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Posets and Closure Operators Relative to Matroids

[1]**Hua Mao and** [2]**Sanyang Liu**
[1]Department of Mathematics, Hebei University
Baoding 071002, China
[2]Department of Mathematics, Xidian University
Xi'an 710071, China
e-mail: [1]yushengmao@263.net, [2] liusanyang@126.com

Abstract Using the closure operator axioms for a matroid presented here, for all the matroids defined on the same ground set E, two posets are provided: one is consisted by the system of the closure operators of all these matroids; another is instructed by the system of all these matorids. Additionally, when the set E is finite, by graph theory, two different skecthes to search out all the matroids defined on E are given.

Keywords Poset; matroid; closure operator; Moore family.

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1 Introduction and Prliminaries

It is well known that there is no single class of structures that one calls infinite matroids [1–3]. This paper will use the definition of matroids of arbitrary cardinality given by Betten and Wenzel [1]. The definition has been used by many researches [1, 4–10]. The main purpose of this paper is to provide an axiom system for a matroid of arbitrary cardinality by its closure operator. Additionally, it provides some applications of the new axiom system as follows: for simplicity, in what follows, let E be an arbitrary–possibly infinite. The objectives of the paper are the following:

- Constructing a poset which is consisted by the system of closure operators of matroids defined on E;
- Presenting a poset construction consisted by all the matroids defined on E;
- When E is finite, two thoughts for searching all the matroids defined on E will be provided. Comparing with that Mao [11], we will know that both of thoughts given in this paper are different from the method given by Mao [11].

In what follows, we assume that $\mathcal{P}(E)$ denotes the powerset of E. We only review some preliminary knowledge here. For others, the knowledge of finite matroids are seen [12, 13]; poset theory come from [14, 15]; order and the axiom of choice etc. are referred to Hungerford [16]; graph theory and combinatorial algorithms are seen from Korte and Vygen [17].

First of all, we recall some knowledge about a matroid of arbitrary cardinality. Definition 1 and Lemma 1 come from Betten and Wenzel [1]; Definition 2 and Lemma 2 are referred to Higuchi [18]; Definition 3 is seen Hungerford [16, p.13].

Definition 1 Assume $m \in \mathbb{N}_0$ and $\mathcal{F} \subseteq \mathcal{P}(E)$. Then the pair $M := (E, \mathcal{F})$ is called a *matroid of rank m with $\mathcal{F}$ as its closed sets*, if the following axioms hold:

(F1) $E \in \mathcal{F}$;

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;

(F3) Assume $F_0 \in \mathcal{F}$ and $x_1, x_2 \in E \setminus F_0$. Then one has either

(i) $\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$ or

(ii) $F_1 \cap F_2 = F_0$ for certain $F_1, F_2 \in \mathcal{F}$ containing $F_0 \cup \{x_1\}$ or $F_0 \cup \{x_2\}$, respectively;

(F4) $m = max\{n \in \mathbb{N}_0 |$ there exist $F_0, F_1, ..., F_n \in \mathcal{F}$ with $F_0 \subset F_1 \subset ... \subset F_n = E\}$.

The *closure operator* $\sigma = \sigma_M : \mathcal{P}(E) \to \mathcal{F}$ of M is defined by $\sigma(A) := \bigcap\limits_{\substack{F \in \mathcal{F} \\ A \subseteq F}} F$.

Obviously, Definition 1 is the extension of that of a finite matroid. For the sake of convenience, in this paper, a matroid of arbitrary cardinality will be simply called a matroid except for special describing.

Lemma 1 Assume $M = (E, \mathcal{F})$ is a matroid with σ as its closure operator. Then

(1) For any family $(F_i)_{i \in I}$ of closed sets in M, $F := \bigcap\limits_{i \in I} F_i \in \mathcal{F}$.

(2) For any $A \subseteq E$, the set $\sigma(A)$ is the smallest set in $\mathcal{F}$ containing A. In particular, $\sigma(A) = A$ if and only if $A \in \mathcal{F}$. Moreover, σ satisfies the following conditions, which characterize a closure operator:

$A \subseteq \sigma(A) = \sigma(\sigma(A))$ for all $A \subseteq E$; for $A \subseteq B \subseteq E$ one has $\sigma(A) \subseteq \sigma(B)$.
Furthermore, σ satisfies the following exchange condition:

For $A \subseteq E$ and $x, y \in E \setminus \sigma(A)$, one has $y \in \sigma(A \cup \{x\})$ if and only if $x \in \sigma(A \cup \{y\})$.

Secondly, some knowledge about closure operator on E and Moore family of subsets of E are reviewed in the following Definition 2 and Lemma 2.

Definition 2 A map $C : \mathcal{P}(E) \to \mathcal{P}(E)$ is called a *closure operator on* E if it satisfies for $A, B \subseteq E$

(i) $A \subseteq C(A)$;

(ii) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$;

(iii) $C(C(A)) = C(A)$.

A family $\mathcal{A}$ of subsets of E is called *Moore family* if it satisfies

(i) $E \in \mathcal{A}$;

(ii) $\emptyset \neq \varphi \subseteq \mathcal{A} \Rightarrow \cap \varphi \in \mathcal{A}$.

We denote by $MF(E)$ the set of all the Moore families of subsets of E. Let $\mathcal{A}_C = \{A \subseteq E | C(A) = A\}$ and $CO(E)$ denote the set of all the closure operators on E.

Lemma 2 If C is a closure operator on E. Then for each $A \subseteq E$, it has $C(A) = \cap\{B \in \mathcal{A}_C | A \subseteq B\}$. The map $\beta : C \mapsto \mathcal{A}_C$ is a bijection of $CO(E)$ onto $MF(E)$.

Let $X \in \mathcal{A} \in MF(E)$ and $\mathcal{B} = \mathcal{A} \setminus \{X\}$. Then

$$X \neq \cap\{B \in \mathcal{B} | X \subseteq B\} \Longleftrightarrow \mathcal{B} \in MF(E).$$

For the need of Theorem 3, we recall the definition of well ordered as follows.

Definition 3 Let $(P, \leq)$ be a poset. If every nonempty subset of P has a least (or minimum) element, then P is said to be *well ordered*.

In this paper, we denote all the matroids on E by $IM(E)$; $CM(E)$ and $FM(E)$ are defined as follows:

(1) $C \in CM(E)$ if and only if there is $M \in IM(E)$ with σ_M as its closure operator such that $\sigma_M = C$.

(2) $\mathcal{P}(E) \supseteq \mathcal{F} \in FM(E)$ if and only if there exists $M \in IM(E)$ such that $\mathcal{F}$ is the set of closed sets of M.

2 Closure Operator Axioms

This section presents a theorem that gathers the closure operator axioms for a matroid. The theorem builds the foundation of the other results in this paper which is essential in the study on matroids.

Theorem 1 *(Closure operator axioms) A function $\sigma : \mathcal{P}(E) \to \mathcal{P}(E)$ is the closure operator of a matroid on $E \Leftrightarrow$ for X, Y subsets of E and $x, y \in E$,*

(S1) $X \subseteq \sigma(X)$;

(S2) $Y \subseteq X \Rightarrow \sigma(Y) \subseteq \sigma(X)$;

(S3) $\sigma(X) = \sigma(\sigma(X))$;

(S4) $y \notin \sigma(X), y \in \sigma(X \cup x) \Rightarrow x \in \sigma(X \cup y)$;

(S5) For any chain satisfying $\sigma(X) \subset \sigma(X_1) \subset \ldots \subset \sigma(X_\alpha) = \sigma(Y)$, then $\alpha \in \mathbb{N}_0$ such that $\alpha < \infty$ where $X_i \subseteq E$, $(i = 1, 2, \ldots, \alpha)$.

Proof ($\Rightarrow$) From (2) in Lemma 1, we may express that (S1)-(S4) are right for σ. According to (F4) in Definition 1, there is $m < \infty$. This follows (S5) to be hold.

($\Leftarrow$) Let $\sigma : \mathcal{P}(E) \to \mathcal{P}(E)$ be a function satisfying (S1)-(S5). Then the collection $\{X \subseteq E \mid \sigma(X) = X\}$ is $\mathcal{F}(\sigma)$ and $\mathcal{F}(\sigma) \in MF(E))$ because Lemma 2. Hence $E \in \mathcal{F}(\sigma)$ and $\bigcap\limits_{\substack{F_i \in \mathcal{F}(\sigma) \\ i \in I}} F_i \in \mathcal{F}(\sigma)$. That is to say, (F1) and (F2) hold for $\mathcal{F}(\sigma)$. In addition $\sigma(A) = \cap\{B \in \mathcal{F}(\sigma)|A \subseteq B\}$. Therefore, both (F1) and (F2) are checked.

Next to prove (F3).

Firstly, we prove (F3)(i).

Assume $F_0 \in \mathcal{F}(\sigma), x_1, x_2 \in E \setminus F_0$. Let $\sigma(F_0 \cup x_1) = F_1$ and $\sigma(F_0 \cup x_2) = F_2$. Then if $\{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_2\} \subseteq F\}$, one has $\sigma(F_0 \cup \{x_1\}) = \cap\{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_1\} \subseteq F\} = \cap\{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_2\} \subseteq F\} = \sigma(F_0 \cup \{x_2\})$.

We first suppose $\sigma(F_0 \cup \{x_1\}) = \sigma(F_0 \cup \{x_2\})$. Let $F_0 \cup \{x_1\} \subseteq F' \in \mathcal{F}(\sigma), F_0 \cup \{x_2\} \subseteq F'' \in \mathcal{F}(\sigma)$. By (S2), $\sigma(F_0 \cup \{x_1\}) \subseteq F', \sigma(F_0 \cup \{x_2\}) \subseteq F''$, and so $\sigma(F_0 \cup \{x_1\}) \subseteq F'', \sigma(F_0 \cup \{x_2\}) \subseteq F'$. Hence $F'' \in \{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_1\} \subseteq F\}$ and $F' \in \{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_2\} \subseteq F\}$. Moreover $\{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_2\} \subseteq F\}$.

Namely, $\sigma(F_0 \cup \{x_1\}) = \sigma(F_0 \cup \{x_2\})$ if and only if $\{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F}(\sigma)|F_0 \cup \{x_2\} \subseteq F\}$.

Secondly, we prove (F3)(ii).

We suppose $\sigma(F_0 \cup \{x_1\}) \neq \sigma(F_0 \cup \{x_2\})$.

By (S4), $x_2 \notin \sigma(F_0 \cup \{x_1\}), x_1 \notin \sigma(F_0 \cup \{x_2\})$. For $y \in \sigma(F_0 \cup \{x_1\}) \cap \sigma(F_0 \cup \{x_2\}) \setminus F_0 \neq \emptyset$, one gets $y \in \sigma(F_0 \cup \{x_1\}) \setminus F_0$. In virtue of (S4), it follows $x_1 \in \sigma(F_0 \cup \{y\})$. Similarly, $x_2 \in \sigma(F_0 \cup \{y\})$. Hence by (S1), (S2) and (S3), one gets $\sigma(F_0 \cup \{x_1\}), \sigma(F_0 \cup \{x_2\}) \subseteq \sigma(F_0 \cup \{y\})$, and further $\sigma(F_0 \cup \{x_1\}) \cap \sigma(F_0 \cup \{x_2\}) \subseteq \sigma(F_0 \cup \{x_i\}) \subseteq \sigma(F_0 \cup \{y\}), (i = 1, 2)$. On the other hand, $y \in \sigma(F_0 \cup \{x_1\}) \cap \sigma(F_0 \cup \{x_2\})$ and (S2) taken together hints $\sigma(F_0 \cup \{y\}) \subseteq \sigma(F_0 \cup \{x_1\}) \cap \sigma(F_0 \cup \{x_2\}) \subseteq \sigma(F_0 \cup \{x_1\}), \sigma(F_0 \cup \{x_2\})$.

Summing up the above two hands, one obtains $\sigma(F_0 \cup \{y\}) = \sigma(F_0 \cup \{x_1\}) \cap \sigma(F_0 \cup \{x_2\}) = \sigma(F_0 \cup \{x_i\}) \, (i = 1, 2)$, a contradiction. That is to say, it should have $F_0 = \sigma(F_0 \cup \{x_1\}) \cap \sigma(F_0 \cup \{x_2\})$, and so (F3) holds for $(E, \mathcal{F}(\sigma))$.

Owing to (S5), it follows that (F4) holds for $(E, \mathcal{F}(\sigma))$.

Consequently, $M = (E, \mathcal{F}(\sigma))$ is a matroid with $\mathcal{F}(\sigma)$ as its closed sets.

Let σ_M be the closure operator of M. The following is to prove that σ_M is in fact σ.

For $A \subseteq E$, by Lemma 1, it leads to $\sigma_M(A) = \bigcap\limits_{A \subseteq F \in \mathcal{F}(\sigma)} F$. However, by the definition of $\mathcal{F}(\sigma)$, $\sigma(A) = \cap\{B \in \mathcal{F}(\sigma)|A \subseteq B\}$. Thus, $\sigma_M(A) = \sigma(A)$. Namely $\sigma_M = \sigma$. □

Based on Theorem 1 and Definition 1, in what follows, it is no different to say that $C \in CM(E)$ from $C \in IM(E)$; and also, no different to say that $\mathcal{F} \in FM(E)$ from $(E, \mathcal{F}) \in IM(E)$ or $\mathcal{F} \in IM(E)$.

Remark 1

Aigner in [19, p.52] and [19, p.256] give the same definition of a matroid which allows infinite sets as follows: "A matroid $M(S)$ is a set S together with a closure $A \to \overline{A}$ such that for all $p, q \in S, A \subseteq S$: (i) $p \notin \overline{A}, p \in \overline{A \cup q} \Rightarrow q \in \overline{A \cup p}$; (ii) $\exists B \subseteq A, |B| < \infty$ with $\overline{B} = \overline{A}$." We believe that this is a definition of an infinite matroid. By Theorem 1 and Mao [10, Theorem 1], we could prove the equivalence between Definition 1 and the definition of Aigner. This also stressed the truth of expression about infinite matroids given by Oxley

[2]: there is no single class of structures that one calls infinite matroids. Rather, various authors with differing motivations have studied a variety of classes of finite matroid-like structures on infinite sets. The precise relationship between particular classes is still not known. In this paper, we highlight the links between posets and closure operators of matroids. Because we indicate the links between two definitions of infinite matroids which we carry out a duty provided by different researches with differing motivations. This analysis also hints an important significance of Theorem 1. The other significance of Theorem 1 will show in the next section.

3 Posets of Closure Operators

On the same ground set E, this section will carry out the three objectives raised in Section 1. We define a binary relation $\preceq$, and further $\ll$, on $IM(E)$ as follows.

Definition 4 Let $\sigma_1, \sigma_2, \sigma \in IM(E)$.

$\sigma_1 \preceq \sigma_2 \Leftrightarrow \sigma_1(A) \subseteq \sigma_2(A)$ for any $A \subseteq E$.

$\sigma_1 \ll \sigma_2 \Leftrightarrow \sigma_1 \preceq \sigma_2, \sigma_1 \neq \sigma_2$ and $\sigma_1 \preceq \sigma \preceq \sigma_2$ implies $\sigma_1 = \sigma$ or $\sigma = \sigma_2$.

Since $\subseteq$ is an obvious partially order, this evidently implies that the relation $\preceq$ is a partial order on $IM(E)$ and $\ll$ is the covering relation of $\preceq$. Thus, it is enough to consider the properties about $\preceq$.

Next we discuss some properties about $(IM(E), \preceq)$ in the following two Lemmas.

Lemma 3 Let $(E, \mathcal{F}_1), (E, \mathcal{F}_2) \in IM(E)$ with σ_1, σ_2 as its closure operator respectively. Then $\sigma_1 \preceq \sigma_2 \Leftrightarrow \mathcal{F}_2 \subseteq \mathcal{F}_1$.

Proof

($\Rightarrow$)

For $A \in \mathcal{F}_2$, by Lemma 1, it follows $A \subseteq \sigma_1(A) \subseteq \sigma_2(A) = A$, and so $A = \sigma_1(A)$, namely $A \in \mathcal{F}_1$. Hence $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

($\Leftarrow$)

For $A \subseteq E$,

$$\sigma_1(A) = \bigcap_{A \subseteq F \in \mathcal{F}_1} F = (\bigcap_{A \subseteq F \in \mathcal{F}_2} F) \cap (\bigcap_{A \subseteq F \in \mathcal{F}_1 \setminus \mathcal{F}_2} F) = \sigma_2(A) \cap (\bigcap_{A \subseteq F \in \mathcal{F}_1 \setminus \mathcal{F}_2} F) \subseteq \sigma_2(A)$$

implies $\sigma_1 \preceq \sigma_2$.$\square$

Lemma 4 If X is a maximal element of $\mathcal{F}_1 \setminus \mathcal{F}_2$ and $\mathcal{B} = \mathcal{F}_1 \setminus \{X\}$ where $\mathcal{F}_1, \mathcal{F}_2 \in MF(E), \mathcal{F}_2 \subseteq \mathcal{F}_1$. Then $\mathcal{B} \in MF(E)$.

Proof First, it is to prove that $\{B \in \mathcal{F}_2 | X \subseteq B\} = \{B \in \mathcal{B} | X \subseteq B\}$.

Because $X \in \mathcal{F}_1 \setminus \mathcal{F}_2$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$ together induces $\mathcal{F}_2 \subseteq \mathcal{B}$. Hence, one has $\{B \in \mathcal{F}_2 | X \subseteq B\} \subseteq \{B \in \mathcal{B} | X \subseteq B\}$. Conversely, assume $X \subseteq B \in \mathcal{B}$. In view of the definition of $\mathcal{B}$, one has $B \neq X$, and so $X \subset B$. The maximality of X hints $B \notin \mathcal{F}_1 \setminus \mathcal{F}_2$. However, $B \in \mathcal{B}$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$ taken together shows $B \in \mathcal{F}_2$. Thus one obtains that $\{B \in \mathcal{B} | X \subseteq B\} \subseteq \{B \in \mathcal{F}_2 | X \subseteq B\}$. Hence $\{B \in \mathcal{F}_2 | X \subseteq B\} = \{B \in \mathcal{B} | X \subseteq B\}$.

Furthermore, $\cap\{B \in \mathcal{B} | X \subseteq B\} = \cap\{B \in \mathcal{F}_2 | X \subseteq B\} \in \mathcal{F}_2$ because of $\mathcal{F}_2 \in MF(E)$ and Lemma 2. Hence $X \neq \cap\{B \in \mathcal{B} | X \subseteq B\}$. Besides $\mathcal{F}_1 \in MF(E)$ holds. Thus using Lemma 2, one gets $\mathcal{B} \in MF(E)$. □

Notice 1 From the proof in Lemma 4, we see that by Lemma 2, $C_{\mathcal{B}} \in CO(E)$ corresponding to $\mathcal{B}$. Then evidently, $C_{\mathcal{B}}$ satisfies (S1)-(S3). If $\mathcal{F}_1 \in IM(E)$, then obviously $C_{\mathcal{B}}$ satisfies (S5) because $\mathcal{B} \subseteq \mathcal{F}_1$ and Definition 1. Hence, one can say that, in order to determine if $\mathcal{B} \in IM(E)$, it only needs to check that (S4) is satisfied or not by $C_{\mathcal{B}}$.

The discussion above leads to the following theorem.

Theorem 2 *Let E be a finite set, $C, D \in CM(E)$ and $C \preceq D$. Then there exists $C \ll \sigma_1 \ll \sigma_2 \ll \ldots \ll \sigma_\alpha = D$ being a chain of $(IM(E), \preceq)$.*

Proof Let $(E, \mathcal{F}_C), (E, \mathcal{F}_D) \in IM(E)$ with C, D as its closure operator respectively. It is easy to know $\mathcal{F}_C, \mathcal{F}_D \in MF(E)$. Let X be any maximal element of $\mathcal{F}_C \setminus \mathcal{F}_D$. By the above analysis, one has $\mathcal{B} = \mathcal{F}_C \setminus \{X\} \in MF(E)$.

If there exists a maximal element of $\mathcal{F}_C \setminus \mathcal{F}_D$ satisfying $\mathcal{B}_1 = \mathcal{F}_C \setminus \{X_1\} \in IM(E)$, then $C \ll \sigma_1$ where σ_1 is the closure operator of $(E, \mathcal{B}_1)$. Otherwise, repeated application of the above process, one yields that $\mathcal{B}_k = \mathcal{F}_C \setminus \{X_1, X_2, \ldots, X_k\} \in IM(E)$ where $\mathcal{F}_C \setminus \{X_{i_1}, X_{i_2}, \ldots, X_{i_j}\} \notin IM(E),\ i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, k-1\}, j < k$ and X_t is a maximal element of $\mathcal{F}_C \setminus (\{X_1, X_2, \ldots, X_{t-1}\} \cup \mathcal{F}_D), t = 2, \ldots, k$. Evidently, $C \ll \sigma_k$ where σ_k is the closure operator of $(E, \mathcal{B}_k)$.

This process is efficient and clear because $|\mathcal{F}_C \setminus \mathcal{F}_D| \leq |E| < \infty, C \preceq D$, Lemma 3 and Lemma 4.□

Using Theorem 2, we provide a sketch to determine the structure of $(IM(E), \preceq)$ when $|E| < \infty$.

By Definition 1 or the definition of a finite matroid in [12, 13], $M_0 = (E, \mathcal{F}_0 = \mathcal{P}(E))$ and $M_M = (E, \mathcal{F}_M = \{\{E\}\})$ belong to $IM(E)$. Let σ_0, σ_M be the closure operator of M_0, M_M respectively. In virtue of Theorem 2, there exists at least one chain $\sigma_0 \ll \sigma_1 \ll \ldots \ll \sigma_n = \sigma_M$ connected σ_0 with σ_M.

In addition, for any $M = (E, \mathcal{F}) \in IM(E)$ with σ as its closure operator, according to the definition of the relation $\preceq$, Lemma 3 and Theorem 2, there exist two chains $\sigma_0 \ll \sigma_1 \ll \ldots \ll \sigma_n = \sigma$ and $\sigma \ll \sigma_{11} \ll \sigma_{21} \ll \ldots \ll \sigma_{m1} = \sigma_M$. It is easy to see that $\sigma_0 \ll \sigma_1 \ll \ldots \ll \sigma_n \ll \sigma_{11} \ll \sigma_{21} \ll \ldots \ll \sigma_{m1} = \sigma_M$ is a maximal chain between σ_0 and σ_M. This states that every matroid in $IM(E)$ is contained in a maximal chain from σ_0 to σ_M. Hence by the language of graph theory, the diagram of $(IM(E), \preceq)$ is connected and the diagram of $(IM(E), \ll)$ is a spanning tree of $(IM(E), \preceq)$.

Combining the above process with Korte and Vygen [17, pp.26-29] presented the Breadth-First Search algorithm, we believe that the whole members of $IM(E)$ can be found.

At the beginning of this paper, we say that we will present two posets relative to $IM(E)$ when E is finite. Now, we see that the above sketch is a method to provide a poset relative to $IM(E)$ if $|E| < \infty$. Next, we will give another method to discover a poset relative to $IM(E)$.

Higuchi [18] presents a way to finding all the lattices of closure operators by the Moore families. Considering Theorem 1 and the relation between the Moore families and closure operators of matroids, similar to the method of Higuchi in [18], we provide an idea to search out all the members of the new poset which is consisted by $IM(E)$. For this purpose, when E is finite, we define a binary relation $<$ of $\mathcal{P}(E)$ as a linear extension of $\subset$ when the extension is well ordered and for $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(E), \mathcal{A} \lessdot \mathcal{B}$, i.e. $\mathcal{B}$ covers $\mathcal{A}$, if and only if there exists $X \in \mathcal{P}(E)$ such that

(i) $\mathcal{A} = \mathcal{B} \setminus \{X\}$;

(ii) X is the minimum element of $(\mathcal{P}(E) \setminus \mathcal{A}, <)$;

(iii) $X \neq \cap\{A \in \mathcal{A} | X \subseteq A\}$.

We will discuss some properties about the new binary relation.

Lemma 5 Let $\mathcal{A} \subseteq \mathcal{P}(E)$ and $(\mathcal{P}(E), <)$ be well ordered. Then $(E, \mathcal{A}) \in IM(E) \Rightarrow \mathcal{A} \lessdot \mathcal{B}$ for some $(E, \mathcal{B}) \in IM(E)$.

Proof Let $(E, \mathcal{A}) \in IM(E)$ with $\sigma_{\mathcal{A}}$ as its closure operator. If $\mathcal{A} = \mathcal{P}(E)$, then $\mathcal{P}(E) \setminus \mathcal{A} = \emptyset$, and further, $(E, \mathcal{P}(E)) \notin IM(E)$ provided $|E| \not< \infty$. Thus supposing $\mathcal{A} \neq \mathcal{P}(E)$.

Let X be the minimum element of $(\mathcal{P}(E) \setminus \mathcal{A}, <)$ and $\mathcal{B} = \mathcal{A} \cup \{X\}$. Then $E \in \mathcal{A}$ implies $E \in \mathcal{B}$. Taking $A_i \in \mathcal{B}\,(i \in I)$. If $X \notin \{A_i\}_{i \in I}$, then $\bigcap_{i \in I} A_i \in \mathcal{A}$ because of $(E, \mathcal{A}) \in IM(E)$ and Lemma 1. Further, $\bigcap_{i \in I} A_i \in \mathcal{B}$ holds. If $X \in \{A_i\}_{i \in I}$. Assuming $\mathcal{A} \not\ni \bigcap_{i \in I} A_i \neq X$. Then one has $X \supset \bigcap_{i \in I} A_i \in \mathcal{P}(E) \setminus \mathcal{A}$, a contradiction with the minimum of X in $(\mathcal{P}(E) \setminus \mathcal{A}, <)$. Thus there is the correct of $\bigcap_{i \in I} A_i \in \mathcal{A}$, or $\bigcap_{i \in I} A_i = X$. No matter which case happens, $\bigcap_{i \in I} A_i \in \mathcal{B}$ is right. Thus, $\mathcal{B} \in MF(E)$ is true. By Lemma 2, $X \neq \cap\{A \in \mathcal{A} | X \subseteq A\}$ is real.

Let $C_{\mathcal{B}} \in CO(E)$ be the closure operator on E corresponding to $\mathcal{B}$ using Lemma 2. Obviously, $C_{\mathcal{B}}$ satisfies (S1)-(S3).

Let $B \in \mathcal{B}, x, y \in E \setminus B$ and $y \in C_{\mathcal{B}}(B \cup x)$.
If $C_{\mathcal{B}}(B \cup x) \neq X$. Then $C_{\mathcal{B}}(B \cup x) \in \mathcal{A}$ is true. This follows $C_{\mathcal{B}}(B \cup x) = \sigma_{\mathcal{A}}(B \cup x)$ where $\sigma_{\mathcal{A}}$ is the closure operator of $(E, \mathcal{A})$. Using Lemma 1, $x \in \sigma_{\mathcal{A}}(B \cup y) \in \mathcal{A}$ holds. Therefore, it obtains $x \in \sigma_{\mathcal{A}}(B \cup y) = C_{\mathcal{B}}(B \cup y) \in \mathcal{B}$.
If $C_{\mathcal{B}}(B \cup x) = X$. Since $X = C_{\mathcal{B}}(B \cup x) = \cap\{A \in \mathcal{B} | B \cup x \subseteq A\} = \cap\{A \in \mathcal{A} | B \cup x \subseteq A\} \cap \{X\} = \sigma_{\mathcal{A}}(B \cup x) \cap \{X\} \subset \sigma_{\mathcal{A}}(B \cup x)$ is correct by $X \neq \cap\{A \in \mathcal{A} | X \subseteq A\}$. Additionally, $x, y \in X$, $B \subset B \cup x \subseteq C_{\mathcal{B}}(B \cup x)$ and $y \in C_{\mathcal{B}}(B \cup x)$ taken together induces $B \cup x, B \cup y \subseteq X$. Suppose $B \cup x \subset X$. Considering with $B \cup x \subset X \subset \sigma_{\mathcal{A}}(B \cup x)$, one has $B \cup x \in \mathcal{P}(E) \setminus \mathcal{A}$.

Thus $B \cup x \subset X$ follows a contradiction with the minimum of X in $(\mathcal{P}(E) \setminus \mathcal{A}, <)$. Hence it should get $B \cup x = X$. However $x, y \in E \setminus B$ and $B \cup y \subseteq C_{\mathcal{B}}(B \cup x) = X = B \cup x$ taken together implies $x = y$. Therefore it earns $x \in C_{\mathcal{B}}(B \cup y) = C_{\mathcal{B}}(B \cup x) = X$.

Summing up the above, it follows that (S4) holds for $C_{\mathcal{B}}$. Furthermore, considering Notice 1, we obtain $(E, \mathcal{B}) \in IM(E)$, and so $\mathcal{A} \lessdot \mathcal{B}$. □

Theorem 3 *Let $|E| < \infty$ and $\mathcal{A} \subseteq \mathcal{P}(E)$. Then $(E, \mathcal{A}) \in IM(E)$ implies $\mathcal{A} = \mathcal{P}(E)$ or $\mathcal{A} \lessdot \mathcal{B}$ for some $(E, \mathcal{B}) \in IM(E)$.*

Proof Based on Definition 2, $(\mathcal{P}(E), <)$ is well ordered. Considering with Lemma 5, the need is straightforward.□

Consequently, when $|E| < \infty$, there exists a chain $\mathcal{A} = \mathcal{A}_0 \lessdot \mathcal{A}_1 \lessdot \ldots \lessdot \mathcal{A}_n = \mathcal{P}(E)$ if $\mathcal{A} \in IM(E)$. Moreover, such a chain, if it exists, is unique for a fixed linear extension.

Therefore, according to the knowledge of graph theory and combinatorial algorithms Korte and Vygen [17], one knows that the diagrams of $(IM(E), \lessdot)$ is a spanning tree of the diagram of $(IM(E), \subset)$. Hence using the Depth-First Search algorithm shown in Korte and Vygen [17, pp.26-29]), one will find out all the members of $IM(E)$ because the diagram of $(IM(E), \lessdot)$ is a tree.

Comparing the two methods given here to search out all the finite matroids defined on the same finite set given by Mao [11], we see that the methods here are different from the way of Mao [11]. The methods here are much more direct perceptive because the method had combined some views and algorithms of graph theory.

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