Balancing numbers; Lucas-balancing numbers; Triangular numbers; Recurrence relation; Balancing Q-matrix; Balancing R-matrix.

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1 Introduction

Behera and Panda [1] recently introduced a number sequence called balancing numbers defined in the following way: A positive integer n is called a balancing number with balancer r, if it is the solution of the Diophantine equation $1 + 2 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r)$. They also proved that the recurrence relation for balancing numbers is

$$B_{n+1} = 6B_n - B_{n-1}, \quad n > 2, \tag{1}$$

where B_n is the n^{th} balancing number with $B_1 = 1$ and $B_2 = 6$.
It is well known that (see [1]), n is a balancing number if and only if n^2 is a triangular number, that is $8n^2 + 1$ is a perfect square. In [2], Lucas-balancing numbers are defined as follows: If n is a balancing number, $C_n = \sqrt{8n^2 + 1}$ is called a Lucas-balancing number. The recurrence relation for Lucas-balancing numbers is same as that of balancing numbers, that is

$$C_{n+1} = 6C_n - C_{n-1}, \quad n > 2, \tag{2}$$

where C_n is the n^{th} Lucas-balancing number with $C_1 = 3$ and $C_2 = 17$. Liptai [3], showed that the only balancing number in the sequence of Fibonacci numbers is 1. In [4], Ray obtain nice product formulas of balancing and Lucas-balancing numbers. Panda and Ray [5], link balancing numbers with Pell and associated Pell numbers. Many interesting properties of balancing numbers and its related sequences are available in the literature [2, 6, 7].

2 Balancing Matrices

Matrices can be used to represent the balancing numbers and can be extended to related sequences. Additionally this turns out to be an elegant way of finding relationships between the balancing and Lucas-balancing numbers.

2.1 Balancing Q-Matrices

The Q-matrix was first studied by Charles H. King [8] in 1960. This matrix has an important role while dealing with Fibonacci numbers. Motivated by this, we introduce balancing Q-matrix which is given by

$$Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

and we prove the following important theorem.

Theorem 1 *Let Q_B be the balancing Q-matrix given in* (3). *Then for every positive integer n,*

$$Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}, \tag{4}$$

where B_n is the n^{th} balancing number.

Proof. The theorem holds for $n = 1$. Suppose it holds for $n = k$, then using (1), observe that

$$\begin{aligned} Q_B^{k+1} &= Q_B^k Q_B \\ &= \begin{pmatrix} B_{k+1} & -B_k \\ B_k & -B_{k-1} \end{pmatrix} \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 6B_{k+1} - B_k & -B_{k+1} \\ 6B_k - B_{k-1} & -B_k \end{pmatrix} \\ &= \begin{pmatrix} B_{k+2} & -B_{k+1} \\ B_{k+1} & -B_k \end{pmatrix}. \end{aligned}$$

By induction, the theorem holds for all natural number n.

Some important known formulas can also be obtained through the matrix Q_B. The following corollaries are immediate consequences of Theorem 2.1.

Corollary 1 For every natural number k,

$$B_k^2 - B_{k-1}B_{k+1} = 1. \tag{5}$$

Proof. If $|Q_B|$ is determinant of matrix Q_B, then

$$\begin{aligned} |Q_B| &= 1 \\ &= |Q_B^k| \\ &= B_k^2 - B_{k-1}B_{k+1}. \end{aligned}$$

Corollary 2 For every integer k,

$$B_{-k} = -B_k. \tag{6}$$

Proof. By virtue of Theorem 2.1 and since $|Q_B^k| = 1$, we have

$$\begin{aligned}\begin{pmatrix} B_{-k+1} & -B_{-k} \\ B_{-k} & -B_{-k-1} \end{pmatrix} &= Q_B^{-k} = (Q_B^k)^{-1} \\ &= \begin{pmatrix} -B_{k-1} & B_k \\ -B_k & B_{k+1} \end{pmatrix},\end{aligned}$$

Equating the second row first column element from both sides, we get the desired result.

One can easily check that the relation (5) holds for negative k too.

Corollary 3 For all integers k, l,

$$B_{k+l} = B_k B_{l+1} - B_{k-1} B_l. \tag{7}$$

Proof. By matrix (4), we get

$$\begin{aligned}\begin{pmatrix} B_{k+l+1} & -B_{k+l} \\ B_{k+l} & -B_{k+l-1} \end{pmatrix} &= Q_B^{k+l} \\ &= Q_B^k Q_B^l \\ &= \begin{pmatrix} B_{k+1} & -B_k \\ B_k & -B_{k-1} \end{pmatrix} \begin{pmatrix} B_{l+1} & -B_l \\ B_l & -B_{l-1} \end{pmatrix}.\end{aligned}$$

Comparing the second row and first column elements from both sides we get the required result.

Equation (1) follows directly from (7) by setting $k = n,\ l = 1$.

2.2 Bilinear Index Reduction Formula for Balancing Numbers

For $a + b = c + d$, the second row and second column entry of $Q_B^a Q_B^b = Q_B^c Q_B^d$ is

$$B_a B_b - B_c B_d = B_{a-1} B_{b-1} - B_{c-1} B_{d-1}.$$

When iteration gives

$$B_a B_b - B_c B_d = B_{a-r} B_{b-r} - B_{c-r} B_{d-r}, \tag{8}$$

which holds for a, b, c, d and $r = 0, 1, 2, \ldots$.

Equation (8) is the index reduction formula for balancing numbers. This formula at once provides a framework for standard bilinear balancing numbers identities. For example, by using (6), the identity (5) follows directly by choosing $a = k+1, b = k-1, c = d = r = k$. Likewise the addition result of (7) is just the case where $a = k, b = -l-1, c = k-1, d = -l, r = -l$ and even the fundamental recurrence (1) can be recovered by choosing $a = k+1, b = 1, c = k, d = r = 2$.

Putting $a = n+k, b = n+k, c = 2n, d = r = 2k$ in (8), we get a known formula due to Panda, [5]

$$B_{n+k}^2 - B_{n-k}^2 = B_{2n} B_{2k}.$$

Also replacing $c = 2n+1, d = r = 2k-1$ in (8) to get the similar square formula

$$B_{n+k}^2 - B_{n-k+1}^2 = B_{2n+1} B_{2k-1}.$$

2.3 Binomial Sums for Balancing Numbers

We know from Cayley-Hamilton theorem that matrix powers are not independent. Indeed, if I is the identity matrix then replacing the first row first column element of (4) by the recurrence relation (1) gives by inspection

$$Q_B^k = B_k Q_B - B_{k-1} I, \quad k = 0, 1, 2, \ldots. \tag{9}$$

Theorem 2 *If* $\binom{n}{r}$ *denote usual notation for combination, then*

$$\begin{aligned} B_{kn} &= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (B_k)^r (B_{k-1})^{n-r} B_r \\ &= \sum_{r=0}^{n} (-1)^{r+1} \binom{n}{r} (B_k)^r (B_{k+1})^{n-r} B_r. \end{aligned}$$

Proof. By virtue of Theorem 2.1 and equation (9), we obtain

$$\begin{aligned} \begin{pmatrix} B_{kn+1} & -B_{kn} \\ B_{kn} & -B_{kn-1} \end{pmatrix} &= Q_B^{kn} \\ &= (B_k Q_B - B_{k-1} I)^n \\ &= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (B_k)^r (B_{k-1})^{n-r} Q_B^r. \end{aligned}$$

Equating second row and first column elements, we get

$$B_{kn} = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (B_k)^r (B_{k-1})^{n-r} B_r. \tag{10}$$

Replace k by $-k$ in (9) and by (6), we get

$$Q_B^{-k} = -B_k Q_B + B_{k+1} I.$$

Again use of Theorem 2.1, yields

$$\begin{aligned} \begin{pmatrix} B_{-kn+1} & -B_{-kn} \\ B_{-kn} & -B_{-kn-1} \end{pmatrix} &= Q_B^{-kn} \\ &= (-B_k Q_B + B_{k+1} I)^n \\ &= \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} (B_k)^r (B_{k+1})^{n-r} Q_B^r. \end{aligned}$$

Comparing second row and first column elements from both sides, we obtain

$$B_{kn} = \sum_{r=0}^{n} (-1)^{r+1} \binom{n}{r} (B_k)^r (B_{k+1})^{n-r} B_r. \tag{11}$$

Observe that, for $k = 2, 3$ in (10) and since $B_0 = 0$, we get the standard results

$$B_{2n} = \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} 6^r B_r$$

$$B_{3n} = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} 35^r B_r.$$

For $k = 2$ in (11) gives the superficially strange result

$$B_{2n} = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} 6^r (35)^{n-r} B_r.$$

2.4 Eigenvalues and Eigenvectors of Balancing Matrices

The roots $\lambda_1 = 3 + \sqrt{8}$, $\lambda_2 = 3 - \sqrt{8}$ of the equation $\lambda^2 - 6\lambda + 1 = 0$ are are eigenvalues of Q_B. It is well known that if λ be an eigenvalue of a matrix A, λ^n is also an eigenvalue of A^n. Therefore obviously λ_1^n and λ_2^n are the eigenvalues of Q_B^n. But to make interesting, we suggest the proof of the following theorem.

Theorem 3 *The eigenvalues of Q_B^n are λ_1^n and λ_2^n, where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.*

Proof. If I be the identity matrix same order as Q_B^n, then characteristic equation of Q_B^n is

$$\begin{aligned} |Q_B^n - \lambda I| &= \begin{vmatrix} B_{k+1} - \lambda & -B_k \\ B_k & -B_{k-1} - \lambda \end{vmatrix} \\ &= \lambda^2 - 2C_n + 1 \\ &= 0, \end{aligned}$$

where $B_{n+1} - B_{n-1} = 2C_n$ and $B_n^2 - B_{n-1}B_{n+1} = 1$. Now solving the characteristic equation and using Binet's formulas for B_n and C_n, we get

$$\begin{aligned} \lambda &= \frac{2C_n \pm \sqrt{4C_n^2 - 4}}{2} \\ &= C_n \pm \sqrt{8} B_n \\ &= \frac{\lambda_1^n + \lambda_2^n}{2} \pm \sqrt{8} \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}}. \end{aligned}$$

Thus the eigenvalues of Q_B^n are λ_1^n and λ_2^n.

Observe that for $n = 1$, the eigenvalues of Q_B are λ_1 and λ_2.

2.5 Generating Functions

The matrix generating function is just the expansion of $(I - sQ_B)^{-1} = \sum_k s^k Q_B^k$ where $|I - sQ_B| = 1 - 6s + s^2$. The second row first column and first row second column entries give at once the balancing generating function

$$B_0 + sB_1 + s^2 B_2 + \cdots = \frac{s}{1 - 6s + s^2}$$

for $|s| < \frac{1}{\lambda_1}$.

3 Lucas-Balancing Matrices

If C_k be the k^{th} Lucas-balancing number, then $2C_k = Tr(Q_B^k) = B_{k+1} - B_{k-1}$ and by (1), we get

$$C_k = 3B_k - B_{k-1}. \tag{12}$$

As we know that, Lucas-balancing numbers obey same recurrence relation as that of balancing numbers and since $C_0 = 1$ and $C_1 = 3$, the matrix containing these elements, call as balancing R-matrix which is defined in the following subsection.

3.1 Balancing R-matrix

We define balancing R-matrix by

$$R_B = \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix}, \tag{13}$$

which can be used to transform Q_B^n into a matrix of Lucas-balancing numbers. Here,

$$\begin{aligned} R_B Q_B^n &= \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 3B_{n+1} - B_n & -(3B_n - B_{n-1}) \\ B_{n+1} - 3B_n & -(B_n - 3B_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{pmatrix}. \end{aligned}$$

Applying the same method as mentioned earlier, one can obtain

$$C_{k+l} = B_k C_{l+1} - B_{k-1} C_l, \tag{14}$$

for all k and l. For $k = 2$, equation (14) recovers the basic recurrence (1) and for $l = 0$ recovers (13).
Also,

$$\begin{aligned} Q_B^k &= \begin{pmatrix} B_{k+1} & -B_k \\ B_k & -B_{k-1} \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} C_{k+1} & -C_k \\ C_k & -C_{k-1} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix} \end{aligned}$$

gives the inverse of (13) as

$$16B_k = C_{k+1} - C_{k-1}.$$

3.2 Index-Reduction Formulas for Lucas-balancing Numbers

Equation (8) together with (13) can obtain the index reduction formula as

$$B_a C_b - B_c C_d = B_{a-r} C_{b-r} - B_{c-r} C_{d-r}, \tag{15}$$

where $a + b = c + d$. For $a = n + 1, b = n - 1, c = d = n$ and $r = n - 1$, (15) becomes

$$B_{n+1} C_{n-1} - B_n C_n = 3. \tag{16}$$

Observe that, interchanging B and C in (16) we get

$$C_{n+1}B_{n-1} - C_nB_n = -3.$$

Similarly there is an index reduction formula with replacing B by C in (15) to get

$$C_aC_b - C_cC_d = C_{a-r}C_{b-r} - C_{c-r}C_{d-r},$$

where a+b=c+d. Using these formulas one can easily obtain

$$C_{k+1}C_{k-1} - C_k^2 = 8,$$

which is the Lucas-balancing version of the well known formula $B_n^2 - B_{n-1}B_{n+1} = 1$.

For a different index reduction formula, observe that the matrix $R_B = \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix}$ obeys $R_B^{-1} = \frac{1}{8}R_B$ and $Q_BR_B = R_BQ_B$. Therefore for $a+b=c+d$, we have not only $Q_B^aQ_B^b = Q_B^cQ_B^d$ but also $8Q_B^aQ_B^b = Q_B^cR_BQ_B^dR_B$, which gives a new result

$$8B_aB_b - C_cC_d = 8B_{a-r}B_{b-r} - C_{c-r}C_{d-r}, \tag{17}$$

for all $r = 0, 1, 2, \ldots$. Noting that for $a = b = c = d = r = n$, (17) reduces to a well known identity $C_n^2 - 8B_n^2 = 1$.

3.3 Balancing and Lucas-balancing Numbers Relationships Revisited

Using matrices we can prove some important identities more elegantly. The following theorem shows, how solving via matrices will give benefit to prove several related identities in one step.

Theorem 4 *If B_n, C_n respectively denote n^{th} balancing and Lucas-balancing numbers, then*

$$\begin{aligned} C_{n+1}B_{m+1} - C_nB_m &= 3B_{m+n+1} - B_{m+n} \\ C_nB_{m+1} - C_{n-1}B_m &= B_{m+n+1} - 3B_{m+n}. \end{aligned}$$

Proof. By virtue of theorem 2.1, equation (3.2) and since $R_BQ_B^n = \begin{pmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{pmatrix}$, we obtain

$$\begin{aligned} \begin{pmatrix} C_{n+1}B_{m+1} - C_nB_m & -C_{n+1}B_m + C_nB_{m-1} \\ C_nB_{m+1} - C_{n-1}B_m & -C_nB_m + C_{n-1}B_{m-1} \end{pmatrix} &= \begin{pmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{pmatrix} \begin{pmatrix} B_{m+1} & -B_m \\ B_m & -B_{m-1} \end{pmatrix} \\ &= R_BQ_B^nQ_B^m \\ &= R_BQ_B^{m+n} \\ &= \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} B_{m+n+1} & -B_{m+n} \\ B_{m+n} & -B_{m+n-1}, \end{pmatrix} \\ &= \begin{pmatrix} 3B_{m+n+1} - B_{m+n} & -(3B_{m+n} - B_{m+n-1}) \\ B_{m+n+1} - 3B_{m+n} & -(B_{m+n} - 3B_{m+n-1}) \end{pmatrix} \end{aligned}$$

By equating corresponding entries from both sides, we get the desired results.

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