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Positive Solutions for a Class of Semipositone Problems

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Abstract Here we consider the autonomous two point boundary value problem:

$$-u''(x) = \lambda f(u(x)) \quad ; \qquad x \in (-1, 1),$$

$$u(-1) = 0 = u(1),$$

where $\lambda > 0$ and $f : [0, \infty) \to R$ is monotonically increasing and concave ($f'' < 0$) with $f(0) < 0$ (semipositone), $f(t) > 0$ for some $t > 0$. We obtain the exact number of positive solutions.

Keywords Semipositone; Two Point Boundary Value Problem; Positive Solutions.

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1 Introduction

We study the positive solutions to the two point boundary value problem

$$-u(x)u''(x) = \lambda f(u(x)), \qquad x \in (-1, 1) \tag{1}$$

$$u(1) = 0 = u(-1), \tag{2}$$

where $\lambda > 0$ and $f : [0, \infty) \to R$ is monotonically increasing and concave ($f'' < 0$) with

$$f(0) < 0 \text{ (semipositone)}, \qquad f(t) > 0 \text{ for some } t > 0. \tag{3}$$

We define g by $g(t) = f(t)/t$ and G by $G(t) = \int_{\epsilon}^{\epsilon+t} g(s)ds$ for any ϵ that $0 < \epsilon < \beta$ where β denote the unique positive zero of f, g and also θ denote the unique positive zero of G. Also, let $\theta > \beta$ and $g(\theta)/\theta < g'(\theta)$. Furthermore let g is such that

$$g'(t) > 0; \qquad t > 0. \tag{4}$$

It can be easily seen that for any positive solution u satisfying (1), (2) we should have $\sup\{u(x) : x \in [-1, 1]\} \geq \theta$. Also, clearly we have $g'(\infty) := \lim_{t\to\infty} g'(t) = 0$ and $g''(t) < 0$ for all $t > 0$. Therefore there exists an $\eta > 0$ such that $\eta g'(\eta) = g(\eta)$; $(g(t)/t)' > 0$ for all $t \in (0, \eta)$ and $(g(t)/t)' < 0$ for all $t \in (\eta, \infty)$ (see[1]).
Let $\phi(t) = (g(t)/t)'$. Then differentiating and using $g'' < 0$ we have

$$\phi'(t) < -2\phi(t)/t,$$

so that $\phi' < 0$ whenever $\phi > 0$. Consequently $\phi(t) \leq 0$ for all t suitably large. Also, by using $\phi'(t) < -2\phi(t)/t$ we conclude that there exists a unique η such that $\phi(\eta) = 0$ (see[1]).

Semipositone problems are not only of mathematical interest but also occur in applications such as population models with constant harvesting effort.

We studied positone and semipositone problems with special functions given in [3, 4]. In [2] semipositone problems with concave nonlinearities have been entensively studied with an additional condition that $f'(\infty) = 0$. We note that this hypothesis is necessary for the existence of positive solutions for large values of λ. It also implies that the supremum norm of positive solutions tends to ∞ as $\lambda \to \infty$ (see [1]). Here we study how the existence and multiplicity of positive solutions. In fact, we establish the exact geometry of the positive solution curves and hence the exact number of positive solutions for any $\lambda > 0$. As a by-product we relax the hypotheses on f in [2, Theorem 1.1(B)] and also establish the exact number of positive solutions for any $\lambda > 0$. Hence our results completely classify semipositone problems with monotonically increasing cocave nonlinearities.

This paper is organised as follows. In section 1 we observe introduction. We prove the Theorems in section 2. In section 3 we give a family of examples which satisfies all the hypotheses of Theorem.

Note that we consider the case when $u(x) > 0$ for $x \in (-1, 1)$, that is, we let the boudary value problem solutions (1),(2) has no interior zeros in $(-1, 1)$.
Our main results are

2 Theorems

Theorem 1 *If f, g satisfy conditions of this paper, then there exist λ_1, λ^* with*

$$0 < \lambda_1 < \lambda^* < \infty$$

such that for $\lambda < \lambda_1$ the problem (1),(2) has no positive solutions. For $\lambda = \lambda_1$ the problem (1),(2) has exactly one positive solution. For $\lambda \in (\lambda_1, \lambda^*]$ the problem (1),(2) has exactly two positive solutions. For $\lambda > \lambda^*$ the problem (1),(2) has exactly one positive solution. If ρ_λ denotes the supremum norm of the positive solution, then

$$\rho_{\lambda^*} = \theta \quad \text{and} \quad \lim_{\lambda \to \infty} \rho_\lambda = +\infty.$$

Proof Dividing (1) by $u(x)$ we have

$$-u''(x) = \lambda f(u(x))/u(x). \tag{5}$$

Multiplying (5) by $u'(x)$ and integrating, we obtain

$$-[u'(x)]^2/2 = \lambda G(u(x) - \epsilon) + c. \tag{6}$$

Since positive solutions are known to be symmetric with respect to $x = 0$ and $u'(x) > 0$ for $x \in (-1, 0)$ we have $\rho := \sup\{u(x)|x \in (-1, 1)\} = u(0)$ and $\rho \geq \theta$. Taking $x = 0$ in (6) implies that

$$u'(x) = \sqrt{2\lambda[G(\rho - \epsilon) - G(u - \epsilon)]}, \qquad x \in [-1, 0]. \tag{7}$$

Now integrating (7) over $[-1, x]$, we obtain

$$\int_0^{u(x)} \frac{du}{\sqrt{G(\rho - \epsilon) - G(u - \epsilon)}} = \sqrt{2\lambda}(x + 1) \qquad x \in [-1, 0], \tag{8}$$

which in turn implies that

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\rho} \frac{du}{\sqrt{G(\rho-\epsilon) - G(u-\epsilon)}} := M(\rho) \tag{9}$$

by taking $x = 0$ in (8). Hence for any λ if there exists a $\rho \in [\theta, \infty)$ with $M(\rho) = \sqrt{\lambda}$, then (1), (2) has a positive solution $u(x)$ given by (8) satisfying $\sup\{u(x) : x \in (-1,1)\} = u(0) = \rho$. In fact, $M(\rho)$ is a continuous function which is differentiable over (θ, ∞) with

$$\frac{d}{d\rho} M(\rho) = \frac{1}{\sqrt{2}} \int_0^1 \frac{H(\rho) - H(\rho v)}{[G(\rho-\epsilon) - G(\rho v - \epsilon)]^{3/2}} dv, \tag{10}$$

where

$$H(t) = G(t-\epsilon) - (t/2) g(t); \qquad t \geq \epsilon. \tag{11}$$

For $\rho \in (\theta, \infty)$ we recall from (9) that

$$M(\rho) = \frac{1}{\sqrt{2}} \frac{\rho}{\sqrt{G(\rho-\epsilon)}} \int_0^1 \frac{dv}{\sqrt{1 - [G(\rho v - \epsilon)/G(\rho - \epsilon)]}}. \tag{12}$$

Note that $g'(\infty) = 0$, therefore we have $H'(t) < 0$ for $t < \eta$, $H'(\eta) = 0$ and $H'(t) > 0$ for $t > \eta$. Since $H(\epsilon) < 0$ we have $H(t) < 0$ for $t \in (0, \eta]$ which, in turn, implies that $M'(\rho) < 0$ for $\rho \leq \eta$. Since $\lim_{t\to\infty} H(t) = +\infty$ we have $H(t) > 0$ for t large and hence $M'(\rho) > 0$ for ρ large. It remains to prove that $\lim_{\rho\to+\infty} M(\rho) = +\infty$. Let $\lim_{t\to+\infty} g(t) = M$; $0 < M \leq +\infty$ then $\lim_{t\to+\infty} G(t)/t = M$ consequently we have

$$\lim_{\rho\to\infty} M(\rho) = (1/\sqrt{2})(\rho/\sqrt{G(\rho)}) \int_0^1 \frac{dv}{\sqrt{1-v}} = \sqrt{2}(\rho/\sqrt{G(\rho)}).$$

But $\lim_{\rho\to\infty} G(\rho) = +\infty$ and $\lim_{\rho\to\infty} \rho^2/G(\rho) = \lim_{\rho\to\infty} 2\rho/g(\rho) = +\infty$. Thus

$$\lim_{\rho\to\infty} M(\rho) = +\infty$$

easily follows. □

Theorem 2 *If we have the boundary value problem*

$$-u''(x) = \lambda f(u(x)), \qquad x \in (-1,1) \tag{13}$$

$$u(1) = 0 = u(-1), \tag{14}$$

where $\lambda > 0$ and $f : [0,\infty) \to R$ is monotonically increasing and concave ($f'' < 0$) with $\lim_{t\to 0} f(t) = -\infty$, $f(t) > 0$ for some $t > 0$, and $\lim_{t\to+\infty} t f'(t) = 0$ and $(f(\theta)/\theta) < f'(\theta)$, then there exist λ^*, λ_1 such that $0 < \lambda_1 < \lambda^* < \infty$ and the problem (13),(14) have no positive solutions for $0 < \lambda < \lambda_1$ and for the case $\lambda = \lambda_1$ has exactly one positive solution. Also, for $\lambda \in (\lambda_1, \lambda^*]$ the problem (13),(14) has exactly two positive solutions and for $\lambda > \lambda^*$ has exactly one positive solution. If ρ_λ denotes the supremum norm of the positive solution, then we have $\rho_{\lambda^*} = \theta$ and $\lim_{\lambda\to\infty} \rho_\lambda = +\infty$.

Proof The proof follows from arguments similar to the ones in Theorem 1. □

3 Examples

Consider $f(t) = -1/(t+1) + t$. Then $f(0) < 0, f'(t) > 0$ for all $t \geq 0$ and $f''(t) < 0$ for all $t \geq 0$. We have $g(t) = \frac{-1}{t(t+1)} + 1$ and $\beta = (\sqrt{5}-1)/2$ and $g'(t) > 0$ for all $t > 0$. Also, $\epsilon = 0.3$ and $G(t) = \int_{0.3}^{0.3+t} g(s)ds = t + \ln(1 + 10/(3+10t)) - \ln(1 + 10/3)$. Since $G(1) > 0$, thus $\theta \in (0,1)$. Also, we have

$$\phi(t) = (g(t)/t)' = (\frac{-1}{t^2(t+1)} + 1/t)' = \frac{-t^3 - 2t^2 + 2t + 2}{t^3(t+1)^2}.$$

Note that $\phi(1) > 0$ and $\phi(2) < 0$ and hence there exists $\eta \in (1,2)$ with $\phi(\eta) = 0$. Thus we have $\eta > \theta > \beta$. We see that the above f, g satisfies all the hypotheses of Theorem. In fact, for any positive real numbers b, c, d, m we get an $f(t) = -b/(ct+d) + mt$ that it can satisfies all the hypotheses of Theorem.

References

[1] Castro A., Gadam S. and Shivaji, R. (2000). Evolution of positive solution curves in semipositone problems with concave nonlinearities. *J. Math. Anal. Appl.* 245: 282-293.

[2] Castro, A. and Shivaji, R. (1988). Non-negative solutions for a class of non-positone problems, *Proc. Roy. Soc.* Edinburgh: Sect. A 108. 291-302,

[3] Jaafari Matehkolaee, M. (2012). *Nonnegative solutions in boundary value problems,* J. Math. Comput. Sci. 2(3): 684-689.

[4] Jaafari Matehkolaee, M. (2012). *Nonnegative solutions in two point boundary value problems.* J. Sci. Art. 1(18): 39-42.