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On Microscopic, Macroscopic, and Kinetic Modeling of Vehicular Traffic

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Abstract A system of vehicular traffic can be modeled in various ways, such as microscopic, macroscopic, and kinetic models. Microscopic models focus on the modeling of individual cars with their deterministic or stochastic interactions. On the other hand, with macroscopic models we can determine the relation between some observable values of a vehicular system, such as density and flux, which usually have the form of partial differential equations of conservation type. In kinetic models, the traffic is resembled as a system of interacting gas particles described by a distribution function with a time evolution. This mesoscopic model is formulated in a Boltzmann-like equation, based on corresponding gain and loss terms of microscopic interactions. These above mentioned modeling approaches have actually interrelations with one another, such that microsopic behaviour of vehicles will imply certain macroscopic state of the traffic system and, by using kinetic model, the detailed transition from microscopic to macroscopic state can be explained. This paper is intended to present a study on this interrelations between microscopic, macroscopic, and kinetic models based on the optimal velocity model. Some numerical simulation results are also presented.

Keywords Vehicular Traffic; Microscopic Interaction; Macroscopic Model; Kinetic Model

2010 Mathematics Subject Classification 90B20

1 Introduction

There are three types of models which can be used to examine the dynamics of traffic flow, namely microscopic, macroscopic, and kinetic models [1,2]. A microscopic model focuses on individual cars and investigate their deterministic or stochastic interactions. Macroscopic model, on the other hand, consists of equations of aggregate quantities like spatial density, average velocity, and velocity moments, which are similar to fluid-dynamic equations. Furthermore, kinetic model describes the statistical distribution of cars with respect to their locations and velocities.

Recently it was suggested that those different types of traffic models may belong to the same universality class, in the sense, that they share qualitatively similar properties. To mention few results are the study given in [3-11]. Based on the microscopic modeling using optimal velocity function studied in [3-5], some relations of microscopic and macroscopic modeling are discussed in [6-7]. Furthermore, some traffic models derived from a gas-kinetic model are discussed in various points of view [8-11]. These reports motivate a further study on mutual relationship between different types of traffic models. This paper is aimed to study the interrelations between microscopic, macroscopic, and kinetic models based on the optimal velocity model.

2 Microscopic Vehicular Traffic Model

The discussion on microscopic model will be focused on the optimal velocity model. This model is of interest because of its ability to explain not only individual behavior of a vehicle, but also it's connectivity to some macroscopic values such as traffic flow and density. The optimal velocity model given in [3] is described by the following equation of motion of car *i*

$$\frac{d^2x_i}{dt^2} = a\left\{V(\Delta x_i) - \frac{dx_i}{dt}\right\}, \tag{1}$$

where $x_i(t)$ is the position of the i-th car at time t, $\Delta x_i = x_{i+1} - x_i$ is the headway of the i-th car to its predecessor x_{i+1}, $V(\Delta x_i)$ is the optimal velocity function on the headway Δx_i, and a is the sensitivity parameter. With this model the behavior of a given car in a traffic system is described as follows. A driver always adjusts the car velocity to approach the optimal velocity, which is determined by the observed headway. Moreover, his movement will also be influenced by the sensitivity a, which describes a time lag τ for the car to reach the optimal velocity, such that $\tau = 1/a$.

The basic properties of an optimal velocity function are as follows. Is should be a monotonic increasing function. Moreover, it should have an upper bound on maximal velocity, denoted by $V_{\max}$, which could be determined by the traffic rule or by the technical limitations of the vehicle. Generally, the graph of an optimal velocity function could take a form given in Figure 1.

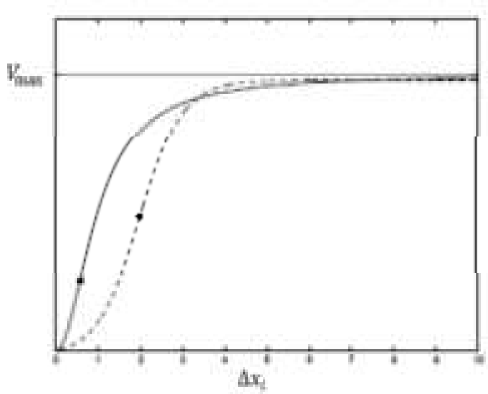

Figure 1 The general optimal velocity functions.

According to these basic properties, the following typical function can be chosen [3]

$$V(\Delta x_i) = \frac{V_{\max}}{2}\left[\tanh(\Delta x_i - x_c) + \tanh(x_c)\right] \tag{2}$$

where $V_{\max}$ is the maximal velocity mentioned above and x_c is a critical headway.

3 Macroscopic Vehicular Traffic Model

Macroscopic models of vehicular traffic according to [2] deal with macroscopic variables, such as density $\rho = \rho(x,t)$, flux $j = j(x,t)$, and average speed $v = v(x,t)$. Those variables are related by the simple identity $j = v\rho$. A flux function can take a form as the following function in terms of density

$$j(\rho) = \begin{cases} \rho V_{\max} & ; \rho < \rho_{\text{crit}} \\ \rho V_{\max} \ln\left(\frac{\rho_{\max}}{\rho}\right) \Big/ \ln\left(\frac{\rho_{\max}}{\rho_{\text{crit}}}\right) & ; \rho \geq \rho_{\text{crit}} \end{cases}$$

where $\rho_{\max}$ is a maximum density and ρ_{crit} is a critical density. A graph of flux is usually called a fundamental diagram, see Figure 2.

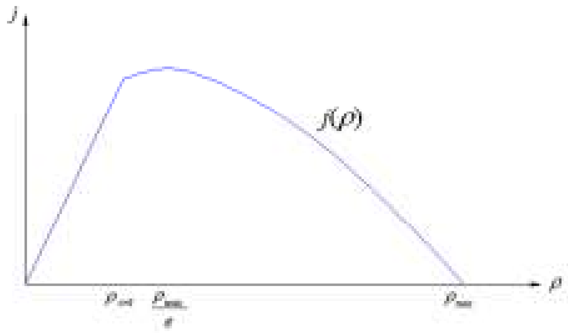

Figure 2 Fundamental diagram of traffic flux with respect to density.

Conservation law can be expressed in first-order partial differential equation with respect to density and flux, which has the following form [2,6]

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}j(x,t) = 0.$$

It is interesting to compare the dispersion relations of microscopic and macroscopic models. In a microscopic description, small perturbations with respect to the homogeneous state can be written as [6]

$$x_n(t) = v_h(t) + \frac{n}{\rho_h} + \delta x \exp(i\kappa n + \gamma t).$$

While, on the other hand, small perturbations in the macroscopic description can be written as [6]

$$\rho(x,t) = \rho_h + \delta\rho \exp(ikx + \omega t)$$
$$v(x,t) = v_h + \delta v \exp(ikx + \omega t).$$

4 Kinetic Vehicular Traffic Model

According to [9], a simple kinetic model of traffic system can be illustrated as follows. Consider a vehicular traffic system on a road with its elements are moving vehicles. Let there are three types of vehicles, which are defined as follows.

Type-0: slow moving vehicles, for example heavy trucks, which travel with velocity u;
Type-1: faster vehicles, but for some reason have to move with velocity u; for example cars behind trucks but there is no opportunity to pass the trucks;
Type-2: faster moving vehicles with velocity $v > u$.

Let further that each type of vehicles has a distribution function $f_i(x,t)$, which is defined as probability that there are certain number of vehicles at position x and at time t which are moving at certain velocity.

In order to simplify the situation, without loss of generality, let there is a constant number of type-0 vehicles. If there is no interaction between vehicles, then the vehicles move freely according its velocities, so that the time evolutions of type-1 and type-2 vehicles follow the Liouville transport equations [9]

$$\begin{aligned}(\partial_t + u \cdot \partial_x) f_1 &= 0 \\ (\partial_t + v \cdot \partial_x) f_2 &= 0.\end{aligned} \tag{3}$$

The interactions between vehicles are defined so that collisions between vehicles are avoided. Therefore, if a fast moving vehicle is approaching a slower moving vehicle, then it must decelerate or brake in order to avoid collision. But, on the other hand, if there is enough space on a multilane road, then it can pass the slower moving vehicle ahead. In the case of simpler model, these interactions are defined on the right hand side of the above transport equation (3) as follows.

i. The rate of type-2 vehicles, which have to brake, is proportional to the number of slower moving vehicles, i.e. proportional to $f_0 + f_1$. Therefore, the right hand side of (3) becomes

$$\alpha(f_0 + f_1) f_2 .$$

ii. The probability of type-1 vehicles to pass the vehicles ahead will be bigger, so that the acceleration is also bigger, when the traffic density is lower. The interaction term in the model can be defined as

$$\beta(\rho_{\max} - \rho) f_1 ,$$

with $\rho(x) = f_0 + f_1(x) + f_2(x)$ and constant maximum density $\rho_{\max}$.

As a result, the proposed simple kinetic model for the vehicular traffic system has the following form :

$$\begin{aligned} f_0 &\equiv const, \\ (\partial_t + u \cdot \partial_x) f_1 &= \alpha(f_0 + f_1) f_2 - \beta(\rho_{\max} - \rho) f_1 \\ (\partial_t + v \cdot \partial_x) f_2 &= \beta(\rho_{\max} - \rho) f - \alpha(f_0 + f_1) f_2. \end{aligned} \tag{4}$$

The domain of the equations has the following constraints:

$$f_i \geq 0, \quad f_0 + f_1 + f_2 \leq \rho_{\max} .$$

The aim of organizing a traffic system has usually concerns about how to regulate the vehicles, such that an optimal number of vehicles can pass through a certain segment of street. In that case, based on the model that has been developed, the aim can be reduced to determining how many faster moving vehicles can pass through a certain street, given there are a certain number of trucks present on the same street. Furthermore, for the sake of simplification, let the traffic system is in a homogeneous case, i.e. the interaction among vehicles is defined to be constant. Therefore, the equations (4) give

$$\alpha(f_0 + f_1) f_2 - \beta(\rho_{\max} - \rho) f_1 = 0. \tag{5}$$

Let $\rho_{\max} = 1$ and $\gamma = \dfrac{\alpha}{\beta}$, then equation (3) will give the density of type-2 vehicles as a function of type-1 vehicles as follows

$$f_2 = \frac{1 - f_0 - f_1}{\gamma\, f_0 + (1+\gamma) f_1} \cdot f_1 .$$

As a result, the mean velocity has the form

$$\frac{uf_1 + vf_2}{f_1 + f_2} = \frac{u\left(\gamma\, f_0 + (1+\gamma) f_1\right) + v(1 - f_0 - f_1)}{\gamma\, f_0 + (1+\gamma) f + (1 - f_0 - f_1)} .$$

Since the velocity of traffic flow is defined as

$$V = uf_1 + vf_2 ,$$

therefore, the resulting value of traffic flow velocity is

$$V = f_1 \left[u + \frac{1 - f_0 - f_1}{\gamma f_0 + (1+\gamma) f_1} v \right].$$

This explains that the traffic flow is mostly influenced by the slow moving vehicles in the system.

5 Simulation Results

Simulation study has been carried out to implement the microscopic, as well as kinetic models of vehicular traffic. Simulation on microscopic model gives the average velocity of the cars in the system on various values of density parameter ρ, and the result given in [10] is presented in Figure 3. Finally, the result of traffic flow velocity on simple kinetic model simulation is given in Figure 4 [11].

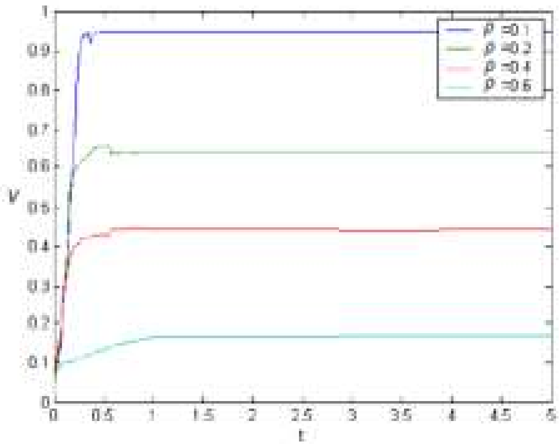

Figure 3 Microscopic model simulation of average velocity on various density ρ.

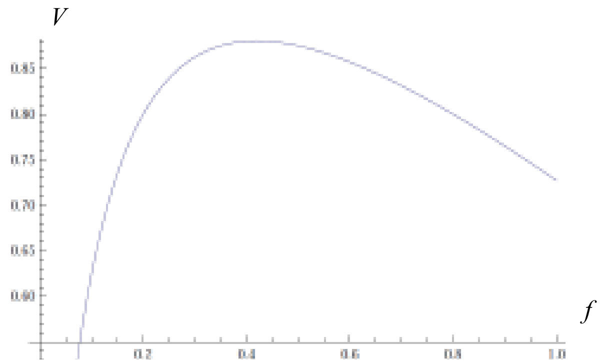

Figure 4 Simulated traffic flow velocity using simple kinetic model.

Fig 3 shows that on a low density, the average velocity of individual car tend to be high, compared to the opposite case that high density will imply low velocity. This explains quite well the real traffic situation. Moreover, Fig 4 shows a simple kinetic model simulation on traffic flow velocity as a function of distribution of slow moving vehicles. This result explains that there is a certain level of slow moving vehicles that will give a maximal velocity of the traffic flow. Afterwards, the higher probability that slower moving vehicles present on the street, the lower traffic velocity will be.

6 Concluding Remarks

This paper shows that vehicular traffic system can be modeled quite well using microscopic, macroscopic, as well as kinetic model. The simulation results can also well explain the real situation of vehicular traffic.

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