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Analytic Approximate Solution for the KdV Equation with the Homotopy Analysis Method

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Abstract The homotopy analysis method (HAM) is applied to obtain the analytic approximate solution of the well-known Korteweg-de Vries (KdV) equation. The HAM is an analytic technique which provides us with a new way to obtain series solutions of such nonlinear problems. HAM contains the auxiliary parameter $\hbar$, which provides us with a straightforward way to adjust and control the convergence region of the series solution. The resulted HAM solution at eighth order approximation is then compared with that of the exact soliton solution of KdV equation, and shown to be in excellent agreement.

Keywords KdV equation; homotopy analysis method; approximate analytic solution; soliton solution; $\hbar$-curve.

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1 Introduction

There are several analytical techniques of studying the integrable nonlinear waves equations that have soliton solutions, where each technique has its own suppositions and areas of usage. For example, the inverse scattering transform (IST) can be used to solve initial value problems, but it uses powerful analytical methods and quantum scattering theory (e.g. [1]), and therefore makes strong assumptions about the nonlinear equations. On the lesser extreme, one can find a travelling wave solution to almost all equations by a simple substitution which reduces the equation to an ordinary differential equation (e.g. [2]). Between these two extremes lies Hirota's direct/bilinear method. Although the transformation was intrinsically inspired by IST, Hirota's method does not need the same mathematical assumption and, as a consequence, the method is applicable to a wider class of equations than IST (e.g. [3]). At the same time, because it does not use such sophisticated techniques, it usually produces a smaller class of solutions, the multi-soliton solutions.

It is difficult customarily to solve nonlinear problems, especially by analytic technique. Therefore, seeking suitable solving methods is an active task in branches of computational physics. Recently, a new analytic approach named homotopy analysis method (HAM) has seen rapid development. The homotopy analysis method (HAM) [4, 5], is an analytic technique for nonlinear problems, which was first introduced by Liao in 1992. Logically it contains Lyapunov's small parameter method, the d-expansion method and Adomian's decomposition method [5]. Without depending on a small parameter such as in a perturbation approach, HAM has a particular advantage in solving strong nonlinear problems. Other advantages associated with HAM over the perturbation technique include greater flexibility in the selection of a proper set of base functions for the solution and a much simpler way in the control of the convergence rate and region. This method has been applied successfully to many nonlinear problems in engineering and science, such as the magnetohydrodynamics flows of non-Newtonian fluids over a stretching sheet [6], boundary-layer flows over an impermeable stretched plate [7], nonlinear model of combined convective and radiative cooling of a spherical body [8], exponentially decaying boundary layers [9], and unsteady boundary-layer flows over a stretching flat plate [10]. Thus the validity, effectiveness and flexibility of the HAM has been verified via all of these successful applications. Also, many types of nonlinear problems were solved with HAM by others [11-21].

Let us consider the celebrated Korteweg-de Vries equation (KdV). This is given by

$$u_t - 6uu_x + u_{xxx} = 0 \ \ , \ x, \, t \in R \tag{1}$$

subjects to the initial condition

$$u\left(x, 0\right) = f\left(x\right) \tag{2}$$

We shall assume that the solution $u(x,\, t)$ and its derivatives tend to zero [22-23] as $|x| \rightarrow \infty$. The nonlinear KdV (1) is an important mathematical model in nonlinear wave's theory and nonlinear surface wave's theory. The same examples are widely used in solid state physics, fluid physics, plasma physics, and quantum field theory [24].

In this paper, we employ the homotopy analysis method (HAM) to obtain the soliton solutions of equation (1). Our main motive is to look into this new analytic approach for physically significant nonlinear problems and compare the results with the existing exact solution of the problems.

2 Important Ideas of HAM

Consider a nonlinear equation in a general form

$$\mathcal{N}\left[u\left(r,t\right)\right] = 0 \tag{3}$$

where $\mathcal{N}$ is a nonlinear operator, $u\left(r,t\right)$ is an unknown function. Let $u_0\left(r,t\right)$ denotes an initial guess of the exact solution $u\left(r,t\right)$, $\hbar \neq 0$ an auxiliary parameter, $\mathcal{H}\left(r,t\right) \neq 0$ an auxiliary function, and ℓ an auxiliary linear operator, $q \in [0, 1]$ as an embedding parameter, and by means of HAM, we construct the so-called zeroth-order deformation equation

$$\left(1-q\right)\ell\left[\phi\left(r,t;q\right) - \ u_0\left(r,t\right)\right] = q\hbar\mathcal{H}\left(r,t\right)\mathcal{N}\left[\phi\left(r,t;q\right)\right] \tag{4}$$

It is very significant that one has great freedom to choose auxiliary objects in HAM. Clearly, when $q = 0, 1$ it holds for $\phi\left(r,t;0\right) = u_0\left(r,t\right)$, $\phi\left(r,t;1\right) = \ u\left(r,t\right)$, respectively. Then as long as q increases from 0 to 1, the solution $\phi\left(r,t;q\right)$ varies from initial guess $u_0\left(r,t\right)$ to the exact solution $u\left(r,t\right)$ Liao [5], by Taylor theorem expanded $\phi\left(r,t;q\right)$in a power series of q as follows

$$\phi\left(r,t;q\right) = \phi\left(r,t;0\right) + \sum_{m=1}^{\infty} u_m\left(r,t\right) q^m \tag{5}$$

where

$$u_m\left(r,t\right) = \frac{1}{m!}\frac{\partial^m \phi\left(r,t;q\right)}{\partial q^m}\Big|_{\mathrm{q=0}}. \tag{6}$$

The convergence of the series (5) depends upon the auxiliary parameter $\hbar$, auxiliary function $\mathcal{H}(r,t)$, initial guess $u_0(r,t)$ and auxiliary linear operator ℓ. If they are chosen properly, the series (5) is convergence at $q=1$, and one has

$$\mathrm{u}(r{,}t) = u_0(r{,}t) + \sum_{m=1}^{\infty} u_m(r{,}t) \quad . \tag{7}$$

According to definition (6), the governing equation can be inferred from the zeroth-order deformation equation (4). Define the vector

$$\bar{u}_n(r,t) = \{u_0(r,t),\, u_1(r,t),\ldots,u_n(r,t)\}$$

and differentiating the zeroth-order deformation equation (4) m-times with respect to q and dividing them by $m!$ and finally setting $q=0$ we obtain the so-called mth $-$ order deformation equation

$$\ell\left[u_m(r{,}t) - \chi_m u_m(r{,}t)\right] = q\hbar\mathcal{H}(r,t)\,\mathcal{R}_m(u_{m-1},r,t) \tag{8}$$

where

$$\chi_m = \begin{cases} 0\ , m \le 1 \\ 1\ ,\ m > 1 \end{cases} \tag{9}$$

and

$$\mathcal{R}_m(u_{m-1},r,t) = \frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}\left[\sum_{m=0}^{\infty} u_m(r{,}t)\,q^m\right]\right\}\Bigg|_{q=0} \quad . \tag{10}$$

Theorem 1 *As long as the series (7) is convergence it is convergence to exact solution of (3).*

The proof of this theorem can be found in Liao [5].

Note that HAM contains auxiliary parameter $\hbar$, which provides us with that control and adjustment of the convergence of the series solution (7).

3 Exact Solution of KdV

The KdV equation describes the theory of water waves in shallow channels, such as canal. It is a non-linear equation which is governed by (1) and (2). We shall suppose that the solution $\mathrm{u}(x{,}t)$ with its derivatives, tend to zero when $|x| \to \infty$.

Wazwaz [25] gave an exact solution of the KdV equation of the form

$$u(x,t) = -\frac{k^2}{2}\mathrm{sech}^2\frac{k}{2}\left(x - k^2t\right) \tag{11}$$

or equivalently

$$u(x,t) = -2k^2\frac{e^{k\left(x-k^2t\right)}}{\left(1+e^{k(x-k^2t)}\right)^2} \tag{12}$$

We will be using this solution form for comparison purposes with the HAM solution.

4 Results and Discussion on HAM Solution

For HAM solution we choose

$$u_0\left(x,t\right)=-2\frac{e^x}{\left(1+e^x\right)^2} \tag{13}$$

as the initial guess and

$$\ell\left[u\left(x,t;q\right)\right]=\frac{\partial u\left(x,t;q\right)}{\partial t} \tag{14}$$

as the auxiliary linear operator satisfying

$$\ell\left[c\right]=0 \tag{15}$$

where c is a constant.

We consider the auxiliary function as

$$\mathcal{H}\left(r,t\right)=1 \tag{16}$$

and the zeroth-order deformation problem

$$\left(1-q\right)\ell\left[u\left(x,t;q\right)-\ u_0\left(x,t\right)\right]=q\hbar\mathcal{N}\left[u\left(x,t;q\right)\right] \tag{17}$$

$$u_0\left(x,t\right)=-2\frac{e^x}{\left(1+e^x\right)^2}, \tag{18}$$

$$\mathcal{N}\left[u\left(x,t;q\right)\right]=\frac{\partial u\left(x,t;q\right)}{\partial t}-6u\left(x,t;q\right)\frac{\partial u\left(x,t;q\right)}{\partial x}+\frac{\partial^3 u\left(x,t;q\right)}{\partial x^3} \tag{19}$$

and the mth-order deformation problem

$$\ell\left[u_m\left(x,t\right)-\chi_m u_m\left(x,t\right)\right]=q\hbar\left[\frac{\partial u_{m-1}}{\partial t}-6\sum_{i=0}^{m-1}u_i\frac{\partial u_{m-1-i}}{\partial x}+\frac{\partial^3 u_{m-1}}{\partial x^3}\right] \tag{20}$$

$$u_m\left(x,0\right)=0,\quad\left(m\geq 1\right) \tag{21}$$

MATHEMATICA is being used to solve the set of linear equations (20) with conditions (21). It is found that the solution in a series form is given by

$$\begin{aligned}u\left(x,t\right)=&-2\frac{e^x}{\left(1+e^x\right)^2}\\&+\frac{2e^x(-1+e^x)\hbar t\mathrm{Log}[e](12e^x+\mathrm{Log}[e]^2-10e^x\mathrm{Log}[e]^2+e^{2x}\mathrm{Log}[e]^2)}{\left(1+e^x\right)^5}+\ldots.\end{aligned} \tag{22}$$

The analytical solution given by (22) contains the auxiliary parameter $\hbar$, which influences the convergence region and rate of approximation for the HAM solution. In Figure 1 the $\hbar$-curves are plotted for $u\left(x,t\right)$, $\ddot{u}\left(x,t\right)$, $\dddot{u}\left(x,t\right)$ against $\hbar$, when $x=t=0.01$ at the 8^{th}-order approximation, i.e. u vs $\hbar$, $\ddot{u}$ vs $\hbar$, $\dddot{u}$ vs $\hbar$.

As pointed out by Liao [5], the valid region of $\hbar$ is a horizontal line segment. It is clear from Figure 1 that the valid region for this case is in the interval $-1.4<\hbar<-0.4$. According to Theorem 1 the solution series (22) must be an exact solution, as long as it

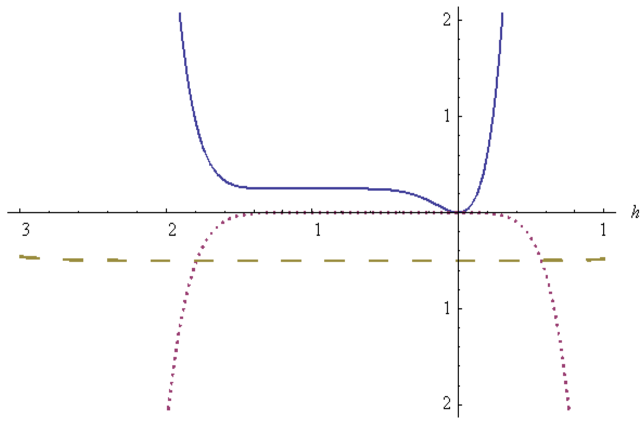

Figure 1: The $\hbar-$curve of at 8^{th}-order Approximation with Dashed point $u(0.01, 0.01)$; Solid Line: $\ddot{u}\,(0.01, 0.01)$, Dashed Line: $\dddot{u}\,(0.01,\ 0.01)$

is convergence. In this case for $-1 < t < 1$ and $\hbar = -1$, the exact soliton solution and HAM solution generated the same profile as are shown in Figure 2. The obtained numerical results for exact and HAM solutions are compared and summarized in Table 1. Clearly, from Table 1, this is shown to be in excellent agreement.

In Figure 3 we have MATHEMATICA graphs depicting $u\,(x, 0.75)$ $u\,(x, 1)\,, u\,(x, 1.25)$ and $u\,(x, 1.5)$ against x. The results obtained by the 8^{th}-order approximation for $\hbar = -0.5, -0.75, -1$. We study the diagrams of the results obtained by HAM for $\hbar = -0.5,\ \hbar = -0.75$ and $\hbar = -1$, and in comparison with the exact soliton solution (1) we conclude that the best value for $\hbar$ in this case is $\hbar = -1$.

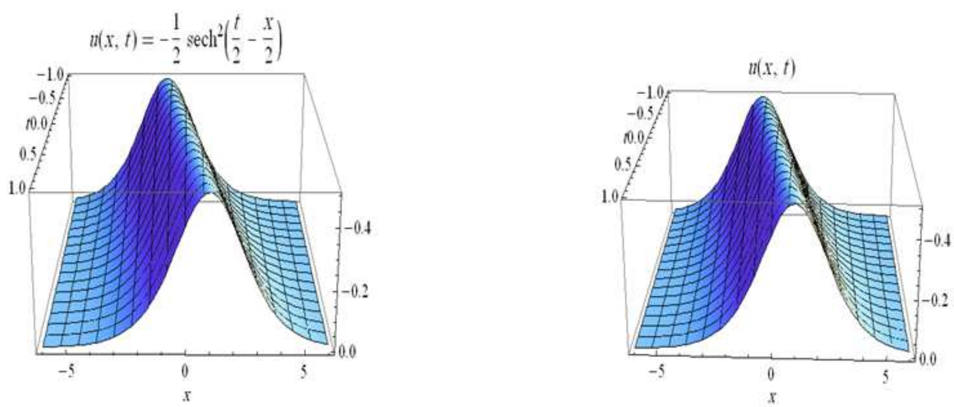

Figure 2: Comparison of the Exact Solution with the HAM Solution of $u(x, t)$, when $\hbar = -1$.

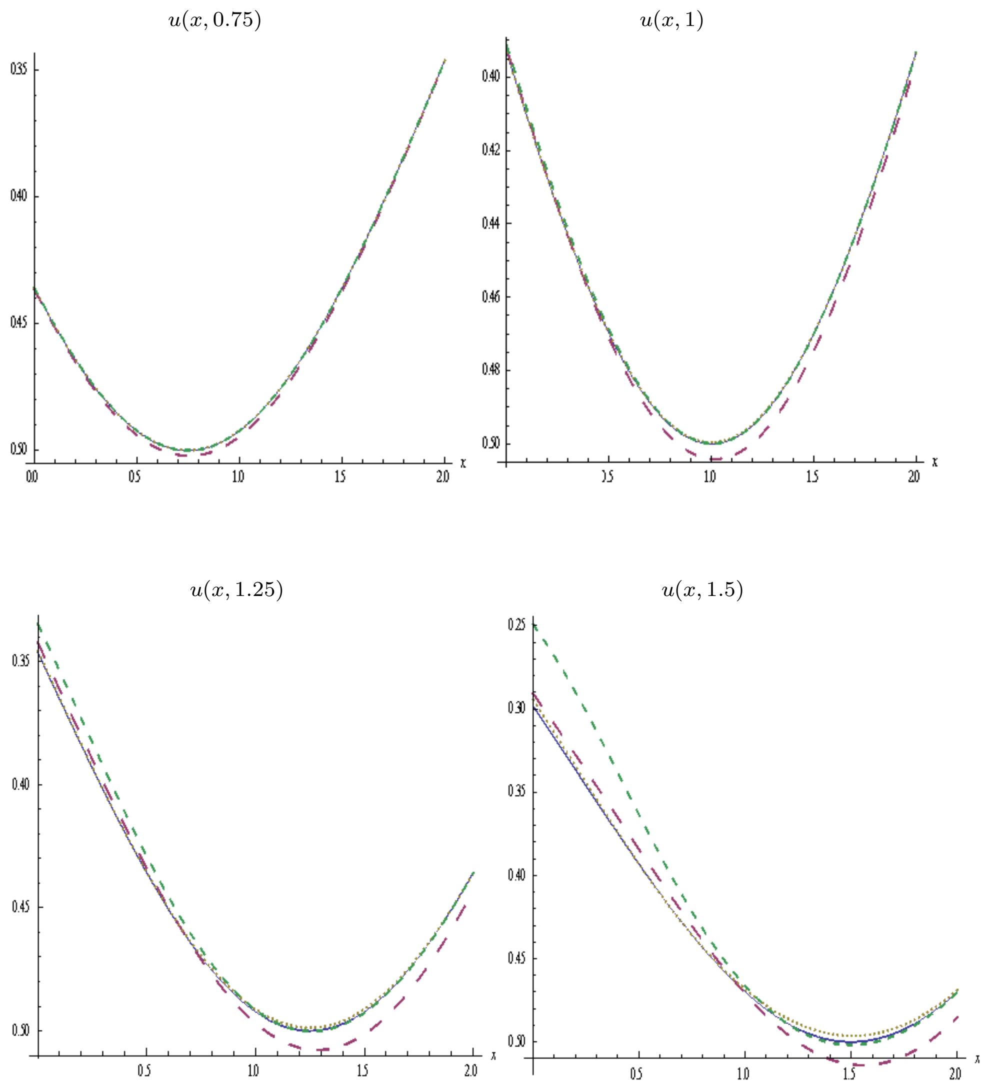

Figure 3: MATHEMATICA Graphs Depicting $u\,(x, 0.75)\,,\; u\,(x, 1)\,,\; u\,(x, 1.25)\,,\; u\,(x, 1.5)$ Against x. Solid-line is for Exact Soliton Solution; Large Dash is for $\hbar = -0.5$; Medium Dash is for $\hbar = -0.75$; Tiny Dash is for $\hbar = -1$.

Table 1: Comparison of the HAM Solution with Exact Solution when $\hbar = -1$ and $t = 0$, $0.25, 0.5, 0.75, 1$, Respectively

t	x	Exact Solution	HAM Solution	Absolute Error
0.05	-6	-0.004693564364	-0.004693564364	$4.805184028455 \times 10^{-16}$
	-3	-0.086345707150	-0.086345707150	$4.066191827689 \times 10^{-15}$
	2	-0.218079638303	-0.218079638303	$5.301314942585 \times 10^{-15}$
	5	-0.013968231613	-0.013968231613	$1.117161918529 \times 10^{-15}$
0.25	-6	-0.003846044713	-0.003846044713	$5.782613970994 \times 10^{-14}$
	-3	-0.071867181662	-0.071867181650	$1.177556663289 \times 10^{-11}$
	2	-0.252258450383	-0.252258450373	$1.313826825111 \times 10^{-11}$
	5	-0.017007824315	-0.017007824315	$5.215099185829 \times 10^{-13}$
0.5	-6	-0.002997857389	-0.002997857417	$2.805728522778 \times 10^{-11}$
	-3	-0.056906053378	-0.056906047759	$5.619168172432 \times 10^{-9}$
	2	-0.298292912976	-0.298292904140	$8.836291531810 \times 10^{-9}$
	5	-0.021732459722	-0.021732459444	$2.780519009482 \times 10^{-10}$
0.75	-6	-0.002336284005	-0.002336285021	$1.0160758805513 \times 10^{-9}$
	-3	-0.044899022133	-0.044898820764	$2.0136902956213 \times 10^{-7}$
	2	-0.346210037691	-0.346209573984	$4.6370695244712 \times 10^{-7}$
	5	-0.0277316940333	-0.027731682875	$1.1158048034143 \times 10^{-8}$
1	-6	-0.0018204295917	-0.001820442360	$1.2768480764455 \times 10^{-8}$
	-3	-0.0353279143484	-0.035325412426	$2.50192187 \times 10^{-7}$
	2	-0.3932319186044	-0.393223866482	$8.0521215122253 \times 10^{-7}$
	5	-0.0353255675256	-0.035325412426	$1.5509909238181 \times 10^{-7}$

5 Conclusion

In this paper, the homotopy analysis method (HAM) as proposed in [4, 5] is applied to obtain the solitary solution of the KdV equation. The resulted HAM solution at the eighth order approximation is then compared with that of the exact soliton solution of KdV equation, and it is shown to be in excellent agreement. Clearly, via the auxiliary parameter $\hbar$, HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. Thus, we are of the opinion that this example enables us to show the flexibility and potential of the homotopy analysis method for solving other possible complicated nonlinear problems in industrial and engineering applications.

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