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Interpretation of Different Approaches to Sensitivity Analysis in Cash Flow Problem

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Abstract Cash is the driving power of all business and cash-flow statement is one of the major issues of institutions, especially in crisis. Optimal cash-flow plan of a company could be one of the most important indicators of that business's financial health and can be considered as its financial analysts' ability and skill. Linear optimization (LO) is one of the mathematical tools in modeling the cash-flow problem and its rich literature helps analysts to device the optimal one when the situation satisfies the requirements of the LO model. However, the situation is always due to variation and the optimal solution arisen from the LO model have to be analyzed according to measurable variation of input data. Sensitivity analysis and parametric programming is the tool to this analysis. Using the Simplex method to find a basic optimal solution and having multiple optimal solutions is one of the reasons that different solvers lead to different optimal solutions. In these situations, sensitivity analysis may produce confusing results. Moreover, there are different points of views to sensitivity analysis such as optimal basis invariancy, optimal partition invariancy, support set invariancy to name some examples. Here, we briefly review different approaches to sensitivity analysis in LO and a short term cash-flow problem of a dummy institution is modeled as an LO problem. It is shown that the problem has multiple optimal solutions which are degenerate, the situation that usually occurs in practice and causes of ambiguous and unclear results. The confusing results in analyzing the sensitivity of these solutions are highlighted in this example. Then, a strictly complementary optimal solution is provided and its useful interpretation in sensitivity analysis is mentioned in a nutshell. In the sequel, the concept of the results arising in different points of views to sensitivity analysis is analyzed.

Keywords Cashflow Problem, Linear Optimization, Sensitivity Analysis.

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1 Introduction

Optimization plays a major role in finance nowadays. Appropriate decision making is of the most important criterion in surviving of all the finance and economic institutes and industrial organizations specially in crises.

Many problems in quantitative finance and risk management such as asset allocation, derivative pricing, value at risk modeling and model fitting, are now efficiently solved using state-of-the-art optimization techniques. Two of the closely related financial problems are the cash-flow and the asset pricing. In this study, we consider the cash-flow problem [1] and review different models, emphasizing the rich in theory and mature in methodology and implementation, the LO Model. Sensitivity analysis of obtained solutions based on ever-changing atmosphere of the world is not an option but is an obligation. Different aspects of sensitivity analysis and parametric programming have divert interpretations. Having rational understanding of the results leads to choose right practical and sustainable financial policies.

In this study, we consider different aspects of sensitivity analysis and parametric programming applied to a cash-flow problem that is formulated as an LO problem. The paper goes as follows. The next section introduce the cash-flow problem in a nutshell and review different possible mathematical models. In Section 3, LO problem is mentioned and the concern turns to different aspects of sensitivity analysis and parametric programming in LO. The LO model of cash-flow problem is devised in Section 4, and privileges and drawbacks of the LO model are mentioned then. Interpretation of sensitivity analysis in this model is the main aim Section 5. Concluding remarks might help the researchers for further studies.

2 Cash Flow Problem and Optimization Models

In its simplest form, owing the actual cash in and out of the business, and being identified both their sources and uses to recognize cash-flow variation over a period is the subject of cash-flow analysis. Cash management, i.e. controlling the cash-flow, is vital to businesses of all sizes. Small businesses are especially vulnerable to cash-flow problems since they tend to operate with inadequate cash reserves or none at all, and worse, tend to miss the implications of a negative cash-flow until it's too late.

Cash-flow analysis could be described in several steps, which allow one to model it as an optimization problem. First list cash inflows (sources). The sources of cash may be limited to: new investment, new debt, sale of fixed assets and operating profits. Then, record cash outflows (uses) and identify when (by date) cash flows in or out. Timing is the next step; that is, cash inflows minus cash outflows. One of the main task is to identify the major consequences of cash as it currently flows and indicate the constraints; inflows or outflows, which cannot be changed. Considering these steps could lead to an appropriate optimization model. Finally establishing a plan for positive cash-flow is the goal of the problem. By solving the problem and doing sensitivity analysis, the inflows and outflows, which can be changed (rescheduled) without considerably influencing the optimality of the plan can be recognized. Observe that positive cash-flow can be considered as a measure of a company's financial health and much positive cash-flow the better.

There are some assumptions inherently accompanied with LO and its success in modelling refers to how closely relatively matches up with these assumptions. **Linearity** of the objective function and constraints are the two important ones. The other is **proportionality** assumption, which means that the contribution to the objective of any decision variable is proportional to the value of the decision variable. Similarly, the contribution of each variable to the left-hand side of each constraint is proportional to the value of the variable. Moreover, the **additivity** assumption asserts that the contribution of a variable to the objective and constraints is independent of the values of the other variables. The other assumption is the **divisibility** assumption, meaning that taking any fraction of any variable is permitted. The final assumption is the **certainty** assumption. In LO, no uncertainty is permitted on the input parameters.

It is obvious that for a cash-flow problem, satisfying all these assumption may not happen simultaneously. Losing the additivity or proportionality assumptions leads to a nonlinear programming model. If the divisibility assumption does not hold, then a technique called integer programming rather than LO is required. This technique takes orders of magnitude more time to find solutions but may be necessary to create realistic solutions. Problems with uncertain parameters can be addressed using stochastic programming or robust optimization approaches.

Though these weak points tempts one to avoid using the LO model, however the rich theory, existence of many efficient solvers and over all these, possibility of sensitivity analysis may persuade us to use it efficiently at least for a short term cash-flow problem. In this paper, we consider an example devised in [1], to describe different points of view to sensitivity analysis on this problem.

3 LO and Different Aspects of Sensitivity Analysis

Let us consider the LO problem in standard form as $\min\left\{c^T x \middle| Ax = b, x \geq 0\right\}$, where A is a matrix of dimension $m \times n$ and c, x, b are of appropriate dimensions. Its dual can be defined as $\max\left\{b^T y \middle| A^T y + s = c, s \geq 0\right\}$. Having feasible solution of primal and dual problems, existence of optimal solutions is guaranteed. A primal-dual solution is denoted by $\left(x^*, y^*, s^*\right)$ that satisfies the complementarity property $\left(x^*\right)^T s^* = 0$. When the given optimal solution is basic, the index set $\{1,2,...,n\}$ is partitioned to two sets $\pi_B = (B, N)$ where B and N are corresponding to the index set of the given basic and non-basic variables, respectively. This partition is referred to as **basis optimal partition**. Observe that degeneracy exists in almost every LO problem and when the problem has multiple (basic) optimal solutions, the basic optimal partition is not unique.

For a vector $x \geq 0$, the support set of x is denoted by $\sigma(x) = \{j | x_j > 0\}$. If we are given an arbitrary primal optimal solution x^*, there is another partition identified. For a given primal optimal solution x^*, let a partition be defined and denoted by $\pi_P = (\sigma, \varsigma)$, where $\sigma = \sigma(x^*)$ and $\varsigma = \{1,2,...,n\} \setminus \sigma(x^*)$ and be referred to as **primal support set partition**. Analogous partition could be defined when a dual optimal solution (y^*, s^*) is given and $\varsigma = \sigma(s^*) = \{j | s_j > 0\}$. In this case $\sigma = \{1,2,...,n\} \setminus \varsigma(s^*)$. This partition might be dented by $\pi_D = (\sigma, \varsigma)$ and referred to as **dual support set partition**. Observe that these two recent partitions are different with each other when the primal and dual optimal solutions are degenerate simultaneously. In this case, it is different from the basic optimal solution as well. Analogous to the basic optimal partition, these partitions are depends to the given optimal solutions and, consequently are not unique.

On the other hand, the primal-dual optimal solution (x^*, y^*, s^*) may satisfy the strictly complementary property, when $x^* + s^* > 0$. By the Goldman-Tucker Theorem [6], the existence of strictly complementary optimal solutions is guaranteed if the primal and dual problems are feasible. This leads to a partition of the index set $\{1,2,...,n\}$ into two sets $\mathcal{B} = \{i | x_i^* > 0,\}$ and $\mathcal{N} = \{i | s_i^* > 0\}$ for an arbitrary primal-dual optimal solution (x^*, y^*, s^*). Observe that the index i belongs to $\mathcal{B}$, when the corresponding variable x_i is positive in an optimal solution. Analogously, i is included in $\mathcal{N}$, when the corresponding dual slack variable s_i is positive in a dual optimal solution. Identifying this partition is possible if the problem is solved by an interior point method [12]. This partition is simply referred to as optimal partition and denoted as $\pi = (\mathcal{B}, \mathcal{N})$. Unlike other partitions, this partition is unique because of the convexity of the optimal solution sets. All the aforementioned partitions are identical with the optimal partition when the given primal and dual optimal solutions are non-degenerate.

There are different points of view towards sensitivity analysis and parametric programming depending on the type of the in-hand optimal solution. Let us consider the perturbed LO problem as $\min\{(c + \alpha\Delta c)^T x | Ax = b + \beta\Delta b\}$, for arbitrary perturbation vectors Δc and Δb of appropriate dimension and real parameters α, β. We are interested to find the region for these parameters where special characteristics holds for a current optimal solution. There are different intervals corresponding and depending the given optimal solution.

Here we review some of them.

- We may have a (degenerate or non-degenerate) primal basic optimal solution. Finding the region for parameters where the associate basis optimal partition remains optimal. As mentioned above, having different basic optimal solutions leads to different confusing intervals (e.g. [12]). A good reference to find details in this point of view for a uni-parametric case is [11].
- A strictly complementary optimal solution is in hand that leads to identify the optimal partition π. Finding the region where this partition remains invariant is the aim of this case. Recall that this interval is independent of the type of primal and dual optimal solutions. Finding this interval only needs to solve the following two problems:

$$\epsilon_l = \min\{\epsilon: Ax - \epsilon\Delta b = b, x_{\mathcal{B}} \geq 0, x_{\mathcal{N}} = 0\},$$
$$\epsilon_u = \max\{\epsilon: Ax - \epsilon\Delta b = b, x_{\mathcal{B}} \geq 0, x_{\mathcal{N}} = 0\},$$

where $\pi = (\mathcal{B}, \mathcal{N})$ is the known optimal partition. A good reference for a uni-parametric case could be [12].

- The goal is to identify the region for parameters, where only positive variables of the given optimal solution remains positive after any change on the parameters in this region. When we are interested to the primal optimal solution, identifying the invariancy interval of the primal support set partition π_P is intended [5] and the corresponding interval (ϵ_l, ϵ_u) can be identified by the following two axillary problems:

$$\epsilon_l = \min\{\epsilon: A_P x_P - \epsilon\Delta b = b, x_P \geq 0\}$$
$$\epsilon_u = \max\{\epsilon: A_P x_P - \epsilon\Delta b = b, x_P \geq 0\}$$

where $P = \sigma(x)$ and x is the given optimal solution.

When the dual optimal solution is of the interest, finding the invariancy interval of the support set partition π_D is the aim. For more details in this case, we refer the interested reader to [4].

We restate that these intervals are identical when the problem has unique non-degenerate primal-dual optimal solution. However, they are different when the solution is not a basic one, or when it is a degenerate basic optimal solution. Ignoring this fact may lead to confusing result (e.g. [7]). In this study, we highlight this misunderstanding in cash-flow problem, and to keep it easy to track, we only consider the uni-parametric case, when either α or β is nonzero.

4 Linear Model for Cash-Flow Problem

To illustrate the main idea of the paper we consider an example from [1].

Example: Consider a company has the following short-term financing problem:

Month	Jan	Feb	Mar	Apr	May	Jun
Net cash-flow	-150	-100	200	-200	50	300

Net cash-flow requirements are given in thousands of dollars. The company has the following sources of funds:

- a line of credit of up to \$100k at an interest rate of 1% per month;
- in any one of the first three months, it can issue 90-day commercial paper bearing a total interest of 2% for the three-month period;
- excess funds can be invested at an interest rate of 0.3% per month.

Following [1], we use the following decision variables: the amount x_i drawn from the line of credit in month i, the amount y_i of commercial paper issued in month i, the excess funds z_i in month i and the companies wealth v in June.

Here we have three types of constraints: (i) cash inflow = cash outflow for each month, (ii) upper bounds on x_i, and (iii) nonnegativity of the decision variables x_i, y_i and z_i. We remind that x_i is the balance on the credit line in month i, not the incremental borrowing in month i. Similarly, z_i represents the overall excess funds in month i. For the detail of formulation we refer to [1]. The LO model of this problem in standard form is as follows;

$$\begin{aligned} \max \quad & v \\ & x_1 + y_1 \qquad\qquad\qquad\qquad -z_1 = 150 \\ & x_2 + y_2 \qquad -1.01x_1 + 1.003z_1 - z_2 = 100 \end{aligned}$$

$$\begin{aligned}
x_3 + y_3 \qquad & -1.01x_2 + 1.003z_2 - z_3 = -200 \\
x_4 - 1.02y_1 \qquad & -1.01x_3 + 1.003z_3 - z_4 = 200 \\
x_5 - 1.02y_2 \qquad & -1.01x_4 + 1.003z_4 - z_5 = -50 \\
-1.02y_3 \qquad & -1.01x_5 + 1.003z_5 - v = -300 \\
& x_1 + w_1 = 100 \\
& x_2 + w_2 = 100 \\
& x_3 + w_3 = 100 \\
& x_4 + w_4 = 100 \\
& x_5 + w_5 = 100 \\
& x_i, y_i, z_i, w_i \geq 0.
\end{aligned}$$

Replacing y_1, y_2, y_3 with x_6, x_7, x_8, respectively; $z_1,\ldots,z_5$ with $x_9,\ldots,x_{13}$; v with x_{14}; and $w_1,\ldots,w_5$ with $x_{15},\ldots,x_{19}$, we only deal with the variable vector $x \in \mathbb{R}^{19}$. Moreover, without loss of generality we can add nonnegativity of v to the problem. To have the standard form, we replace the objective function withe minimization of the negative of $v = x_{14}$. In this way the matrix A is of dimension 11×19 and the index set is {1,…,19}.

Solving this problem with the EXCEL solver leads to the following basic solution;

$$\begin{array}{lll}
x_1 = x_3 = x_4 = x_5 = 0, & x_2 = 50.98, & \\
y_1 = 150, & y_2 = 49.02, & y_3 = 203.43 \\
z_1 = z_2 = z_4 = z_5 = 0, & z_3 = 351.94, & v = 92.50 \\
w_1 = w_3 = w_4 = w_5 = 100, & w_2 = 49.02. &
\end{array} \tag{1}$$

It should be restated that there is another basic optimal solution as

$$\begin{array}{lll}
x_1 = x_2 = x_3 = x_4 = 0, & x_5 = 52, & \\
y_1 = 150, & y_2 = 100, & y_3 = 151.94 \\
z_1 = z_2 = z_4 = z_5 = 100, & z_3 = 351.94, & v = 92.50 \\
w_1 = w_2 = w_3 = w_4 = 100, & w_5 = 48, &
\end{array} \tag{2}$$

and consequently strictly optimal solution exists, say

$$\begin{array}{lll}
x_1 = x_3 = x_4 = 0, & x_2 = 18.07, & x_5 = 33.57 \\
y_1 = 150, & y_2 = 81.93, & y_3 = 170.19 \\
z_1 = z_2 = z_4 = z_5 = 0, & z_3 = 351.94, & v = 92.50 \\
w_1 = w_3 = w_4 = 100, & w_2 = 81.93, & w_5 = 66.43.
\end{array} \tag{3}$$

Having multiple optimal solutions means that the problem is dual degenerate and a strictly optimal solution reveals the optimal partition $\pi = (\mathcal{B}, \mathcal{N})$, where

$\mathcal{B}$ = {2, 5, 6, 7, 8, 11, 14, 15, 16, 17, 18, 19} and $\mathcal{N}$ = {1, 3, 4, 9, 10, 12, 13}.

The interpretation of the solution is easy. In all optimal solutions, the companies wealth v in June will be $92,500. To achieve this goal, in the basic optimal solution (1) for example, the company will issue \$150,000 in commercial paper in January, \$49,020 in February and \$203,430 in March. In addition, it will draw \$50,980 from its line of credit in February. Excess cash of \$351,940 in March will be invested for just one month. Analogous interpretation can be considered for other optimal solutions.

If we denote the dual variable by t_j, $j = 1,\ldots,11$, the unique dual optimal solution is,

$$t_1 = -1.03729 \qquad t_2 = -1.03020 \qquad t_3 = -1.02000 \qquad t_4 = -1.01695$$
$$t_5 = -1.01000 \qquad t_6 = -1 \qquad t_7 = \ldots = t_{11} = 0$$

These values are known as **shadow prices**. The nonzero dual slack variables are $s_1 = 0.00321$, $s_3 = 0.00712$, $s_4 = 0.00315$, $s_9 = 0.00400$, $s_{10} = 0.00714$, $s_{12} = 0.00392$ and $s_{13} = 0.007$, which are in complementarity with (all) primal optimal solutions. These values are known as reduced costs. Shadow prices and reduced costs play important rules in sensitivity analysis and the interpretations.

Most of solvers have tremendous information on sensitivity analysis and they have useful explanation on the problem in question such as allowable increase and decrease for the Right-Hand-Side (RHS) of each individual constraint and objective function coefficient of individual decision variable (e.g. [1]). However, there are a little published explanations when the variation of input data, say the RHS of the constraints occurs simultaneously. This can be categorized as parametric programming. In the next section we consider such cases and mention some financial description of the parametric programming.

5 Description of Parametric Programming in LO Model

To realize a concrete case, consider there is an option offered to the company as follows:

Option 1. *Reduce the net cash-flows in months January, February and April by the rate of 2, 1 and 2 respectively, and pay back them equivalently in months March, May and June with the fixed interest rates % 1.75 percent in these months.*

In this way the perturbation vector of the RHS of constraints could be

$$\Delta b = (2, 1, -1.0175, 2, -1.0175, -1.0175, 0, 0, 0, 0, 0)^T,$$

and we are interested to identify the range for the parameter value β, in the following cases.

(1) The basic optimal solution (1) is given and finding the basis invariancy interval for this solution is aimed. Simple calculation reveals that this range is [-49.14, 44.996] and the range of variation for objective value is from 193.512 to 0. It means that less negative cash-flow in months January, February and April is allowed, i.e., accepting this option increases the final companys wealth \$ 193.512 at the end of this 6-month period.

(1) When (2) is the known basic optimal solution, the basis invariancy interval is [-50.473, 44.996]. It has analogous interpretation as the other basic optimal solution. Observe that this interval is different from the obtained interval for the basic optimal solution (1). In this case, the objective function value starts from \$ 196,251 and decreases to \$ 0 when the parameter goes to the right end of the interval.

(2) When a strictly complementary primal optimal solution say (3) is given (and consequently the optimal partition π is known), and we are interested in finding the optimal partition invariancy interval. In this case, the interval is (-62.915, 44.996) and the objective function value decreases from 221,828 to 0 when the parameter value increases. The important issue here is that, the invariancy interval of the optimal partition is greater than of the basis

invariancy interval in both basic optimal solutions (1) and (2). Moreover, this interval is an open one [12], while the two others are closed.

Remark: For all parameter values in these interval specially the largest one, the dual optimal solution (i.e., the shadow prices) is valid. There are useful interpretations are mentioned in [1].

One of the interpretation of the results of sensitivity analysis is the information arises from shadow prices. We mention one of this information from [1] and make the analysis deeper.

Option 2. *Assume that the negative net cash-flow in January is due to the purchase of a machine worth \$150,000. The vendor allows the payment to be made in June at an interest rate of 3% for the five-month period. Would the companys wealth increase or decrease by using this option? What if the interest rate for the 5-month period was 4%?*

Since the shadow price of the January constraint is 1.0373, reducing cash requirements in January by \$1 increases the wealth in June by \$1.0373. In other words, the break-even interest rate for the five-month period is 3.73%. So, if the vendor charges less than this amount, we should accept, but if he/she charges more, we should not. For the exact interest value 3.73%, acceptance or rejection of the proposal has identical result.

We restate that this analysis is valid when the amount of change in the RHS of the corresponding constraint is within the allowable decrease. Observe that for this analysis the perturbed vector is $\Delta b = (1,0,0,0,0,0,0,0,0,0,0)^T$. For this perturbation vector, if the given optimal solution is (1) the basis invariancy interval is [-89.172,150]. However, if the in hand optimal solution is (2), this interval reduces to [-89.172,149.411]. It seems that the aforementioned analysis is not valid in the later case. However, the optimal partition invariancy interval for this perturbation is identical with interior of the basis partition invariancy interval (-89.172,150). Thus we can ensure the management on his decision.

Example is revisited: We saw that in this example the primal problem has multiple optimal solutions those are not degenerate. Let us change the model modestly as follows. “The credit line up to \$50k at an interest rate of 1% per month is available only at February.” With this option, only the constraints of upper bound for variables changes accordingly. In this way the problem has degenerate optimal solution:

$$\begin{array}{lll} x_1 = x_3 = x_4 = x_5 = 0, & x_2 = 50 & \\ y_1 = 150, & y_2 = 50 & y_3 = 203.438 \\ z_1 = z_2 = z_5 = 0, & z_3 = 352.938, & z_4 = 0.997 \\ w_1 = w_2 = w_3 = w_4 = w_5 = 0 & v = 92.493 & \end{array} \tag{4}$$

Observe that in this optimal solution, we only have 7 positive variables, while a basic solution needs to have 11 in this example. Therefore, 4 other variables among zero ones must be chosen to extend these positive variables to a basic one. This means that there are many finitely corresponding basis'. Let us again consider the Option 1. In this case, there might be different basis optimal partition invariancy intervals. For example, when one let corresponding basis' to basic optimal solutions (1) and (2), strangely the optimal basis invariancy for these two ones is $(-\infty,\infty)$. However, the primal support set invariancy interval of this solution is (-50,94.994), that is clearly an open interval [5].

Moreover, there is a primal strictly complementary optimal solution, say,

$$\begin{array}{llll} x_1 = x_3 = x_4 = x_5 = 0, & x_2 = 50 & & \\ y_1 = 150.356, & y_2 = 49.643, & y_3 = 203.438, & \\ z_2 = z_5 = 0, & z_1 = 0.356 & z_3 = 352.938, & z_4 = 0.634, \\ w_1 = w_2 = w_3 = w_4 = w_5 = 0, & v = 92.493. & & \end{array} \tag{5}$$

It is clear the the optimal partition for this problem is: $\pi = (\mathcal{B}, \mathcal{N})$, where

$\mathcal{B} = \{2, 6, 7, 8, 9, 11, 12, 14\}$ and $\mathcal{N} = \{1, 3, 4, 5, 10, 13, 15, 16, 17, 18, 19\}$.

Observe that this optimal solution is not basic again and not surprisingly, the optimal partition invariancy interval is exactly identical with the primal support set invariancy interval.

There is an interesting interpretation for the support set invariancy interval. Recall that in primal support set invariancy, we want to keep positivity of positive variables in the given optimal solution, while the parameter varies in the interval. In the optimal solution (4), the company will issue \$150,000 in commercial paper in January, \$50,000 in February and \$203,438 in March. In addition, it will draw \$50,000 (the maximum possible) from its line of credit in February. Excess cash of \$352,938 and \$997 in March and April will be invested, respectively.

If the company wants to keep this policy along with Option 1, i.e., investing in two months March and April whatever possible, issuing commercial paper in January, February and March in addition to drawing from the credit line in February, allowable change of the parameter is only the interval (-50, 94.994) but not the whole real line as obtained for the two corresponding basic optimal solutions.

There is another finding. For the variation of the parameter in this interval, there is no change in reduce costs and shadow prices. Because the dual optimal solution is invariant in the optimal partition invariancy interval [12]. Thus similar analysis to the Option 2 can be carried out again.

6 Conclusion

It is obvious that the inherent uncertainty is not the thing that can be answered only by the LO. However, its rich theory in sensitivity analysis and parametric programming have many things to say to economists. In this study, we only interpreted the parametric programming results when the RHS is perturbed. Analogous analysis can be carried out for the case when the objective function is perturbed [12]. Uni-parametric case can be considered when the parameter presents in both the objective function and the RHS of constraints [2]. Moreover Multi-parametric programming in LO has been studied by many authors (e.g., [3, 8, 9]) with different points of views. Developing of multi-parametric analysis of LO could be a tool to tackle some of the difficulties in many financial problems whose can be modelled as an LO problem. Interpretation of the result may clear some facts in cash-flow problem as well as other financial problems which can be formulate as an LO.

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