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Solving a Paraboloidally Constrained Quadratic Programming Using Parabola-Quadratic Programming

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Abstract In this paper, we discuss the state-of-the-art models in estimating, evaluating, and selecting among non-linear mathematical models for obtaining the optimal solution of the optimization problems which involve the nonlinear functions in their constraints. We review theoretical and empirical issues including Newton's method, linear programming, quadratic programming, quadratically constrained programming, parabola, circle and the relation between parabola and circle. Finally, we outline our method called parabola-quadratic programming which is useful for solving economic forecasting and financial time-series with non-linear models.

Keywords Parabola; Circle; Quadratic; Optimization; Algorithm.

2010 Mathematics Subject Classification 90C30

1 Introduction

Optimization means selecting the best from a set of alternatives by using **optimization techniques** according to well defined objective criteria. Mathematical methods are used to choose the values of variables that give the maximum or minimum value of the objective function.

Optimization methods are extremely important for management and design. The methods by themselves do not guarantee that the optimal alternative will be selected. To ensure selection of the optimal alternative, it must be included in the set of available choice of methods. The development of a set of the best apparent practical alternatives is the crucial first step.

The objective function explains the essential characteristics of what is to be optimized. The function combines the essential descriptive quantitative variables. The limits of the values of variable for each alternative can be expressed as constraints on the range of values that may be used by an optimal alternative. The maximum or minimum criteria are chosen by the nature of the variables and objectives. As an example, costs are minimized, and profits are maximized.

The alternatives may be described by continuous or discrete variables. Many situations consist of choices between discrete courses of action. Discrete problems may be combinatorial and large problems may become intractable. Continuous variables may be required to describe situations that can produce a large number of alternatives by mixing various proportions. These types of problems are tractable if the variables can be described by continuous functions.

An optimization process selects values of independent variables which result in the maximum (or minimum) value of one or more dependent variable(s) of a value or measure of merit relationship. The relationship or objective function is usually written as $f(x)$, where f is a function of x.

The independent variables are usually subjected to a number of constraints in the form of other relationships and the whole group is often written as $g(x)$. Note that, in this context g may be an array of functions, equations or inequalities.

The symbol x is a vector or array of independent variables which describes the process. The optimum (or optima) are groups of values of x which satisfy the optimal conditions of the objective function f and the constraints g.

The most frequently used methods for searching the optimum value of a mathematical function are

(a) differential calculus

(b) search methods

(c) direct method

(d) mathematical (linear and nonlinear) programming

(e) classical matrix method

(f) calculus of variation

(g) Bellman's dynamic programming

(h) Pontryagin's maximum principle

Accounting systems tend to treat costs as fixed and variable, i.e. fixed costs are not sensitive to the quantity of activity while variables are directly related. These definitions lead to nonlinear unit costs and choosing an economic quantity by minimizing cost. Even if the objective is to maximize profits and revenues are described by Quantity multiplied by Price relationships, the costs remain non linear.

Quite often the essential problem can be made linear if the fixed costs essentially remain constant over the range of alternatives. The problem becomes one of optimizing the differences.

Another notion is choosing the cost items that are fixed and ones that tend to increase per unit. This is sometimes stated as Costs that vary inversely with quantity versus Costs that vary directly with quantity.

There is a function for estimating inventory, order and production quantity problems. The fixed cost is the amount to place an order, change tooling, bring in equipment for an operation, etc. which is unrelated to quantity. The variable costs are those of storage, interest on inventory, etc. which are related to the maximum amount held at any one. The function assumes that inventory is averaged and that production and consumption rates are uniform with working days. For problems that fit the restrictions of the linear programming (LP) model MAXIMIZE or MINIMIZE use the simplex method to solve the general LP problem.

In this paper, a new method so-called parabola-quadratic programming for solving the special type nonlinear programming problem will be proposed. However this new method can be extended or modified for solving the other types of problems. In order to understand how this new method is proposed, we orderly arranged all the materials in several sections as follows. In Section 2, we briefly provide an explanation about Newton's method to be used in this paper. Linear programming and quadratic programming will be described in Section 3 and Section 4 respectively. Section 5 contains one of the quadratic programming problem where its constraints consist of quadratic function and a set of linear system. In Section 6, we need to expose to the reader about the parabola in great detail, and this is very useful in solving the problem which involves the conics. We continue the explanation about the circle in Section 7. The relationship between parabola and circle is given in Section 8. Our new method will be explained in Section 9 and some numerical results will be displayed in Section 10 where its computation is done by using the algorithm given in Section 11. Conclusion given in Section 12 will end our paper.

2 Newton's Method

Newton's method (or **Newton–Raphson method**) [1] defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \dots), \tag{1}$$

is perhaps the best known method for finding successively better approximations to the zeros (or **roots**) of a **real**-valued **function**. Newton's method can often converge remarkably quickly, especially if the iteration (1) begins with x_0 by "sufficiently near" the desired root.

3 Linear Programming

A linear programming (LP) problem [2] is one in which the objective and all of the constraints are linear functions of the decision variables.

Since all linear functions are **convex**, linear programming problems are intrinsically easier to solve than general nonlinear (NLP) problems, which may be non-convex. In a **non-convex**, NLP there may be more than one feasible region and the optimal solution might be found at any point within any such region. In contrast, an LP has at most one feasible region with "flat faces" (i.e. no curves) on its outer surface, and the optimal solution will always be found at a vertex (corner point) on the surface where the constraints intersect.

LP problems are usually solved using the Simplex method which originally developed by Dantzig in 1948, and has been dramatically enhanced in the last decade, using advanced methods from numerical linear algebra. This has made it possible to solve LP problems with up to hundreds of thousands or millions of decision variables and constraints. An alternative to the Simplex method, called the Interior Point or Newton-Barrier method, was developed by Karmarkar in 1984. Also in the last decade, this method has been dramatically enhanced with advanced linear algebra methods so that it is often competitive with the Simplex method, especially on very large scale problems.

Primal and Dual Simplex Method

The standard Microsoft Excel Solver uses a basic implementation of the primal Simplex method to solve LP problems. It is limited to 200 decision variables.The Premium Solver uses an improved primal Simplex method with two-sided bounds on the variables. It handles up to 1,000 decision variables. The Premium Solver Platform uses an extended

LP/Quadratic version of this Simplex Solver to handle problems of up to 2,000 decision variables. It optionally uses a dual Simplex method to solve LP subproblems in a mixed-integer (MIP) problem. However, this Simplex algorithm does not exploit sparsity in the model.

The Large-Scale LP Solver for the Premium Solver Platform uses a state-of-the-art implementation of the primal and dual Simplex method, which fully exploits sparsity in the LP model to save time and memory. It uses advanced strategies for matrix updating and refactorization, multiple and partial pricing and pivoting, and overcoming degeneracy. This Solver engine is available in three versions, handling up to 8,000, 32,000, or an unlimited number of variables and constraints, subject to available time and memory.

The MOSEK Solver includes a state-of-the-art primal and dual Simplex method that also exploits sparsity and uses advanced strategies for matrix updating and refactorization. It handles problems of unlimited size, and has been tested on linear programming problems of over a million decision variables.

The XPRESS Solver Engine uses a highly tuned, state-of-the-art implementation of the primal and dual Simplex method, with its own advanced strategies for matrix updating and refactorization, multiple and partial pricing and pivoting, and overcoming degeneracy. Its dual Simplex method is probably the best in the world. The XPRESS Solver engine can handle an unlimited number of variables and constraints, subject to available time and memory.

4 Quadratic Programming

A quadratic programming (QP) is the problem [3] of optimizing (minimizing or maximizing) a quadratic function of the decision variables, and subject to constraints which are all linear functions of the variables.

A widely used QP problem is the Markowitz mean-variance portfolio optimization problem, where the quadratic objective is the portfolio variance (sum of the variances and covariances of individual securities), and the linear constraints specify a lower bound for portfolio return.

If $x \in R^n$, the $n \times n$ matrix Q is symmetric, and c is any $n \times 1$ vector then QP is the problem which minimize

$$f(x) = \frac{1}{2}x^TQx + c^Tx$$

subject to

$$Ax \leq b, \text{ and } Ex = d,$$

where the superscript "T" indicates the vector transpose.

QP problems, like LP problems, have only one feasible region with "flat faces" on its surface (due to the linear constraints), but the optimal solution may be found anywhere within the region or on its surface. The quadratic objective function may be **convex** which makes the problem easy to solve or **non-convex**, which makes it very difficult to solve.

If Q is a **positive semidefinite matrix,** then $f(x)$ is a **convex function** [4, 5]. In this case the quadratic program has a global minimizer if there exists at least one vector x satisfying the constraints and $f(x)$ is bounded below on the feasible region. If the matrix Q is **positive definite matrix,** then this global minimizer is unique. Portfolio optimization problems are usually of this type. If Q is zero, then the problem becomes a **linear program**.

From optimization theory, a necessary condition for a point x to be a global minimizer is for it to satisfy the **Karush-Kuhn-Tucker** (KKT) conditions. The KKT conditions are also sufficient when $f(x)$ is convex.

If there are only equality constraints, then the QP can be solved by a **linear system**. Otherwise, a variety of methods for solving the QP are commonly used, including **interior point**, **active set**, **exploration**, and **conjugate gradient** methods.

Convex quadratic programming is a special case of the more general field of **convex optimization**.

Dual Programming

The dual of a QP is also a QP. To see that let us focus on the case where $c = 0$ and Q is positive definite. We write the **Lagrangian**

$$L(x, \lambda) = \frac{1}{2} x^T Q x + \lambda^T (Ax - b).$$

To calculate the dual function $g(\lambda)$, defined as

$$g(\lambda) = \inf \lim_x L(x, \lambda),$$

we find the **infimum** of L, using

$$\nabla_x L(x, \lambda) = 0, \quad x* = -Q^{-1} A^T \lambda,$$

the dual function is

$$g(\lambda) = -\frac{1}{2} \lambda^T A Q^{-1} A^T \lambda - b^T \lambda$$

hence the dual of the QP is to maximize

$$-\frac{1}{2} \lambda^T A Q^{-1} A^T \lambda - b^T \lambda$$

subject to

$$\lambda \geq 0.$$

Complexity

For **positive definite** Q, the **ellipsoid method** solves the problem in **polynomial time**. If, on the other hand, Q is **negative definite**, then the problem is **NP-hard** [5, 6]. In fact, even if Q has only one negative **eigenvalue**, the problem is **NP-hard** [5,7]. If the objective function is purely quadratic, negative semidefinite and has fixed rank, then the problem can be solved in **polynomial time** [8].

5 Quadratically Constrained Quadratic Programming

In mathematics, a quadratically constrained quadratic programming (QCQP) is the problem of **optimizing** a quadratic objective function of the decision variables, and subject to constraints which are quadratic and linear functions of the variables [9]. The problem is to minimize

$$\frac{1}{2}x^T P_0 x + q_0^T x$$

subject to

$$x^T P_i x + q_i^T x + r_i \leq 0 \text{ for } i = 1, ..., m,$$

$$Ax = b,$$

where P_0, $\ldots P_n$ are $n \times n$ matrices and $x \in R^n$ is the optimization variable. If P_1, $\ldots P_n$ are all zero, then the constraints are in fact linear and the problem is a **quadratic programming**.

Hardness

Solving the general case is an **NP-hard** problem. To see this, note that the two constraints $x_1(x_1 \quad - 1) \leq 0$ and $x_1(x_1 - 1) \geq 0$ are equivalent to the constraint $x_1(x_1 - 1) = 0$, which is in turn equivalent to the constraint $x_1 \in \{0, 1\}$. Hence, any **0-1 integer programming** (in which all variables have to be either 0 or 1) can be formulated as a quadratically constrained quadratic programming. But 0–1 integer programming is NP-hard, so QCQP is also NP-hard.

Example 1 **Max Cut** is a problem in graph theory, which is NP-hard. Given a graph, the problem is to divide the vertices in two sets, so that as many edges as possible go from one set to the other. Max Cut can be easily formulated as a QCQP, which allows obtaining good lower bounds using SDP realization of the dual.

Special Cases

There are two main relaxations of QCQP: using **semidefinite programming** (SDP), and using the reformulation-linearization technique (RLT).

Semidefinite Programming

When P_0, $\ldots P_n$ are all **positive-definite matrices**, the problem is **convex** and can be readily solved using **exploration method**, as done with **semidefinite programming**.

6 Parabola

A parabola [10–15] is the set of all points in the **plane** equidistant from a given **line** L (the **conic section directrix**) and a given point F(**the focus)** not on the line as shown in Figure 1 and Figure 2. The **focal parameter** (that is, the distance between the directrix and focus) is therefore given by $p = 2a$, where a is the distance from the vertex to the directrix or focus. The **surface of revolution** obtained by rotating a parabola about its axis of symmetry is called a **paraboloid**. Gregory and Newton considered the **catacaustic** properties of a parabola that bring parallel rays of light to a focus (MacTutor Archive), as illustrated in Figure 3.

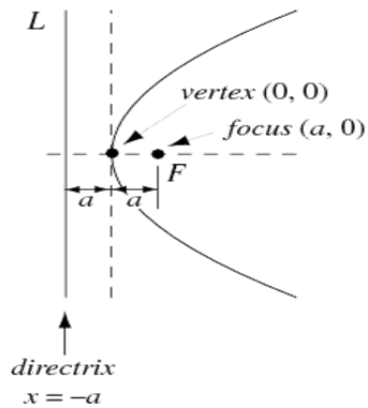

Figure 1: Dimension

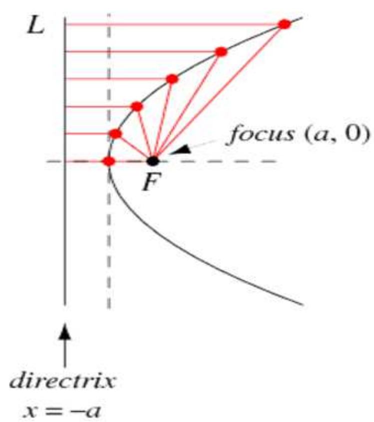

Figure 2: Set of Points

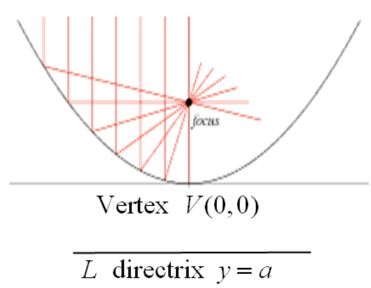

Figure 3: Upward Parabola

For a parabola opening to the upward (u-parabola) with vertex at (0,0), the equation in Cartesian coordinates is

$$\sqrt{(y-a)^2+x^2} = y+a \quad \Rightarrow \quad x^2 = 4ay \quad \Rightarrow y = \frac{1}{4a}x^2.$$

The quantity $4a$ is known as the **latus rectum**. If the vertex is at (x_0, y_0) instead of (0,0), the equation of the u-parabola is $(x-x_0)^2 = 4a(y-y_0)$.

By putting $k = 1/4a$, the equation of the u-parabola is given by

$$(y-y_0) = k(x-x_0)^2, \ \ (k > 0).$$

Three points uniquely determine one parabola with directrix parallel to the x-axis and one with directrix parallel to the y-axis as seen in Figure 6. If these parabolas pass through the three points (see Figure 6) (x_1, y_1), (x_2, y_2), and (x_3, y_3), they are given by equations

$$\begin{vmatrix} x^2 & x & y & 1 \\ x_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} y^2 & x & y & 1 \\ y_1^2 & x_1 & y_1 & 1 \\ y_2^2 & x_2 & y_2 & 1 \\ y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

In **polar coordinates** as shown in Figure 4 and Figure 6, the equation of a parabola with parameter a and center (0,0) is given by

$$r = -\frac{2a}{1+\cos\theta}.$$

The equivalence with the Cartesian form can be seen by setting up a coordinate system $(x_s, y_s) = (x, y-a)$ and plugging in $r = \sqrt{x_s^2 + y_s^2}$ and $\theta = \tan^{-1}(x_s/y_s)$ to obtain

$$y = \frac{1}{4a}x^2,$$

which is the parabola equation. The parabola can be written parametrically as

$$y = at^2, \quad x = 2at \ \text{ or } \ y = \frac{t^2}{4a}, \ x = t.$$

A parabola may be generated as the envelope of two concurrent line segments by connecting opposite points on the two lines [16] as drawn in Figure 7 and Figure 8.

In the following discussion, we will consider the u-parabola defined by $y = kx^2, (k > 0)$. Due to their simplicity, we do not provide any proof for the theorems given.

Theorem 1 (Figure 9)

The ***tangent*** *to the graph of* $y = kx^2$ *at the point* $P_1(x_1, kx_1^2)$ *has the equation*

$$y - kx_1^2 = 2kx_1(x-x_1), \ \ \textit{or} \ \ y = kx_1(2x - x_1)$$

which intersects the y-axis, the x-axis, and the directrix at the points

$$A(0, -kx_1^2), \quad B(\frac{x_1}{2}, 0), \ \textit{and} \ C\left(\frac{x_1}{2} - \frac{1}{8k^2x_1}, \frac{-1}{4k}\right)$$

respectively, provided for the last point, $x_1 \neq 0$.

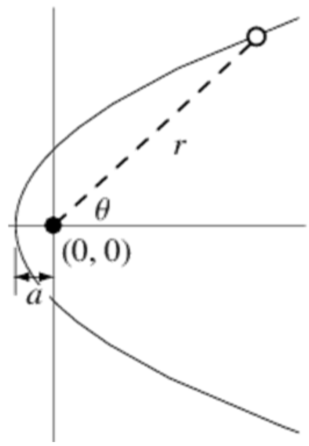

Figure 4: Polar Coordinates

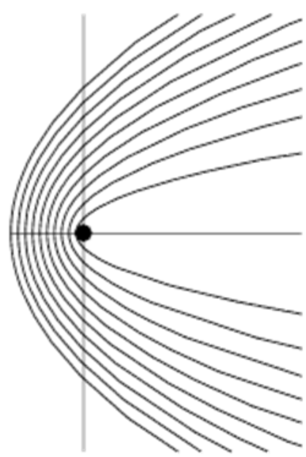

Figure 5: Concentric Parabolas

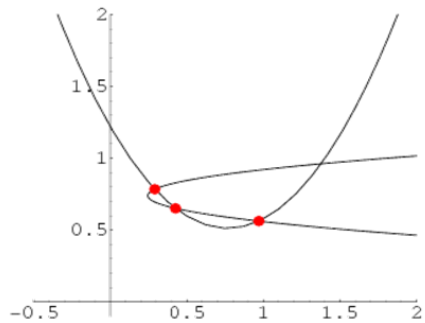

Figure 6: Uniquely Determined

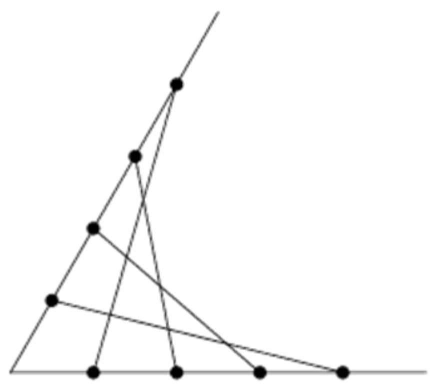

Figure 7: Envelope Parabola

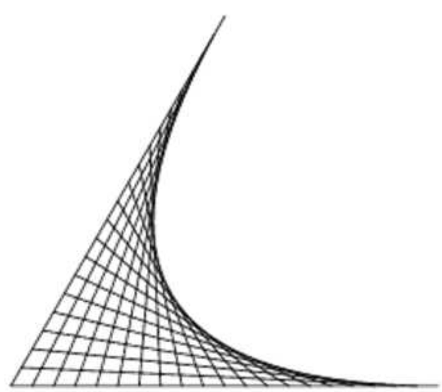

Figure 8: Envelope Parabola

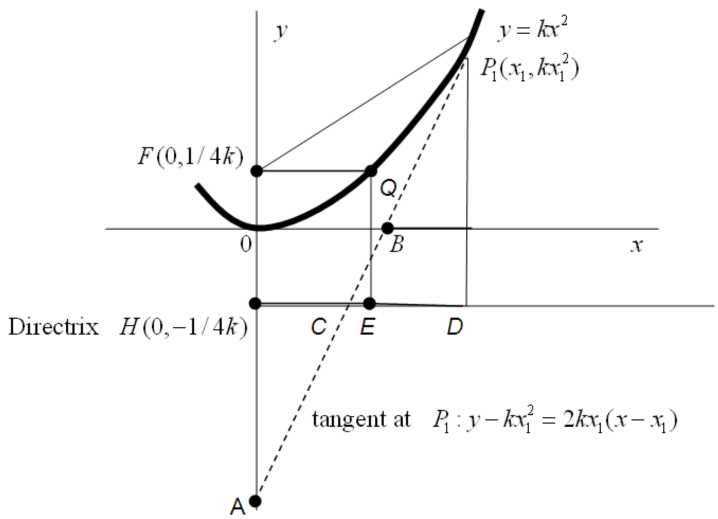

Figure 9: Property of Tangent to the Parabola

Theorem 2 (Figure 10)

The ***normal*** *to the graph of* $y = kx^2$ *at the point* $P_1(x_1, kx_1^2)$ *when* $x_1 \neq 0$, *has the equation*

$$y - kx_1^2 = -\frac{1}{2kx_1}(x - x_1) \quad (x_1 \neq 0)$$

which intersects the y-axis, the x-axis, and the directrix at the points

$$A^*(0, \frac{1}{2k} + kx_1^2), \quad B^*(x_1 + 2k^2x_1^3, 0), \quad and \quad C^*\left(\frac{3x_1}{2} + 2k^2x_1^3, -\frac{1}{4k}\right)$$

respectively.

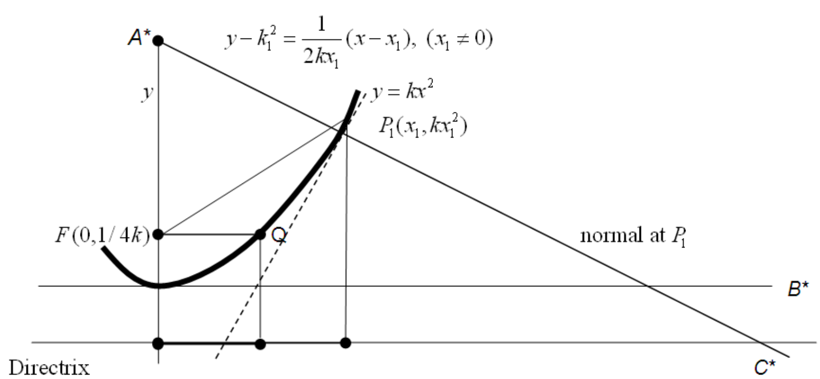

Figure 10: Property of Normal to the Parabola

Theorem 3 (Figure 11)

The coordinates of the intersection R_1 of the y-axis and the line through two points $P_1\left(x_1, y_1\right)$ and $P_2\left(x_2, y_2\right)$ which lie on $y = kx^2$ can be put in the form $R_1\left(0, -kx_1x_2\right)$.

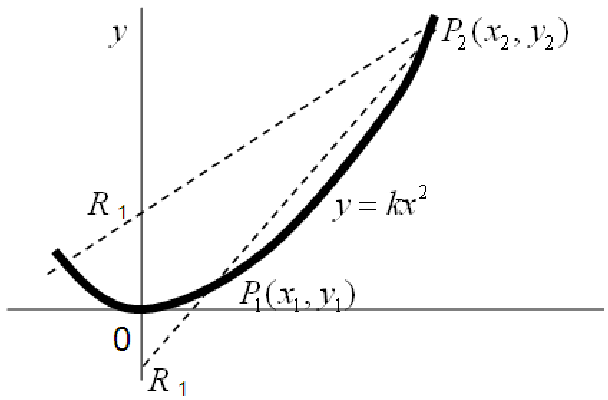

Figure 11: Intersection of Segment Line with y-axis

Theorem 4 (Figure 12)

The coordinates of the intersection Q of the tangents to the parabola $y = kx^2$ at two points $P_1\left(x_1, y_1\right)$ and $P_2\left(x_2, y_2\right)$ can be put in the form $Q((x_1 + x_2)/2, kx_1x_2)$.

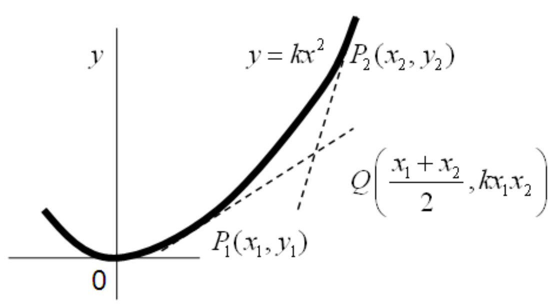

Figure 12: Intersection of Two Tangents of the Parabola

Theorem 5 (Figure 13)

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie on $y = kx^2$. Let Q be the intersection of the tangents at P_1 and P_2. Let R be the intersection of the line P_1P_2 and the axis of the parabola. Then P, the mid-point of the segment QR lies on the line tangent to the parabola at the vertex.

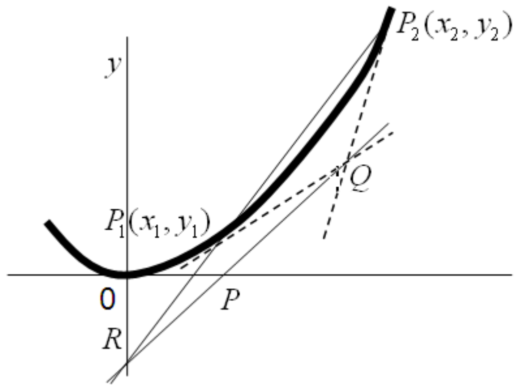

Figure 13: Intersection between Vertex Line, Tangent and Line Segment of the Parabola

Theorem 6 (Figure 14)

Let W be the intersection of the tangents to the parabola $y = kx^2$ at $P_1(x_1, y_1)$ which is not the vertex V(0,0). Then the line from the focus F to W is perpendicular to the line WP_1.

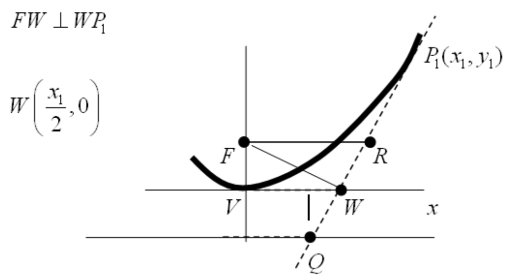

Figure 14: Normal to the Parabola Passes the Focus of Parabola

Theorem 7 (Figure 15)

Let $P_1(x_1, y_1)$ be a point on a parabola $y = kx^2$ which is not the vertex V. Then the tangent to the parabola at P_1 meets the directrix and the line through the focus F parallel to the directrix at two points Q and R respectively that are equidistant from F.

Theorem 8 (Figure 16)

Let $x_1 > 0$. Then (a) the line through $P_1(x_1, kx_1^2)$ parallel to the parabola axis intersects the directrix at the point $D_1(x_1, -1/4k)$, (b) the tangent to the parabola at $P_1(x_1, kx_1^2)$ intersects the parabola axis at the point $Q_1(0, -kx_1^2)$, (c) the quadrilateral $Q_1D_1P_1F$ is a rhombus (an equilateral parallelogram), and (d) the diagonals of this rhombus are perpendicular to each other at point $(x_1/2, 0)$.

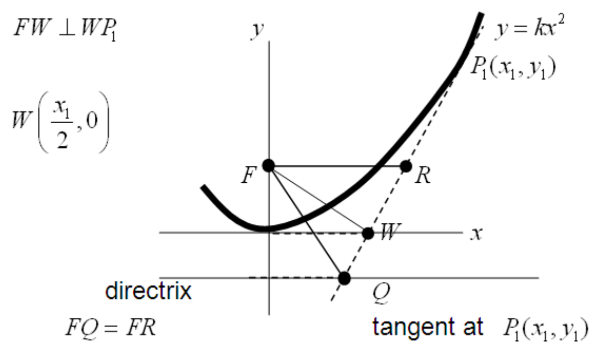

Figure 15: Property of Normal, Tangent and Directrix of the Parabola

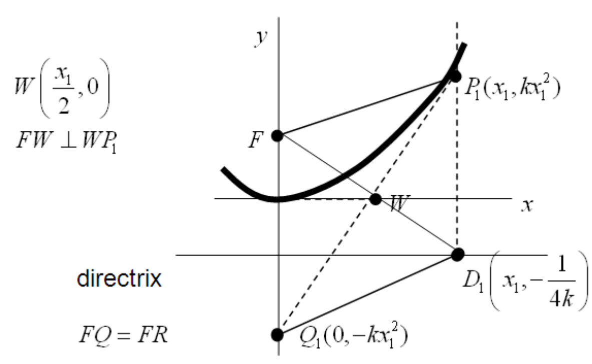

Figure 16: Rhombus

Theorem 9 (Figure 17)

Let define a focal chord of a parabola be a line segment which contains the focus and has its ends at points on the parabola. Supposing that $x_2 < 0 < x_1$. Then the two points $P_1(x_1, kx_1^2)$ and $P_2(x_2, kx_2^2)$ on the graph of $y = kx^2$ are end points of a focal chord if and only if $(2kx_1)(2kx_2) = -1$ and hence if and only if the tangents to the parabola at $P_1(x_1, kx_1^2)$ and $P_2(x_2, kx_2^2)$ are perpendicular.

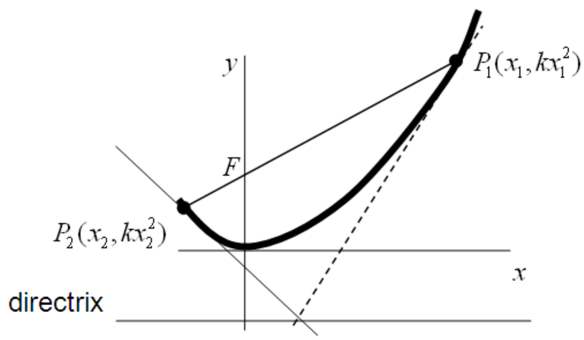

Figure 17: Focal Line and Two Tangents of the Parabola

Theorem 10 *Two different tangents to a parabola intersect on the directrix if and only if the tangents are perpendicular and hence if and only if the points of tangency are ends of a focal chord.*

7 Circle

Concept

This section is about the shape and mathematical concept of circle as shown in Figure 18, Figure 19 and Figure 20 ([17]).

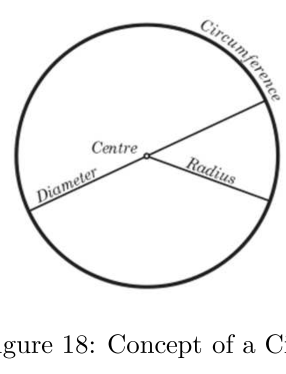

Figure 18: Concept of a Circle

Circles are simple **shapes** of **Euclidean geometry** consisting of those **points** in a **plane** which are at a constant **distance**, called the **radius**, from a fixed point, called the **centre**. A **chord** of a circle is a line segment whose both endpoints lie on the circle. A **diameter** is a chord passing through the center. The length of a diameter is twice the radius. A diameter is the largest chord in a circle. Circles are simple closed **curves** which divide the plane into an interior and an exterior. The **circumference** of a circle is the perimeter of the circle, and the interior of the circle is called a **disk**. An **arc** is any **connected** part of a circle. A circle is a special **ellipse** in which the two **foci** are coincident. Circles are **conic sections** attained when a **right circular cone** is intersected with a plane perpendicular to the axis of the cone.

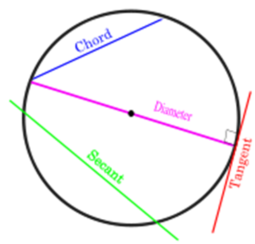

Figure 19: Definition of Tangent

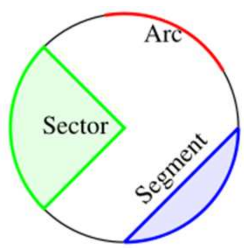

Figure 20: Definition of Segment

8 Parabola-Circle Relationship

Theorem 11 *Let $k > 0$ and $a > 0$.*

(i) *if f(a) is the y coordinate of the centre of the circle tangent to the graph of $y = kx^2$ at the points for which $x = a$ and $x = -a$, then*

$$f(a) = ka^2 + \frac{1}{2k},$$

(ii) *if (g(a),h(a)) is the centre of the circle which is tangent to the graph of $y = kx^2$ at the point (a, ka^2) and which contains (or passes through) the origin, then*

$$g(a) = -k^2a^3 \text{ and } h(a) = \frac{3}{2}ka^2 + \frac{1}{2k}.$$

Theorem 12

Let several circles which have centres on the positive y-axis and be tangent to the x-axis at the origin. Suppose that for k >0 we have a parabola having the equation $y = kx^2$. *The circle with centre at (0,a) and radius a intersects the parabola only at the origin if and only if* $a \leq 1/2k$.

By Theorem 12, the biggest one of those circles which lies completely on or inside the parabola has a radius equal to the distance from the focus to the directrix of the parabola.

We can locate the set S which contains a point $P(x, y)$ if and only if the point is equidistant from the x axis and the circle with centre at the origin and radius a. We can observe that S contains some points inside the circle as well as some points on and some points outside the circle. Whether a point $P(x, y)$ lies inside or on or outside the circle, it will be in the set S if and only if

$$\left|\sqrt{x^2 + y^2} - a\right| = |y| \tag{2}$$

and hence if

$$\sqrt{x^2 + y^2} = a \pm y. \tag{3}$$

If (3) holds, then

$$x^2 + y^2 = a^2 \pm 2ay + y^2 \tag{4}$$

and hence either

$$y = -\frac{a}{2} + \frac{x^2}{2a} \tag{5}$$

or

$$y = \frac{a}{2} - \frac{x^2}{2a}. \tag{6}$$

It can be shown that if (5) or (6) holds, then (2) holds. It follows that S is the sum (or union) of two parabolas of which one has the equation (5) and the other has the equation (6). Each parabola has its focus at the origin, and the directrices are the tangents to the circle that are parallel to the x-axis. The parabolas intertsect the x-axis where the circle does.

9 Parabola-Quadratic Programming

In this paper, we would like to consider a programming so-called parabola-quadratic programming (**PQP**) which minimize

$$(x - p)^2 + (y - q)^2$$

subject to

$$y - \left(\alpha x^2 + \beta x + \gamma\right) \geq 0,$$

where $x, y, \alpha, \beta, \gamma \in R$ and the optimization variables x and y are nonnegative. If α is zero, then the constraints are in fact linear and the problem is a **quadratic programming**. Figure 21, Figure 22 and Figure 23 show some of the configurations of PQP.

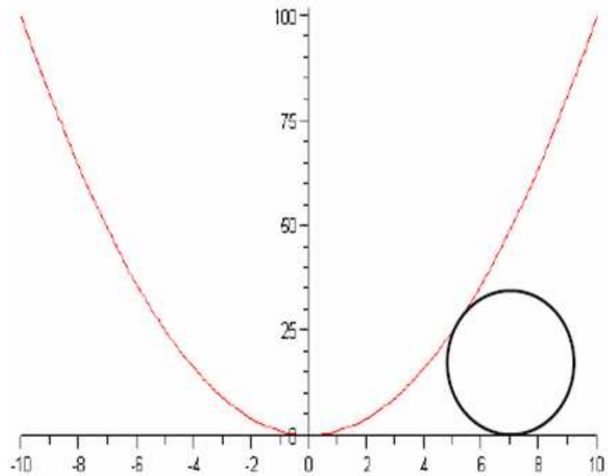

Figure 21: Outside Parabola

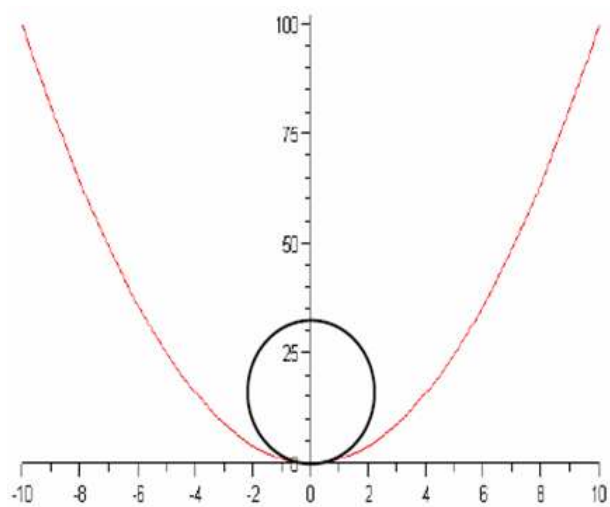

Figure 22: Inside Parabola

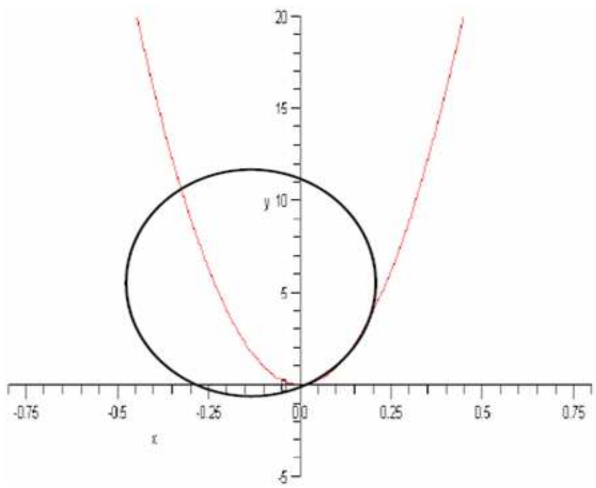

Figure 23: Intersection

10 Solution of the PQP Problem

In this section, we provide some examples how to solve the PQP problem.

Example 2

Suppose that we want to minimize

$$(x-p)^2+(y-q)^2$$

subject to

$$y-x^2 \geq 0 \quad \text{and} \quad x, y \geq 0.$$

Any point on the parabola $y = x^2$ for $x \geq 0$ has a chance to be a solution and this will depend on p and q. If $(0.5, 0.25)$ is its solution, then by using

$$(0.5-p)^2+(0.25-q)^2=q^2$$

and the tangent at $(0.5, 0.25)$, we will obtain

$$(p, q)=\left(\left(1+\sqrt{2}\right)/4, \left(2-\sqrt{2}\right)/4\right)$$

the centre of the circle. By further computation, we definitely have the line

$$y=\left(\sqrt{2}-1\right)x-\frac{\sqrt{2}-1}{4}$$

which passes the point (p, q) and the intersection point of the tangent at $(0.5, 0.25)$and x-axis.

Example 3

Analogous to Example 2, the solution of the general parabola-quadratic programming (PQP) which minimizes

$$(x-p)^2 + (y-q)^2$$

subject to

$$y - kx^2 \geq 0 \quad (k > 0) \text{ and } x, y \geq 0,$$

is given by $\left(x_1, kx_1^2\right)$ if and only if

$$p = \frac{x_1}{2}\left(1 + \sqrt{1 + 4k^2x_1^2}\right) \text{ and } q = kx_1^2 + \frac{1}{4k}\left(1 - \sqrt{1 + 4k^2x_1^2}\right).$$

Furthermore, the line which passes the centre of the circle and the intersection point of x-axis and the tangent at $\left(x_1, kx_1^2\right)$, is given by

$$y = \frac{1}{2kx_1}\left(\sqrt{1 + 4k^2x_1^2} - 1\right)\left(x - \frac{x_1}{2}\right).$$

Example 4 The solution of the general parabola-quadratic programming (PQP) which minimizes

$$(x-p)^2 + (y-q)^2$$

subject to

$$y - kx^2 \geq 0 \quad (k > 0) \text{ and } x, y \geq 0$$

without knowing the line containing the centre of the objective function, can be obtained by solving the equation

$$2k^2\xi^3 - (2kq - 1)\,\xi - p = 0 \tag{7}$$

using Newton's method where the equation (7) is derived through pythagoras property among the points $\left(\xi, k\xi^2\right)$, (p, q), and $(\xi/2, 0)$.

11 Algorithm of the PQP Problem

Suppose that we would like to minimize

$$(x-p)^2 + (y-q)^2$$

subject to standard form

$$y - kx^2 \geq 0, \quad (k > 0) \quad \text{and} \quad x, y \geq 0.$$

The brief algorithm for solving the PQP problem is as follows.

Algorithm PQP

Data : $p, q, k, \max \in \mathrm{R}$. $f(\xi) = 2k^2\xi^3 - (2kq-1)\xi - p$

1. $i = 0$
2. while $i < \max$ do

    2.1. $f\xi i = f(\xi_i)$

    2.2. $fd\xi i = f'(\xi_i)$

    2.3. $\xi_{i+1} = \xi_i - \frac{f\xi i}{fd\xi i}$

    2.4. if ξ_{i+1} follows the Newton stopping criterium

        then

        2.4.1. if ξ_i satisfies $p = \frac{\xi_i}{2}\left(1 + \sqrt{1+4k^2\xi_i^2}\right)$ and $q = k\xi_i^2 + \frac{1}{4k}\left(1 - \sqrt{1+4k^2\xi_i^2}\right)$ do

            2.4.1.1. stop

        else

        2.4.2. $i = i + 1$

    3. return.

12 Conclusion

We have shown that both parabola and circle or more generally quadratic have some relationship features which can be exploited for obtaining the solution(s) of the economic problems of the parabola-quadratic programming.

Although we can find this relationship precisely, we still use the approximated method (in this paper Newton's method) to obtain the solution, and therefore in order to obtain more precise result we need to seek the best criterium for stopping the routine in Newton's method.

Our method can be extended to the problem with more than one constraint and we prefer to explain in another paper.

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