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# Verifications of Pointwise Multipliers of Relations between Weighted Bergman Spaces and Hardy Spaces

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**Abstract** The justification of Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces which are Hilbert are characterized by using Bloch type spaces, BMOA type spaces, weighted Bergman spaces and tent spaces.

**Keywords** Pointwise multipliers; Hardy spaces; Bergman spaces; Bloch type spaces; BMOA type spaces; tent spaces.

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## 1 Introduction

In this paper we follow the same literature and methods of Ruhan Zhao [1] with a little change.
Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane, and let $\partial D = \{z : |z| = 1\}$ be the unit circle. Let $H(D)$ be the space of all analytic functions on the unit disk $D$. For $o < p < \infty$ , let $H^p$ denote the Hardy space which contains $f \in H(D)$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

For $0 < p < \infty$ and $-1 < \alpha < \infty$, let $A^{p,\alpha}$ denote the weighted Lebesgue spaces which contain measurable functions $f$ on $D$ such that

$$\|f\|^p_{p,\alpha} = \int_D |f(z)|^p dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z) = (1 - |z|^2)^\alpha \frac{dxdy}{\pi}$. We also denote by
$A^{p,\alpha}_a = A^{p,\alpha} \cap H(D)$, the weighted Bergman space on $D$, with the same norm as above. If $\alpha = 0$, we simply write them as $A^p$ and $A^p_a$, respectively. Let $\mathbf{g}$ be an analytic function on $D$ , let $X$ and $Y$ be two spaces of analytic functions . We say that $\mathbf{g}$ is a pointwise multipliers from $X$ into $Y$ if $\mathbf{g}f \in Y$ for any $f \in X$. The space of all pointwise multipliers from the space $X$ into the space $Y$ will be denoted by $M(X, Y)$ . Let $M_{\mathbf{g}}$ be the multiplication operator defined by $M_{\mathbf{g}}f = f\mathbf{g}$. A simple application of the closed graph theorem shows that $\mathbf{g}$ is a pointwise multiplier between two weighted Bergman spaces or between a Hardy space and a weighted Bergman space if and only if $M_{\mathbf{g}}$ is a bounded operator between the same spaces. Pointwise multipliers are closely related to Toeplizt operators and Hankel operators. They have been studied by many authors [1–4]for a few examples. The pointwise multipliers between the Hardy space $H^2$ and the unweighted Bergman spaces $A^2_a$ were characterized, [2].

The pointwise multipliers between the Hardy space $H^2$ and the unweighted Bergman spaces $A_a^2$ were characterized by using the Carleson measure, [3]. their results are generalize in, [1]. In order to state the results, we need notation of various other function spaces. For $0 < \alpha < \infty$, we say an analytic function $f$ on $D$ is in the $\alpha$-Bloch space $B^\alpha$ , if

$$\sup_{z\in D} |f^{'}(z)|(1-|z|^2)^\alpha < \infty.$$

As $\alpha = 1$, $B^1 = B$ the well-known Bloch space. As $0 < \alpha < 1$, the space $B^\alpha = Lip_{1-\alpha}$, the analytic Lipshitz space which contains analytic functions $f$ on $D$ satisfying

$$|f(z) - f(w)| \leq c|z-w|^{1-\alpha},$$

for any $z$ and $w$ in $D$ , [5]. If $\alpha > 1$, it is known that $f \in B^\alpha$ if and only if

$$\sup_{z\in D} |f(z)|(1-|z|^2)^{\alpha-1} < \infty,$$

or the antiderivative of $f$ is in $B^{\alpha-1}$. We define a general family of function spaces. We will use a special Mobius transformation $\Phi_\alpha(z) = \frac{(\alpha-z)}{(1-\bar{\alpha}z)}$, which exchange 0 and $a$, and has derivative $\Phi_a^{'}(z) = -\frac{(1-|a|^2)}{(1-\bar{a}z)^2}$. Let $p, q$ and $s$ be real numbers such that $0 < p < \infty$ , $-2 < q < \infty$ and $0 < s < \infty$. We say that an analytic function $f$ on $D$ belongs to the space $F(p, q, s)$, if

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in D} \int_D |f^{'}(z)|^p(1-|z|^2)^q(1-|\Phi_a(z)|^2)^s dA(z) < \infty.$$

The spaces $F(p, q, s)$ were introduced in, [6]. They contains, as special cases, many classical function spaces. It was proved that, for $-1 < \alpha < \infty, F(p, p\alpha - 2, s) = B^\alpha$ for any $p > 0$ and any $s > 1$, [6, 7].When $s = 1$ we define BMOA type spaces as follows: $BMOA_p^\alpha = F(p, p\alpha - 2, 1)$. Unlike the $\alpha$ -Bloch spaces, the spaces $BMOA_p^\alpha$ are different for different values of $p$, [6]. It is known that, $BMOA_2^1 = BMOA$, the classical space of analytic functions of bounded mean oscillation. Let $\mu$ be a Borel measure on $D$ ; we say that an analytic function $f$ is in the tent space $T_p^q(d\mu)$ if

$$\|f\|_{T_p^q(d\mu)} = \left( \int_0^{2\pi} \left( \int_{\Gamma(\theta)} |f(z)|^q \frac{d\mu(z)}{(1-|z|^2)} \right)^{\frac{p}{q}} d\theta \right)^{\frac{1}{p}} < \infty,$$

Where $\Gamma(\theta)$ is the Stolz angle at $\theta$ , which is defined for real $\theta$ as the convex hull of the set $\{e^{i\theta}\} \cup \{z : |z| < \sqrt{\frac{1}{2}}\}$. The tent spaces were introduced in, [8, 9].

## 2 Carleson Type Measures

Carleson type measures, [10–13]. are the main tools of the investigation. Let X be a space of analytic functions on $D$. Following the notations in [14], we say a Borel measure $d\mu$ on $D$ is an $(X, q)$ -Carleson measure if

$$\int_D |f|^q d\mu(z) \leq c\|f\|^q_X$$

for any function $f \in X$ . Let $I \subset \partial D$ be an arc. Denote by $|I|$ the normalized arc length of $I$ so that $|\partial D| = 1$. Let $S(I)$ be the Carleson box defined by

$$S(I) = \{z : 1 - |I| < |z| < 1, \frac{z}{|z|} \in I\}.$$

There are many different versions of Carleson type theorems. Here we collect those results we need later. The first result is the classical result for the case $p = q$, [15]. for the case $p < q$, [16]. A proof of the equivalence of $(ii)$ and $(iii)$ can be found in, [17].

**Theorem 1** *For $\mu$ a positive Borel measure on $D$ and $0 < p \leq q < \infty$, the following statements are equivalent:*
*(i) The measure $\mu$ is an $(H^p, q)$ -Carleson measure.*
*(ii) There is a constant $c_1 > 0$ such that, for any arc $I \subset \partial D$,*

$$\mu(S(I)) \leq c_1 |I|^{\frac{q}{p}}.$$

*(iii) There is a constant $c_2 > 0$ such that, for every $a \in D$,*

$$\int_D |\Phi_a^{'}(z)|^{\frac{q}{p}} d\mu(z) \leq c_2.$$

*For the case $0 < q < p < \infty$ , the following result is in, [18, 19].*

**Theorem 2** *For $\mu$ a positive Borel measure on $D$ and $0 < q < p < \infty$, the following statements are equivalent:*
*(i) The measure $\mu$ is an $(H^p, q)$ -Carleson measure.*
*(ii) The function $\theta \to \int_{\Gamma(\theta)} \frac{d\mu}{1-|z|^2}$ belongs to $A^{\frac{p}{(p-q)}}$, where $\Gamma(\theta)$ is the Stolz angle at $\theta$. For the weighted Bergman spaces $A_a^{p,\alpha}$, the following result was obtained by several authors and can be found in, [20]. The equivalence of $(i)$ and $(ii)$ is the same as the equivalence of (ii) and $(iii)$ in Theorem 1*

**Theorem 3** *For $\mu$ a positive Borel measure on $D$ , $0 < p \leq q < \infty$ and $-1 < \alpha < \infty$, the following statements are equivalent:*
*(i) The measure $\mu$ is an $(A_a^{p,\alpha}, q)$ -Carleson measure.*
*(ii) There is a constant $c_1 > 0$ such that, for any arc $I \subset \partial D$,*

$$\mu(S(I)) \leq c_1 |I|^{(2+\alpha)\frac{q}{p}}.$$

*(iii)There is a constant $c_2 > 0$ such that, for every $\alpha \in D$,*

$$\int_D |\Phi_a^{'}(z)|^{(2+\alpha)\frac{q}{p}} d\mu(z) \leq c_2.$$

*We denote by $D(z) = D(z, \frac{1}{4}) = \{w : |\Phi_z(w)| < \frac{1}{4}\}$. For the case $0 < q < p < \infty$, the following result for the case $\alpha = 0$, [9, 21]. For $-1 < \alpha < \infty$ , the result can be similarly proved as in, [9].*

**Theorem 4** *For $\mu$ a positive Borel measure on $D$ , $0 < q < p < \infty$ and $-1 < \alpha < \infty$ , the following statements are equivalent:*
*(i) The measure $\mu$ is an $(A_a^{p,\alpha}, q)$ -Carleson measure.*
*(ii) The function $z \to \mu(D(z))(1 - |z|^2)^{-2-\alpha} \in A^{\frac{p}{(p-q)},\alpha}$.*

## 3 Proofs of the Verifications Theorems

We first give a simple integral criterion for $H^{\infty}$.

**Lemma 1** Let $p > 0$ and let $f \in H(D)$ . Then the following conditions are equivalent:
(i) $f \in H^{\infty}$.
(ii) $(fo\Phi_a)$ is a bounded subset of $A_a^{p,\alpha}$ for some $\alpha > -1$
(iii) $(fo\Phi_a)$ is a bounded subset of $A_a^{p,\alpha}$ for all $\alpha > -1$
(iv) $\sup_{a\in D}\int_D |f(z)|^p(1-|z|)^{-2}(1-(|\Phi_a(z)|^2))^s dA(z) < \infty$ for some $s > 1$.
(v) $\sup_{a\in D}\int_D |f(z)|^p(1-|z|)^{-2}(1-(|\Phi_a(z)|^2))^s dA(z) < \infty$ for all $s > 1$.

**Proof** Let $f \in H^{\infty}$ . Then

$$\sup_{a\in D}\int_D |fo\Phi_a(z)|^p(1-|z|)^{\alpha}dA(z) \le \|f\|_{H^\infty}^p \int_D (1-|z|^2)^{\alpha}dA(z) < \infty$$

for any $\alpha > -1$ . This $(i)$ implies $(iii)$ . It is trivial that $(iii)$ implies $(ii)$. Let $\{fo\Phi_a\}$ be a bounded subset of $A_a^{p,\alpha}$ for $\alpha > -1$ . If $\alpha \ge 0$, we fix an $r \in (0,1)$. By subharmonicity of $|fo\Phi_a|^p$, we get

$$\begin{aligned} |f(a)|^p = |fo\Phi_\alpha(0)|^p &\le \tfrac{1}{r^2}\int_{D(0,r)} |fo\Phi_a(z)|^p dA(z) \\ &\le \tfrac{1}{r^2(1-r^2)^{\alpha}}\int_{D(0,r)} |fo\Phi_a(z)|^p(1-|z|^2)^{\alpha}dA(z) \end{aligned} \tag{3.1}$$

Thus

$$\sup_{a\in D}|f(a)|^p \le c(r)\sup_{a\in D}\int_D |fo\Phi_a(z)|^p(1-|z|^2)^{\alpha}dA(z) < \infty.$$

So $f \in H^{\infty}$. For the case $-1 < \alpha < 0$ , we notice that

$$\int_D |fo\Phi_a(z)|^p dA(z) \le \int_D |fo\Phi_a(z)|^p(1-|z|^2)^{\alpha}dA(z).$$

Thus this reduces the problem to the case $\alpha = 0$ .Thus $(ii)$ implies $(i)$ . If we change the variable $\Phi_\alpha(z)$ by $w$ and let $s = \alpha + 2$ , then it is easy to see that $(iv)$ is equivalent to $(ii)$ , and $(v)$ is equivalent to $(iii)$ . The proof is complete .□

Replacing $f$ by $f'$ , we immediately have an integral criterion for the space $B^0 = \{f \in H(D), f' \in H^{\infty}\}$.

**Lemma 2** Let $p > 0$ and let $f \in H(D)$ . Then the following conditions are equivalent:
(i) $f \in B^o$ .
(ii) $\{f'o\Phi_a\}$ is a bounded subset of $A_\alpha^{p,\alpha}$, for some $a > -1$.
(iii) $\{f'o\Phi_a\}$ is a bounded subset of $A_\alpha^{p,\alpha}$, for all $a > -1$.
(iv) $f \in F(p,-2,s)$ for some $s > 1$.
(v) $f \in F(p,-2,s)$ for all $s > 1$.

We also need the following lemma.

**Lemma 3** Let $0 < p < \infty, q < -2$ and $s > 0$. Let $f \in H(D)$ . If

$$\sup_{a\in D}\int_D |f(z)|^p(1-|z|)^q(1-|\Phi_a(z)|^2)^s dA(z) < \infty, \tag{3.2}$$

then $f = 0$.

**Proof** Let $0 < p < \infty, q < -2$ and $s > 0$ . Let $f \in H(D)$ and satisfy (3.2). Fix $r \in (0,1)$. Similarly as in the proof of Lemma 1, by subharmonicity of $|fo\varphi_a|^p$ , we get

$$\begin{aligned}|f(a)|^p = |fo\Phi_a(0)|^p &\le \tfrac{1}{r^2}\textstyle\int_{D(0,r)} |fo\Phi_a(w)|^p dA(w)\\ &= \tfrac{1}{r^2}\textstyle\int_{D(a,r)} |f(z)|^p|\Phi_a^{'}(z)|^2 dA(z)\\ &\le \tfrac{16}{r^2(1-|a|^2)^2}\textstyle\int_{D(a,r)} |f(z)|^p dA(z),\end{aligned}$$

where $D(a,r) = \{z : |\Phi_a(z)| < r\}$ . It is known that, for $z \in D(a,r)$,
$1-|z|^2 \sim 1-|a|^2$, [22]. Thus, for a suitable constant $c$

$$\begin{aligned}|f(a)|^p(1-|a|^2)^{q+2} &\le \tfrac{16c}{r^2}\textstyle\int_{D(a,r)} |f(z)|^p(1-|z|^2)^q dA(z)\\ &\le \tfrac{16c}{r^2(1-r^2)^s}\textstyle\int_{D(a,r)} |f(z)|^p(1-|z|^2)^q(1-|\Phi_a(z)|^2)^s dA(z).\end{aligned}$$

Thus, if (3.2) holds then

$$\sup_{a\in D}|f(a)|(1-|a|^2)^{q+2} \le M < \infty,$$

Where $M$ is an absolute constant. Thus $|f(a)| \le M(1-|a|^2)^{-q-2}$. When
$q < -2, -q-2 > 0$ . Letting $|a| \to 1$ we see that $\lim_{|a|\to 1}|f(a)| = 0$ . By the maximal principle, we get that $f(z) = 0$ for any $z \in \partial D$ .□

The main verification results are the following two theorems.

**Theorem 5** *Let* $\mathbf{g}$ *be an analytic function on* $D$ *, let* $-1 < \alpha, \beta < \infty, \beta > 3\alpha$ *and let* $\gamma^* = \frac{\beta+2}{2} - \frac{\alpha+2}{2-\varepsilon}$*, for* $0 \le \varepsilon < 2$ *or* $\varepsilon > 2$.
*(i) If* $\gamma^* > 0$ *then* $M(A_a^{2-\varepsilon,\alpha}, A_a^{2,\beta}) = B^{1+\gamma}$
*(ii) If* $\gamma^* = 0$ *then* $M(A_a^{2-\varepsilon,\alpha}, A_a^{2,\beta}) = H^\infty$
*(iii) If* $\gamma^* < 0$ *then* $M(A_a^{2-\varepsilon,\alpha}, A_a^{2,\beta}) = \{0\}$
*(iv) If* $\frac{1}{s} = \frac{1}{2} - \frac{1}{2-\varepsilon}$ *and* $\frac{\delta}{s} = \frac{\beta}{2} - \frac{\alpha}{2-\varepsilon}$*, then* $M(A_a^{2-\varepsilon,\alpha}, A_a^{2,\beta}) = A_a^{s,\delta}$.

### 3.1 Proof of Theorem 5

**Proof** By definition, an analytic function $\mathbf{g} \in M(A_a^{2-\varepsilon,\alpha}, A_a^{2,\beta})$ if and only if, for any $f \in A_a^{2-\varepsilon,\alpha}$,

$$\int_D |f(z)\mathbf{g}(z)|^2 dA_\beta(z) \le c\|f\|^2_{2-\varepsilon,\alpha}. \tag{3.3}$$

Let $d\mu_{\mathbf{g}}(z) = |\mathbf{g}(z)|^2 dA_\beta(z)$. Then (3.3) means that $d\mu_{\mathbf{g}}$ is an $(A_a^{2-\varepsilon,\alpha}, 2)$-Carleson measure. Now we will prove $(i), (ii)$ and $(iii)$ at the same time. By Theorem 3, if $\varepsilon \ge 0$, (3.3) is equivalent to the fact that

$$\sup_{a\in D}\int_D |\Phi_a'(z)|^{\frac{2(2+\alpha)}{2-\varepsilon}} d\mu_{\mathbf{g}}(z) < \infty,$$

which is the same as

$$\sup_{a\in D}\int_D |\mathbf{g}(z)|^2(1-|z|^2)^{\beta-\frac{2(2+\alpha)}{(2-\varepsilon)}}(1-|\Phi_a(z)|^2)^{\frac{2(2+\alpha)}{2-\varepsilon}} dA(z) < \infty. \tag{3.4}$$

From, [10]. $\Phi_a(z) = \frac{a-z}{1-\bar{a}z} \Rightarrow \Phi_a'(z) = -\frac{(1-|a|^2)}{(1-\bar{a}z)^2}$

therefore $|\Phi_a'(z)|^2 = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^4}$

$$1-|\Phi_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\Phi_a'(z)|$$

therefore

$$|\Phi_a'(z)| = (1-|z|^2)^{-1}(1-|\Phi_a(z)|^2) \tag{3.5}$$

$$d\mu_{\mathbf{g}}(z) = |\mathbf{g}(z)|^2 dA_\beta(z) = |\mathbf{g}(z)|^2(\beta+1)(1-|z|^2)^\beta dA(z) \tag{3.6}$$

$$\sup_{a\in D}\int_D |\Phi_\alpha'(z)|^{\frac{2(2+\alpha)}{(2-\varepsilon)}} d\mu_{\mathbf{g}}(z) < \infty \tag{3.7}$$

From (3.5), (3.6), (3.7) we have

$$\sup_{a\in D}\int_D (1-|z|^2)^{-\frac{2(2+\alpha)}{(2-\varepsilon)}}(1-|\Phi_a(z)|^2)^{\frac{2(2+\alpha)}{(2-\varepsilon)}}|\mathbf{g}(z)|^2(1-|z|^2)^\beta.(\beta+1)dA(z) < \infty$$

therefore

$$\sup_{a\in D}\int_D |\mathbf{g}(z)|^2(1-|z|^2)^{\beta-\frac{2(2+\alpha)}{(2-\varepsilon)}}(1-|\Phi_a(z)|^2)^{\frac{2(2+\alpha)}{(2-\varepsilon)}}(\beta+1)dA(z) < \infty$$

therefore

$$\sup_{a\in D}\int_D |\mathbf{g}(z)|^2(1-|z|^2)^{\beta-\frac{2(2+\alpha)}{(2-\varepsilon)}}(1-|\Phi_a(z)|^2)^{\frac{2(2+\alpha)}{(2-\varepsilon)}} dA(z) < \infty.$$

Notice that, as $0 \le \varepsilon < 2$ or $\varepsilon > 2$, $\frac{2(2+\alpha)}{(2-\varepsilon)} > 1$. Let $G$ be an antiderivative of $\mathbf{g}$. If $\gamma^* > 0$, then $\beta - \frac{2(2+\alpha)}{(2-\varepsilon)} > -2$ . (3.4) means $G \in B^{\frac{\beta}{2}-\frac{(2+\alpha)}{(4-\varepsilon)}} = B^{\frac{(\beta+2)}{2}-\frac{(2+\alpha)}{(2-\varepsilon)}}$,which is equivalent to the fact that $\mathbf{g} = G' \in B^{1+\frac{(\beta+2)}{2}-\frac{(2+\alpha)}{(2-\varepsilon)}}$ see, [6, 7, 23–25]. Thus $(i)$ is proved. If $\gamma^* = 0$ then $\beta - \frac{2(2+\alpha)}{(2-\varepsilon)} = -2$. By Lemma 1, (3.4) is equivalent to that $\mathbf{g} \in H^\infty$, which proves $(ii)$. If $\gamma^* < 0$, then $\beta - \frac{2(2+\alpha)}{(2-\varepsilon)} < -2$. By Lemma 3, (3.4) implies $\mathbf{g} = 0$ , which proves $(iii)$. For proving $(iv)$, we use Theorem 4. Let $\varepsilon < 0$ . By Theorem 4, (3.3) is equivalent to the fact that

$$\int_D [\mu_{\mathbf{g}}(D(z,\frac{1}{4}))(1-|z|^2)^{-2-\alpha}]^{\frac{(2-\varepsilon)}{-\varepsilon}} dA_\alpha(z) < \infty,$$

Where $d\mu_{\mathbf{g}}$ is given above. Thus, [10]:

$$\int_D [\frac{1}{(1-|z|^2)^{2+\alpha}} \int_{D(z,\frac{1}{4})} |\mathbf{g}(w)|^2 dA_\beta(w)]^{\frac{(2-\varepsilon)}{-\varepsilon}} dA_\alpha(z) < \infty. \tag{3.8}$$

For: (3.3) Means that $d\mu_{\mathbf{g}}$ is an $(A_a^{2-\varepsilon,\alpha}, 2)$ -Carleson measure and from Theorem 4, and (3.3) the function $z \to \mu_{\mathbf{g}}(D(z))(1-|z|^2)^{-2-\alpha} \in A^{\frac{2-\varepsilon}{-\varepsilon},\alpha}$ which implies that

$$\int_D [\mu_{\mathbf{g}}(D(z))(1-|z|^2)^{-2-\alpha}]^{\frac{2-\varepsilon}{-\varepsilon}} dA_\alpha(z) < \infty$$

from Theorem 3

$$D(z) = D(z, \frac{1}{4}) = \{w : |\Phi_z(w)| < \frac{1}{4}\}$$

therefore

$$\int_D [\mu_{\mathbf{g}}(D(z, \frac{1}{4}))(1-|z|^2)^{-2-\alpha}]^{\frac{(2-\varepsilon)}{-\varepsilon}} dA_\alpha(z) < \infty$$

$$\int_D [\mu_{\mathbf{g}}(w)(1-|z|^2)^{-2-\alpha}]^{\frac{(2-\varepsilon)}{-\varepsilon}} dA_\alpha(z) < \infty. \tag{3.9}$$

$$d\mu_{\mathbf{g}}(w) = |\mathbf{g}(w)|^2 dA_\beta(w)$$

therefore

$$\mu_{\mathbf{g}}(w) = \int_{D(z,\frac{1}{4})} |\mathbf{g}(w)|^2 dA_\beta(w) \tag{3.10}$$

From (3.9) and (3.10) we have

$$\int_D [\frac{1}{(1-|z|^2)^{2+\alpha}} \int |\mathbf{g}(w)|^2 dA_\beta(w)]^{\frac{(2-\varepsilon)}{-\varepsilon}} dA_\alpha(z) < \infty.$$

By subharmonicity of $|\mathbf{g}|^{2-\varepsilon}$, it is easy to see that (by proof of Lemma 3),

$$|\mathbf{g}(z)|^2(1-|z|^2)^{\beta+2} \le c \int_{D(z,\frac{1}{4})} |\mathbf{g}(w)|^2 dA_\beta(w).$$

For the proof of Lemma 3 we have

$$\begin{aligned} |\mathbf{g}(a)|^{2-\varepsilon} &\le \tfrac{16}{r^2(1-|a|^2)^2} \textstyle\int_{D(a,r)} |\mathbf{g}(z)|^{2-\varepsilon} dA(z) \\ &\le \tfrac{16}{r^2(1-|a|^2)^2} \textstyle\int_{D(a,r)} \tfrac{|\mathbf{g}(z)|^{2-\varepsilon} dA_\beta(z)}{(1-|z|^2)^\beta} \end{aligned}$$

$$|\mathbf{g}(a)|^{2-\varepsilon}(1-|z|^2)^\beta(1-|a|^2)^2 \le \frac{16c_1}{r^2} \int_{D(a,r)} |\mathbf{g}(z)|^{2-\varepsilon} dA_\beta(z)$$

Let $a = z, r = \frac{1}{4}, z = w$

$$|\mathbf{g}(z)|^{2-\varepsilon}(1-|z|^2)^{\beta+2} \le 16(16)c_1 \int_{D(z,\frac{1}{4})} |\mathbf{g}(w)|^{2-\varepsilon} dA_\beta(w)$$

therefore

$$|\mathbf{g}(z)|^{2-\varepsilon}(1-|z|^2)^{\beta+2} \leq c\int_{D(z,\frac{1}{4})}|\mathbf{g}(w)|^{2-\varepsilon}dA_\beta(w). \qquad (3.11)$$

When $c=(16)^2c_1$. Thus (3.8) implies that

$$\begin{aligned}&\int_D[|\mathbf{g}(z)|^2(1-|z|^2)^{\beta-\alpha}]^{\frac{(2-\varepsilon)}{-\varepsilon}}dA_\alpha(z)\\ &=\int_D|\mathbf{g}(z)|^{\frac{2(2-\varepsilon)}{-\varepsilon}}(1-|z|^2)^{\frac{2\beta-\varepsilon\beta-2\alpha}{-\varepsilon}}dA(z)<\infty.\end{aligned} \qquad (3.12)$$

i.e. from (3.8) and (3.11) we have

$$\int_D[\frac{1}{(1-|z|^2)^{2+\alpha}}|\mathbf{g}(z)|^2(1-|z|^2)^{\beta+2}]^{\frac{(2-\varepsilon)}{-\varepsilon}}dA_\alpha(z)<\infty$$

then

$$\begin{aligned}&\int_D[|\mathbf{g}(z)|^2(1-|z|^2)^{\beta-\alpha}]^{\frac{(2-\varepsilon)}{-\varepsilon}}dA_\alpha(z)\\ &=\int_D|\mathbf{g}(z)|^{\frac{2(2-\varepsilon)}{-\varepsilon}}(1-|z|^2)^{\frac{(2\beta-\varepsilon\beta-2\alpha)}{-\varepsilon}}(1-|z|^2)^{-\alpha}dA_\alpha(z)\\ &=\int_D|\mathbf{g}(z)|^{\frac{2(2-\varepsilon)}{-\varepsilon}}(1-|z|^2)^{\frac{(2\beta-\varepsilon\beta-2\alpha)}{-\varepsilon}}dA(z)<\infty\end{aligned}$$

Let $\frac{1}{s}=\frac{1}{2}-\frac{1}{2-\varepsilon}$ and $\frac{\delta}{s}=\frac{\beta}{2}-\frac{\alpha}{2-\varepsilon}$. Then $s=\frac{2(2-\varepsilon)}{(-\varepsilon)}$ and $\delta=\frac{(2\beta-\varepsilon\beta-2\alpha)}{(-\varepsilon)}$. Thus (3.12) means $\mathbf{g}\in A_a^{s,\delta}$ (by definition of $(A_a^{p,\alpha})$). Conversely, if $\mathbf{g}\in A_a^{s,\delta}$, then an easy application of Hölder's inequality shows that $\mathbf{g}\in M(A_a^{(2-\varepsilon),\alpha},A_a^{2,\beta})$.
The proof is complete.□

We need some preliminary results.

**Lemma 4** Let $f\in H(D)$ and $0<p<\infty$. Then for any integer $n>0$;
(i) if $T_nf\in A^{p,\alpha}$ then $\int_0^1 M_p^p(r,T_nf)dr\leq K\|T_nf\|_{p,\alpha}^p$;
(ii) if $\int_0^1 M_p^p(r,T_nf)(1-r^2)^\alpha dr<\infty$; then $T_nf\in A^{p,\alpha}$ and

$$\|T_nf\|_{p,\alpha}^p\leq K\int_0^1 M_p^p(r,T_nf)(1-r^2)^\alpha dr.$$

For the proof see, [26–28].

**Proposition 1** Let $f\in H(D)$ and $0<p<\infty$. Then $f\in A_\alpha^{p,\alpha}$ if and only if $f^n(z)(1-|z|^2)^n\in A^{p,\alpha}$, and $\|f\|_{p,\alpha}$ is comparable to

$$\sum_{k=1}^{n-1}|f^k(0)|+\|f^n(z)(1-|z|^2)^n\|_{p,\alpha}.$$

For the case $1\leq p<\infty$ , a proof is given in, [22]. When $0<p<1$, the unweighted case($\alpha=0$) was proved in, [26]. The proof of the weighted case is similar to that in, [26]. We sketch the proof here for completion. Denote by $T_nf(z)=f^n(z)(1-|z|^2)^n$ and $M_p^p(r,f)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^pd\theta$

**Proof** Let $T_n f(z) = f^n(z)(1-|z|^2)^n$. Let $f \in A^{p,\alpha}_{\alpha}$. Then by, [26]. and Lemma 4,

$$\begin{aligned} \|T_n f\|^p_{p,\alpha} &\leq K \int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha dr \\ &= K \int_0^1 M_p^p(r, f^n)(1-r^2)^{np+\alpha} dr \\ &\leq K \int_0^1 M_p^p(r, f)(1-r^2)^\alpha dr \\ &\leq K \|f\|^p_{p,\alpha}. \end{aligned}$$

This proved that $T_n f \in A^{p,\alpha}$ and $\|T_n f\|^p_{p,\alpha} \leq K\|f\|^p_{p,\alpha}$. On the other hand, by Proposition 1 in, [22]. we see that $|f^n(0)| \leq K\|f\|_{p,\alpha}$. Thus

$$\sum_{k=1}^{n-1} |f^k(0)| + \|T_n f\|_{p,\alpha} \leq K\|f\|_{p,\alpha}.$$

Conversely, let $T_n f \in A^{p,\alpha}$. Then by, [26]. and Lemma 4, we get

$$\begin{aligned} \|f\|^p_{p,\alpha} &\leq K \int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha dr \\ &\leq K(\textstyle\sum_{k=1}^{n-1} |f^n(0)|^p + \int_0^1 M_p^p(r, T^n f)(1-r^2)^{np+\alpha} dr) \\ &\leq K(\textstyle\sum_{k=1}^{n-1} |f^n(0)|^p + \|T_n f\|^p_{p,\alpha}). \end{aligned}$$

which implies that $\|f\|_{p,\alpha} \leq K(\sum_{k=1}^{n-1} |f^k(0)|^p + \|T_n f\|^p_{p,\alpha})$. The proof is complete.□

**Proposition 2** Let $f \in H(D)$. Let $0 < p < \infty, -2 < q < \infty$, and $n \in N$. Then $f \in F(p, q, 1)$ if and only if

$$\sup_{a \in D} \int_D |f^n(z)|^p (1-|z|^2)^{(n-1)p+q} (1-|\Phi_a(z)|^2) dA(z) < \infty.$$

**Remark 1** Since $BMOA^\alpha_p = F(p, p\alpha - 2, 1)$, Proposition 2 says that, for $0 < p < \infty$ and $0 < \alpha < \infty$, $f \in BMOA^\alpha_p$ if and only if

$$\sup_{a \in D} \int_D |f^n(z)|^p (1-|z|^2)^{(n-1+\alpha)p-2} (1-|\Phi_a(z)|^2) dA(z) < \infty.$$

Using Proposition 1, the proof of Proposition 2 is exactly the same as the proof in, [29]. and so is omitted here. Note that, however, the proof cannot go through for the general space $F(p, q, s)$ when $0 < s < 1$ and $0 < p < 1$ even with Proposition 1. Now we prove the second main Theorem.

**Theorem 6** *Let* **g** *be an analytic function on* $D$ *, let* $0 < \beta < \infty, \beta < \infty, \beta > 3\alpha$ *and* $\gamma^* = \frac{\beta+2}{2} - \frac{1}{2-\varepsilon}$*, for* $0 \leq \varepsilon < 2$ *or* $\varepsilon > 2$.
*(i) If* $\gamma^* > 0$ *then* $M(H^2, A^{2,\beta}_a) = B^{1+\gamma}$
*(ii) If* $\gamma^* = 0$ *then* $M(H^2, A^{2,\beta}_a) = H^\infty$
*(iii) If* $\gamma^* < 0$ *then* $M(H^2, A^{2,\beta}_a) = \{0\}$
*(iv) If* $\frac{1}{s} = \frac{1}{2} - \frac{1}{2-\varepsilon}$ *then* $(H^2, A^{2,\beta}_a) = T^2_s(dA_\beta)$
*(v) If* $s \to \infty$*, then* $M(H^2, A^{2,\beta}_a) = BMOA_2^{1+\frac{(1+\beta)}{2}}$.

### 3.2 Proof of Theorem 6

**Proof** We will prove $(i), (ii), (iii)$ and $(v)$ at the same time , by using Theorem 1. The proof is similar to the proof of Theorem 5. Let $\mathbf{g} \in M(H^{2-\varepsilon}, A_a^{2;\beta})$ . This means, for any $f \in H^{2-\varepsilon}$ ,

$$\int_D |f(z)\mathbf{g}(z)|^2 dA_\beta(z) \leq c\|f\|^2_{H^{2-\varepsilon}} \tag{3.13}$$

Let $d\mu_{\mathbf{g}}(z) = |\mathbf{g}(z)|^2 dA_\beta(z)$. Then (3.13) says that $\mu_{\mathbf{g}}$ is an $(H^{2-\varepsilon}, 2)$-Carleson measure. If $\varepsilon \geq 0$, by Theorem 1, this is equivalent to the fact that

$$\sup_{a\in D} \int_D |\Phi'_\alpha(z)|^{\frac{2}{(2-\varepsilon)}} d\mu_{\mathbf{g}}(z) < \infty, \tag{3.14}$$

which is the same as

$$\sup_{a\in D} \int_D |\mathbf{g}(z)|^2 (1-|z|^2)^{\beta-\frac{2}{2-\varepsilon}} (1-|\Phi_a(z)|^2)^{\frac{2}{2-\varepsilon}} dA(z) < \infty. \tag{3.15}$$

$$\sup_{a\in D} \int_D [(1-|z|^2)^{-1}(1-|\Phi_a(z)|^2)]^{\frac{2}{2-\varepsilon}} |\mathbf{g}(z)|^2 (\beta+1)(1-|z|^2)^\beta dA(z) < \infty$$

$$\sup_{a\in D} \int_D |\mathbf{g}(z)|^2 (1-|z|^2)^{\beta-\frac{2}{2-\varepsilon}} (1-|\Phi_a(z)|^2)^{\frac{2}{2-\varepsilon}} (\beta+1) dA(z) < \infty$$

therefore

$$\sup_{a\in D} \int_D |\mathbf{g}(z)|^2 (1-|z|^2)^{\beta-\frac{2}{2-\varepsilon}} (1-|\Phi_a(z)|^2)^{\frac{2}{2-\varepsilon}} dA(z) < \infty$$

IF $\varepsilon > 0$ let $G$ be an antiderivative of $\mathbf{g}$ . By, [7]. if $\gamma^* > 0$, then $\beta - \frac{2}{(2-\varepsilon)} > -2$ and so (3.15) means $G \in B^{\frac{(\beta-\frac{2}{4-\varepsilon})}{2}} = B^{\frac{(\beta+2)}{2}-\frac{1}{(2-\varepsilon)}}$ ,which is equivalent to the fact that $\mathbf{g} = G' \in B^{1+\frac{\beta+2}{2}-\frac{1}{2-\varepsilon}}$ thus $(i)$ is proved . If $\gamma^* = 0$ then $\beta - \frac{2}{(2-\varepsilon)} = -2$. By Lemma 1, (3.15) equivalent to that $\mathbf{g} \in H^\infty$ , which proves $(ii)$. If $\gamma^* < 0$ then $\beta - \frac{2}{(2-\varepsilon)} < -2$. by Lemma 3, (3.15) implies $\mathbf{g} = 0$, which proves $(iii)$. If $\varepsilon = 0$ , then (3.15) is the same as

$$\sup_{a\in D} \int_D |\mathbf{g}(z)|^2 (1-|z|^2)^{\beta-1} (1-|\Phi_a(z)|^2) dA(z) < \infty. \tag{3.16}$$

Applying Proposition 1 to the antiderivative $G$ of $\mathbf{g}$ with $n = 2$ and $\beta > -1$ , we see that (3.16) is equivalent to

$$\sup_{a\in D} \int_D |\mathbf{g}'(z)|^2 (1-|z|^2)^{\beta+1} (1-|\Phi_a(z)|^2) dA(z) < \infty.$$

Thus,

$$\mathbf{g} \in F(2, 1+\beta, 1) = F(2, 2[1+\frac{(\beta+1)}{2}]-2, 1) = BMOA_p^{1+\frac{(\beta+1)}{2}}.$$

This proves $(v)$ . For proving $(iv)$, we use Theorem 2, the fact that $\mu_{\mathbf{g}}$ is an $(H^{2-\varepsilon}, -2)$-Carleson measure is equivalent to that the function

$$\theta \to \int_{\Gamma(\theta)} \frac{d\mu_{\mathbf{g}}(z)}{1-|z|^2}$$

belongs to $A^{\frac{2-\varepsilon}{-\varepsilon}}$ , where $\Gamma(\theta)$ is the Stolz angle at $\theta$ , and $d\mu_{\mathbf{g}}$ is given above . Thus

$$\int_0^{2\pi}\int_{\Gamma\theta}(\frac{d\mu_{\mathbf{g}}(z)}{1-|z|^2})^{\frac{(2-\varepsilon)}{-\varepsilon}}d\theta<\infty,$$

or

$$\int_0^{2\pi}(\int_{\Gamma\theta}(\frac{|\mathbf{g}(z)|^2dA_\beta(z)}{1-|z|^2})^{\frac{2-\varepsilon}{-\varepsilon}}d\theta<\infty,$$

which means $\mathbf{g}\in T_s^2(dA_\beta)$, where $\frac{1}{s}=\frac{1}{2}-\frac{1}{2-\varepsilon}$. Thus $(iv)$ holds and the proof is completed.$\square$

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