Co-prime Probability for Nonabelian Metabelian Groups of Order Less than 24 and Their Related Graphs

Nurfarah Zulkifli[*] **and Nor Muhainiah Mohd Ali**

Department of Mathematical Sciences, Universiti Teknologi Malaysia
81310 UTM Johor Bahru, Malaysia
[*]Corresponding author: nurfarahzulkifli@yahoo.com

Abstract Let G be a finite group. The probability of a random pair of elements in G are said to be co-prime when the greatest common divisor of order x and y where x and y in G, is equal to one. Meanwhile the co-prime graph of a group is defined as a graph whose vertices are elements of G and two distinct vertices are adjacent if and only if the greatest common divisor of order x and y is equal to one. In this paper, the co-prime probability and its graphs such as the types and the properties of the graph are determined.

Keywords Co-prime probability; co-prime graph; number of edges; types of graph; dominating number; independent number.

Mathematics Subject Classification 20K99.

1 Introduction

A simple definition on probability is how likely something is to happen. Mathematically, the probability of an event to happen is a number of ways it happens divided by total number of the outcomes.

For the past few years, researches on commutativity degree focuses on nonabelian metabelian groups have been done. It started by Erdos and Turan [1] in 1968, whereby the research of commutativity degree focuses only on symmetric group, S_n. Later, this research has been further explored by studying the nth commutativity degree, the multiplicative degree, the submultiplicative degree, the relative commutativity degree and many more. In 2017, Abd Rhani [2] is then extended the commutativity degree of G to co-prime probability of G. This probability is defined as the probability of the order of a random pair of elements in the group are relatively prime or co-prime. In Abd Rhani [2], the co-prime probability for all $p-$ groups and for some dihedral groups has been determined.

As for the graph part, the study of the co-prime graph for nonabelian metabelian groups of order less than 24 started with prime graph of G introduced by Williams [3] in 1981. Later, the study of prime graph is then continuing to determine the co-prime graph of integers by

Erdos and Sarkozy [4] and Sarkozy [5]. In 2014, Ma *et al.* [6] then extended the research by introducing the co-prime graph of G for finite group. The types and some properties of the graphs were then determined. Unfortunately, none did a research on both co-prime probability and its graphs for nonabelian metabelian groups of order less than 24.

In this paper, the co-prime probability for nonabelian metabelian groups of order less than 24 and their related graphs are discussed. The application of this research can also be seen in so many ways. It will give some contributions to the field of group and graph theory as this is a continuation study from the previous researchers. Graph theory also can be very useful in real life whereby the graph can be used as a solution to a problem. For example, in 1735, Konisberg bridge problem is solved by using the concept of planar and Eulerian graph. Another example of the application of graph is in business management whereby the star graph is related to a model named hub and spoke model. Therefore, in this paper, the co-prime probability for nonabelian metabelian groups of order less than 24 are discussed. Then, the graph for each group is identified. For each graph, the total number of edges, the types of graph and the properties of the graph such as the dominating number and the independent number are determined.

Hence, this paper is structured as follows: the first part is the introduction. The second part states the basic concepts, definitions, propositions and theorems that are used for this paper and lastly, the main results and the conclusion of this paper on both the co-prime probability and its graphs for nonabelian metabelian groups of order less than 24 are discussed.

2   Preliminaries on Groups and Graphs

In this section, some basic concepts on group and graph theory are stated.

Definition 1   [7] *A group G is metabelian if there exists a normal subgroup A such that both A and G/A are abelian.*

Definition 2   [8] *Dihedral groups of degree n*
For each $n \in \mathbb{Z}$ and $n \geq 3$, D_n denoted as the set of symmetries of a regular $n-gon$. Furthermore, the order of D_n is $2n$ or equivalently $|D_n| = 2n$. The Dihedral groups, D_n can be represent in a form of generators and relations given in the following representation:

$$D_n = \langle a, b | a^n = 1, b^2 = 1, ab = a^{-1}b \rangle.$$

Definition 3   [9] *$p-group$*
A $p-group$ is defined as the group that has its order as prime power, labeled as p^n, where p is prime and n is non-negative integers.

Definition 4   [2] *Let G be a finite group. For any $x, y \in G$, the co-prime probability of G, denoted as $P_{copr}(G)$ is defined as:*

$$P_{copr}(G) = \frac{|\{(x,y) \in G \times G, (|x|, |y|) = 1\}|}{|G|^2}.$$

Proposition 1 [2] *Let G be a finite p-group of order p^n where $n \geq 1$. Then*

$$P_{copr}(G) = \frac{2p^n - 1}{p^{2n}}.$$

Proposition 2 [2] *Let D_n be a dihedral group of order 2n, where $n \geq 3$ and n is odd. Then*

$$P_{copr}(G) = \frac{2n^2 + 2n - 1}{4n^2}.$$

Definition 5 [10] *A graph Γ is connected if each pair of the vertices are joined by a path.*

Definition 6 [10] *The degree of vertex x, $deg(x)$ is the number of edges incident with x, each loop counting as two edges.*

Definition 7 [10] *The dominating set $X \subseteq V(\Gamma)$ is a set where for each v outside W there exist $x \in X$ such that v is adjacent to x. The minimum size of x is called the dominating number and it is denoted by $\gamma(\Gamma)$.*

Definition 8 [10] *A non-empty S of $V(\Gamma)$ is called an independent set of Γ if there is no adjacent between two elements of S in Γ. Thus the independent number is the number of vertices in maximum independent set and it is denoted as $\alpha(\Gamma)$.*

Definition 9 [11] *A star graph $K_{1,n}$ is a graph which consists of a single vertex with n neighbours.*

Definition 10 [11] *A planar graph is a graph that can be drawn in a plane, so that no two lines intersect except at the vertices; otherwise it is said to be non-planar.*

Definition 11 [11] *A bipartite graph or a bigraph, $K_{m,n}$ is a set of vertices partitioned in to subsets such that there is no adjacent between two graph vertices within the same set.*

Definition 12 [12] *The complete tripartite graph, $K_{r,s,t}$ consist of three sets (of sizes r, s, t) with edges joining two vertices if and only if they lie in different set.*

Definition 13 [6] *The co-prime graph of G denoted as Γ is a graph whose vertices are elements of G and two distinct vertices u and v are adjacent if and only of $(|u|, |v|) = 1$.*

Proposition 3 [6] *Let G be a group. Then, G is not a $p-$groups if and only if Γ_G is not bipartite.*

Proposition 4 [6] *Let G be a group. If G is cyclic with order 2p for some odd prime p, then Γ_G is planar.*

Proposition 5 [6] *Let p and q be two distinct prime numbers and G be a non-cyclic group with order pq. Then Γ_G is a complete tripartite graph.*

Theorem 1 [6] *Let $\Gamma_{D_{2n}}$ be the co-prime graph of D_{2n} and let n be odd. Then*
(1) $\deg_{\Gamma_{D_{2n}}}(sr^i) = n$ *for any* $1 \le i \le n$;
(2) $\deg_{\Gamma_{D_{2n}}}(r^i) \le n$ *for any* $1 \le i \le n$;
(3) $\Gamma_{D_{2n}}$ *is not Eulerian;*
(4) $\Gamma_{D_{2n}}$ *is Hamiltonian;*
(5) $\Gamma_{D_{2n}}$ *is not planar.*

Proposition 6 [6] *Let G be a group with order greater than 2. Then, e is unique dominating set of size 1 of Γ_G. In particular, $\gamma(\Gamma) = 1$ and $\deg_{\Gamma_G}(e) = |G| - 1$.*

3 Co-prime Probability with Their Related Graphs for Nonabelian Metabelian Groups of Order Less than 24

The first part of this section focuses on computing the co-prime probability for nonabelian metabelian groups of order less than 24 meanwhile the second part explains the co-prime graph, their types and the properties of the graph. As for the properties of the graph, the focus is on the dominating and independent number only.

3.1 Co-prime Probability for Nonabelian Metabelian Groups of Order Less than 24

In this subsection, the co-prime probability for nonabelian metabelian groups of order less than 24 are determined. Firstly, Cayley table generated by Maple 2016 software is used to find the order for each element in the group. Then, the co-prime probability is identified by applying Definition 4. The following four lemmas explain the finding of co-prime probability for nonabelian metabelian groups of order 12, 18, 20 and 21, respectively.

Lemma 1 *Let G be a nonabelian metabelian group of order 12. Then,*

$$
P_{copr}(G) = \begin{cases} ^{51}/_{144}, & \text{if } G = \mathbb{Z}_3 \rtimes \mathbb{Z}_4 \text{ and } D_6, \\ ^{71}/_{144}, & \text{if } G = A_4. \end{cases} \tag{1}
$$

Proof Let $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$. Then, $G = \langle a, b | a^4 = b^3 = 1, bab = a \rangle = \{e, a, a^2, a^3, b, ab, a^2b, a^3b, ba, aba, a^2ba, a^3ba\}$. Firstly, the Cayley table of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ is formed to determine the order for each element in $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. The Cayley table of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ is shown in Figure 1.

It can be found that $|e| = 1$, $|a^2| = 2$, $|b| = |a^3ba| = 3$, $|a| = |a^3| = |ab| = |a^3b| = |ba| = |a^2ba| = 4$ and $|a^2b| = |aba| = 6$. In order to find the $P_{copr}(G)$, three cases that should be considered are given as follows:
Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. So, $|N_1| = 12$. Now, let $y = e$, then y is also co-prime to each element x in $G\backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G\backslash e \times G\}$. So, $|N_2| = 11$.
Case 2 : Let $N_3 = \{(x, y) \in G \times G| \ |x| = 2, |y| = 3\}$, then $x = a^2$ and $y \in \{b, a^3ba\}$. This implies $|N_3| = 2$. Next, let $N_4 = \{(x, y) \in G \times G| \ |x| = 3, |y| = 2\}$, then $x \in \{b, a^3ba\}$ and $y = a^2$. Thus, $|N_4| = 2$.
Case 3 : Let $N_5 = \{(x, y) \in G \times G| \ |x| = 3, |y| = 4\}$, then $x \in \{b, a^3ba\}$ and $y \in$

Figure 1: The Cayley Table of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

$\{a, a^3, ab, a^3b, ba, a^2ba\}$. This implies $|N_5| = 12$. Now, let $N_6 = \{(x, y) \in G \times G| \ |x| = 4, |y| = 3\}$, then $x \in \{a, a^3, ab, a^3b, ba, a^2ba\}$ and $y \in \{b, a^3ba\}$. Thus, $|N_6| = 12$.
Hence,

$$P_{copr}(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) = \frac{|N_1| + |N_2| + |N_3| + |N_4| + |N_5| + |N_6|}{|\mathbb{Z}_3 \rtimes \mathbb{Z}_4|^2} = \frac{51}{144}.$$

Now, let $G = D_6$. Then, $G = \langle a, b|a^6 = b^2 = 1, bab = a^{-1} \rangle = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, \ a^5b\}$. By referring to Cayley table of D_6, it can be found that $|e| = 1, |a^3| = |b| = |ab| = |a^2b| = |a^3b| = |a^4b| = |a^5b| = 2, |a^2| = |a^4| = 3$ and $|a| = |a^5| = 6$. In order to find $P_{copr}(G)$, below are the cases that should be considered:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 12$. Now, let $y = e$, then y is also co-prime to all x in $G\backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G\backslash e \times G\}$. Then, $|N_2| = 11$.
Case 2 : Let $N_3 = \{(x, y) \in G \times G| \ |x| = 2, |y| = 3\}$, then $x \in \{a^3, b, ab, a^2b, a^3b, a^4b, a^5b\}$ and $y \in \{a^2, a^4\}$. This implies $|N_3| = 14$. Next, let $N_4 = \{(x, y) \in G \times G| \ |x| = 3, |y| = 2\}$, then $x \in \{a^2, a^4\}$ and $y \in \{a^3, b, ab, a^2b, a^3b, a^4b, a^5b\}$. Thus, $|N_4| = 14$.
Hence,

$$P_{copr}(D_6) = \frac{|N_1 + N_2 + N_3 + N_4|}{|A_4|^2} = \frac{51}{144}.$$

Next, let $G = A_4$. Then, $G = \langle a, b, c|a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac \rangle = \{e, a, b, c, c^2, ab, ac, ac^2, bc, bc^2, abc, abc^2\}$. By referring to Cayley table of A_4, the order of each element is $|e| = 1, |a| = |b| = |ab| = 2$ and $|c| = |c^2| = |ac| = |ac^2| = |bc| = |bc^2| = |abc| = |abc^2| = 3$. The co-prime probability of G is calculated as follows:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 12$. Now, if $y = e$, then y is also co-prime to all x in $G\backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G\backslash e \times G\}$. Then, $|N_2| = 11$.

Case 2 : Let $N_3 = \{(x,y) \in G \times G | \, |x| = 2, |y| = 3\}$, then $x \in \{a, b, ab\}$ and $y \in \{c, c^2, ac, ac^2, bc, bc^2, abc, abc^2\}$. This implies $|N_3| = 24$. Next, let $N_4 = \{(x,y) \in G \times G | \, |x| = 3, |y| = 2\}$, then $x \in \{c, c^2, ac, ac^2, bc, bc^2, abc, abc^2\}$ and $y \in \{a, b, ab\}$. Thus, $|N_4| = 24$. Hence,

$$P_{copr}(A_4) = \frac{|N_1| + |N_2| + |N_3| + |N_4|}{|A_4|^2} = \frac{71}{144}. \qquad \qquad \square$$

Lemma 2 *Let G be nonabelian metabelian group of order 18. Then,*

$$P_{copr}(G) = \begin{cases} 83/324, & \text{if } G = S_3 \times \mathbb{Z}_3, \\ 179/324, & \text{if } G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2. \end{cases} \qquad (2)$$

Proof Let $G = S_3 \times \mathbb{Z}_3$. Then, $G = \langle a, b, c | a^3 = b^2 = c^3 = 1, bab = a^{-1}, ac = ca, bc = cb \rangle = \{e, a, a^2, \quad b, c, c^2, ab, a^2b, ac, ac^2, a^2c, a^2c^2, bc, bc^2, abc, abc^2, a^2bc, a^2bc^2\}$. By referring to Cayley table of $S_3 \times \mathbb{Z}_3$, the order for each element in $S_3 \times \mathbb{Z}_3$ are $|e| = 1$, $|b| = |ab| = |a^2b| = 2$, $|a| = |a^2| = |c| = |c^2| = |ac| = |ac^2| = |a^2c| = |a^2c^2| = 3$ and $|bc| = |bc^2| = |abc| = |abc^2| = |a^2bc| = |a^2bc^2| = 6$. Below are the cases that should be considered in order to find $P_{copr}(G)$:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 18$. Now, if $y = e$, then y is also co-prime to all x in $G \backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G \backslash e \times G\}$. Then, $|N_2| = 17$.

Case 2 : Let $N_3 = \{(x,y) \in G \times G | \, |x| = 2, |y| = 3\}$, then $x \in \{b, ab, a^2b\}$ and $y \in \{a, a^2, c, c^2, ac, ac^2, a^2c, a^2c^2\}$. This implies $|N_3| = 24$. Next, let $N_4 = \{(x,y) \in G \times G | \, |x| = 3, |y| = 2\}$, then $x \in \{a, a^2, c, c^2, ac, ac^2, a^2c, a^2c^2\}$ and $y \in \{b, ab, a^2b\}$. Thus, $|N_4| = 24$. Hence,

$$P_{copr}(S_3 \times \mathbb{Z}_3) = \frac{|N_1| + |N_2| + |N_3| + |N_4|}{|S_3 \times \mathbb{Z}_3|^2} = \frac{83}{324}.$$

Next, let $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Then, $G = \langle a, b, c | a^2 = b^3 = c^3 = 1, bc = cb, bab = a, cac = a \rangle = \{e, a, b, aba, c, aca, ab, ba, ac, ca, bc, baca, abac, abca, \, abc, abaca, bac, bca\}$. So, from the Cayley table of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, it can be found that $|e| = 1$, $|a| = |ab| = |ba| = |ac| = |ca| = |abc| = |abaca| = |bac| = |bca| = 2$ and $|b| = |aba| = |c| = |aca| = |bc| = |baca| = |abac| = |abca| = 3$. Two cases that need to be considered to find the co-prime probability of G are as follows:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|x|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 18$. Now, if $y = e$, then y is also co-prime to all x in $G \backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G \backslash e \times G\}$. Then, $|N_2| = 17$.

Case 2 : Let $N_3 = \{(x,y) \in G \times G | \, |x| = 2, |y| = 3\}$, then $x \in \{a, ab, ba, ac, ca, abc, abaca, bac, bca\}$ and $y \in \{b, aba, c, aca, bc, baca, abac, abca\}$. This implies $|N_3| = 72$. Next, let $N_4 = \{(x,y) \in G \times G | \, |x| = 3, |y| = 2\}$, then $x \in \{b, aba, c, aca, bc, baca, abac, abca\}$ and $y \in \{a, ab, ba, ac, ca, abc, abaca, bac, bca\}$. Thus, $|N_4| = 72$. Hence,

$$P_{copr}((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) = \frac{|N_1| + |N_2| + |N_3| + |N_4|}{|(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2|^2} = \frac{179}{324}. \qquad \square$$

Lemma 3 *Let G be nonabelian metabelian group of order 20. Then,*

$$P_{copr}(G) = \frac{127}{400}, \ if \ G = D_{10}, Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4 \ and \ \mathbb{Z}_5 \rtimes \mathbb{Z}_5. \tag{3}$$

Proof Suppose G is nonabelian metabelian group of order 20. Now, let $G = D_{10}$ and $G = \langle a, b | a^{10} = b^2 = 1, bab = a^{-1} \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7, a^8b, a^9b\}$. By referring to Cayley table of D_{10}, the order for each element is $|e| = 1$, $|a^5| = |b| = |ab| = |a^2b| = |a^3b| = |a^4b| = |a^5b| = |a^6b| = |a^7b| = |a^8b| = |a^9b| = 2$, $|a^2| = |a^4| = |a^6| = |a^8| = 5$ and $|a| = |a^3| = |a^7| = |a^9| = 10$. In order to find $P_{copr}(G)$, below are the cases that should be considered:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 20$. Now, if $y = e$, then y is also co-prime to all x in $G \backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G \backslash e \times G\}$. Then, $|N_2| = 19$.
Case 2 : Let $N_3 = \{(x, y) \in G \times G | |x| = 2, |y| = 5\}$, then $x \in \{a^5, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b\}$ and $y \in \{a^2, a^4, a^6, a^8\}$. This implies $|N_3| = 44$. Next, let $N_4 = \{(x, y) \in G \times G | |x| = 5, |y| = 2\}$, then $x \in \{a^2, a^4, a^6, a^8\}$ and $y \in \{a^5, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7, a^8b, a^9b\}$. Thus, $|N_4| = 44$. Hence,

$$P_{copr}(D_{10}) = \frac{|N_1| + |N_2| + |N_3| + |N_4|}{|D_{10}|^2} = \frac{127}{400}.$$

Next, let $G = Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. So, $G = \langle a, b | a^5 = b^4 = 1, aba = b \rangle = \{e, a, a^2, a^3, a^4, b, b^2, b^3, ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3, a^3b, a^3b^2, a^3b^3, a^4b, a^4b^2, a^4b^3\}$. Then, the order for each element is $|e| = 1$, $|b^2| = 2$, $|b| = |b^3| = |ab| = |ab^3| = |a^2b| = |a^2b^3| = |a^3b| = |a^3b^3| = |a^4b| = |a^4b^3| = 4$, $|a^2| = |a^4| = |a| = |a^3| = 5$ and $|ab^2| = |a^2b^2| = |a^3b^2| = 10$. Below are the cases that should be considered:

Case 1 : If $x = e$, then x is co-prime to each element y in G, since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 20$. Now, if $y = e$, then y is also co-prime to all x in $G \backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G \backslash e \times G\}$. Then, $|N_2| = 19$.
Case 2 : Let $N_3 = \{(x, y) \in G \times G | |x| = 2, |y| = 5\}$, then $x \in \{b^2\}$ and $y \in \{a^2, a^4, a, a^3\}$. This implies $|N_3| = 4$. Next, let $N_4 = \{(x, y) \in G \times G | |x| = 5, |y| = 2\}$, then $x \in \{a^2, a^4, a, a^3\}$ and $y \in \{b^2\}$. Thus, $|N_4| = 4$.
Case 3 : Let $N_5 = \{(x, y) \in G \times G | |x| = 4, |y| = 5\}$, then $x \in \{b, b^3, ab, ab^3, a^2b, a^2b^3, a^3b, a^3b^3, a^4b, a^4b^3\}$ and $y \in \{a^2, a^4, a, a^3\}$. This implies $|N_5| = 40$. Now, let $N_6 = \{(x, y) \in G \times G | |x| = 5, |y| = 4\}$, then $x \in \{a^2, a^4, a, a^3\}$ and $y \in \{b, b^3, ab, ab^3, a^2b, a^2b^3, a^3b, a^3b^3, a^4b, a^4b^3\}$. Thus, $|N_6| = 40$. Hence,

$$P_{copr}(Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4) = \frac{|N_1| + |N_2| + |N_3| + |N_4| + |N_5| + |N_6|}{|Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4|^2} = \frac{127}{400}.$$

Now, let $G = \mathbb{Z}_5 \rtimes \mathbb{Z}_5$ where $G = \langle a, b | a^5 = b^4 = 1, aba = b \rangle = \{e, a, a^2, a^3, b, b^2, a^3b^2a, a^3ba, ab, ab^2, b^2a, ba, a^2b, a^2b^2, ab^2a, aba, a^3b, a^3b^2, a^2b^2a, a^2ba\}$. Therefore, from the Cayley table of $\mathbb{Z}_5 \rtimes \mathbb{Z}_5$, it can be found that $|e| = 1$, $|a^2| = 2$, $|a| = |a^3| = |ab| = |ab^2| = |ab^3| = |ab^4| = |a^3b| = $

$|a^3b^2| = |a^3b^3| = |a^3b^4| = 4$, $|b^2| = |b| = |a^3ba| = |a^3b^2a| = 5$ and $|a^2b| = |a^2b^2| = |ab^2a| = |aba| = 10$. In order to find $P_{copr}(G)$, below are the cases that should be considered:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 20$. Now, if $y = e$, then y is also co-prime to each element x in $G \backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G \backslash e \times G\}$. Then, $|N_2| = 19$.

Case 2 : Let $N_3 = \{(x, y) \in G \times G| \ |x| = 2, |y| = 5\}$, then $x \in \{a^2\}$ and $y \in \{b^2, b, a^3ba, a^3b^2a\}$. This implies $|N_3| = 4$. Next, let $N_4 = \{(x, y) \in G \times G| \ |x| = 5, |y| = 2\}$, then $x \in \{b^2, b, a^3ba, a^3b^2a\}$ and $y \in \{a^2\}$. Thus, $|N_4| = 4$.

Case 3 : Let $N_5 = \{(x, y) \in G \times G| \ |x| = 4, |y| = 5\}$, then $x \in \{a, a^3, ab, ab^2, ab^3, ab^4, a^3b, a^3b^2, a^3b^3, a^3b^4\}$ and $y \in \{b^2, b, a^3ba, a^3b^2a\}$. This implies $|N_5| = 40$. Now, Let $N_6 = \{(x, y) \in G \times G| \ |x| = 5, |y| = 4\}$, then $x \in \{b^2, b, a^3ba, a^3b^2a\}$ and $y \in \{a, a^3, ab, ab^2, ab^3, ab^4, a^3b, a^3b^2, a^3b^3, a^3b^4\}$. Thus, $|N_6| = 40$. Hence,

$$P_{copr}(\mathbb{Z}_5 \rtimes \mathbb{Z}_5) = \frac{|N_1| + |N_2| + |N_3| + |N_4| + |N_5| + |N_6|}{|\mathbb{Z}_5 \rtimes \mathbb{Z}_5|^2} = \frac{127}{400}. \qquad \square$$

Lemma 4 *Let G be nonabelian metabelian group of order 21. Then,*

$$P_{copr}(G) = \frac{209}{441}. \tag{4}$$

Proof Suppose G is nonabelian metabelian group of order 21 where $G = Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Then, $G = \langle a, b | a^3 = b^7 = 1, ba = ab^2 \rangle = \{e, a, \ a^2, b, a^2ba, a^2bab, aba^2, aba^2b, ababa, ab, ba, bab, a^2ba^2, a^2ba^2b, a^2baba, a^2b, aba, abab, ba^2, ba^2b, baba\}$. By referring to Cayley table of $Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, it can be seen that $|e| = 1$, $|a| = |a^2| = |ab| = |ba| = |bab| = |a^2ba^2| = |a^2ba^2b| = |a^2baba| = |a^2b| = |aba| = |abab| = |ba^2| = |ba^2b| = |baba| = 3$ and $|b| = |a^2ba| = |a^2bab| = |aba^2| = |aba^2b| = |ababa| = 7$. The co-prime probability of G are calculated as follows:

Case 1 : If $x = e$, then x is co-prime to each element y in G since $(|x|, |y|) = (|e|, |y|) = 1$. Let $N_1 = \{(e, y) \in G \times G\}$. Then, $|N_1| = 21$. Now, if $y = e$, then y is also co-prime to all x in $G \backslash e$ since $(|x|, |y|) = (|x|, |e|) = 1$. Let $N_2 = \{(x, e) \in G \backslash e \times G\}$. Then, $|N_2| = 20$.

Case 2 : Let $N_3 = \{(x, y) \in G \times G| \ |x| = 7, |y| = 3\}$, then $x \in \{b, a^2ba, a^2bab, aba^2, aba^2b, ababa\}$ and $y \in \{a, a^2, ab, ba, bab, a^2ba^2, a^2ba^2b, a^2baba, \ a^2b, aba, abab, ba^2, ba^2b, baba\}$. This implies $|N_3| = 84$. Next, let $N_4 = \{(x, y) \in G \times G| \ |x| = 3, |y| = 7\}$, then $x \in \{a, a^2, ab, ba, bab, a^2ba^2, a^2ba^2b, \ a^2baba, a^2b, aba, abab, ba^2, ba^2b, baba\}$ and $y \in \{b, a^2ba, a^2bab, aba^2, aba^2b, ababa\}$. Thus, $|N_4| = 84$.

Hence,

$$P_{copr}(Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3) = \frac{|N_1| + |N_2| + |N_3| + |N_4|}{|Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3|^2} = \frac{209}{441}. \qquad \square$$

Next, from the previous lemmas and propositions by Abd Rhani in [2], Theorem 2 is then giving the co-prime probability for nonabelian metabelian groups of order less than 24.

Theorem 2

Let G be nonabelian metabelian group of order at most 24. Then,

$$
P_{copr}(G) = \begin{cases}
23/36, & \text{if } |G| = 6, \\
15/64, & \text{if } |G| = 8, \\
59/100, & \text{if } |G| = 10, \\
51/144 \ \text{or } 71/144, & \text{if } |G| = 12, \\
111/196, & \text{if } |G| = 14, \\
31/256, & \text{if } |G| = 16, \\
179/324 \ \text{or } 83/324, & \text{if } |G| = 18, \\
127/400, & \text{if } |G| = 20, \\
209/441, & \text{if } |G| = 21, \\
263/484, & \text{if } |G| = 22.
\end{cases} \tag{5}
$$

Proof The co-prime probability of these groups is proved by using Proposition 1 and 2 which have been determined by Abd Rhani in [2] whereas Lemma 1, 2, 3 and 4 are proved by using Definition 4 which is also has been determined by Abd Rhani [2]. Therefore, when $|G| = 6$, 10, 14, and 22, the co-prime probability of G are proven by Proposition 2. When $|G| = 8$ and 16, Proposition 1 proved the co-prime probability of G. Next, when $|G| = 12$, 20 and 21, the co-prime probability of G are proven by Lemma 1, Lemma 2 and Lemma 4 respectively. Lastly, when $|G| = 18$, the co-prime probability for groups is proven by using Lemma 3 and Proposition 2. □

3.2 Co-prime Graph for Nonabelian Metabelian Groups of Order Less than 24

In 2014, Ma et al. [6] have proven the co-prime graph, their types and properties, where the focus is on the finite group. So, a thorough understanding can be attained as the focus of this paper concentrates only on nonabelian metabelian groups of order less than 24. Therefore, the co-prime graph, their types and the dominating number are explained. Apart from that, the independent number will also be determined as Ma *et al.* [6] did none on the independent number. The co-prime graph, the types of the graph and the properties of the graph is explained next when $G = D_4$, D_5 and A_4. Here, D_4, D_5 and A_4 are chosen as examples of types of groups where D_4, D_5 and A_4 is a $p-$group, dihedral group or neither $p-$group nor dihedral group respectively. These groups are used to determine the graphs, the types of graph, the number of edges, the dominating and the independent number.

3.2.1 The Co-prime Graph of D_4

Let $D_4 = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ and $|D_4| = 8 = 2^3$. Therefore, the co-prime graph is illustrated below.

Hence, the total number of edges that connect to every vertex is 7. It can also be concluded that the graph is a star which is also a bipartite graph from Proposition 3 as well as planar by Definition 10. Also, for the dominating number, $\gamma(\Gamma_{copr}(D_4)) = 1$ as it follows from Proposition 6. As for the independent number, from Definition 8, the possible independent sets are as
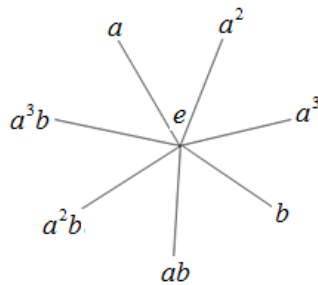
Figure 2: The Co-prime Graph of D_4

follows;

Size 2 :$\{a, a^3\}, \{a, a^2\}, \{a, b\}, \ldots$

Size 3 :$\{a, a^2, a^3\}, \{a, a^2, b\}, \{a, a^2, ab\}, \ldots$

Size 4 :$\{a, a^2, a^3, b\}, \{a, a^2, a^3, a^2b\}, \{a, a^2, a^3, a^3b\}, \ldots$

Size 5 :$\{a, a^2, a^3, b, ab\}, \{a, a^2, a^3, b, a^2b\}, \{a, a^2, a^3, b, a^3b\}, \ldots$

Size 6 :$\{a, a^2, a^3, b, ab, a^2b\}, \{a, a^2, a^3, b, ab, a^3b\}, \{a^2, a^3, b, ab, a^2b, a^3b\}, \ldots$

Size 7 :$\{a, a^2, a^3, b, ab, a^2b, a^3b\}$.

Therefore, $\alpha(\Gamma) = 7$ since the largest independent set is $\{a, a^2, a^3, b, ab, a^2b, a^3b\}$. The same steps are repeated when $|G| = 8$ and 16 as G is a $p-$ group of order p^n, where $n \geq 1$.

3.2.2   The Co-prime Graph of D_5

Let $D_5 = \langle a, b | a^5 = b^2 = 1, bab = a^{-1} \rangle = \{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ and D_5 is a dihedral group with $n = 5$. Therefore, the co-prime graph is illustrated below.
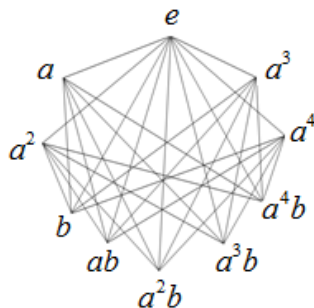

Figure 3: The Co-prime Graph of D_5

Hence, the total number of edges that connect to every vertex is 29. It can also be concluded that the graph is a complete tripartite graph by using Proposition 5. Also, for the dominating number, $\gamma(\Gamma_{copr}(D_5)) = 1$ as it follows from Proposition 6. As for the independent number, from Definition 8, the possible independent sets are as follows;

Size 2 :$\{a, a^3\}, \{a, a^4\}, \{a, a^2\}, \ldots$

Size 3 :$\{b, ab, a^2b\}, \{b, ab, a^3b\}, \{b, ab, a^4b\}, \ldots$

Size 4 :$\{b, ab, a^2b, a^3b\}, \{b, ab, a^2b, a^4b\}, \{a, a^2, a^3, a^4\}, \ldots$

Size 5 :$\{b, ab, a^2b, a^3b, a^4b\}$.

Therefore, $\alpha(\Gamma) = 5$ since the largest independent set is $\{b, ab, a^2b, a^3b, a^4b\}$. The same steps are repeated when $|G| = 6$, 10, 14, 18 and 22 as G is a dihedral group, D_n, whereby $n \geq 3$ and n is odd.

3.2.3 The Co-prime Graph of A_4

Let $A_4 = \langle a, b, c | a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac \rangle = \{e, a, b, c, c^2, ab, ac, ac^2, bc, bc^2, abc, abc^2\}$. Therefore, the co-prime graph is illustrated below.
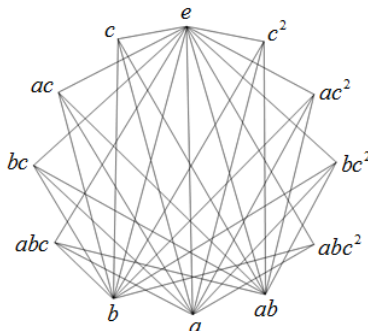

Figure 4: The Co-prime graph of A_4

Hence, the total number of edges that connect to every vertex is 35. It can also be concluded that the graph is a complete tripartite graph based on Proposition 5. Also, for the dominating number, $\gamma(\Gamma_{copr}(A_4)) = 1$ as it follows from Proposition 6. As for the independent number, from Definition 8, the possible independent sets are as follows;

Size 2 :$\{c, ac\}, \{c, bc\}, \{a, b\}, \ldots$
Size 3 :$\{c, ac, bc\}, \{c, ac, abc\}, \{c, ac, c^2\}, \ldots$
Size 4 :$\{c, ac, bc, abc\}, \{c, ac, bc, c^2\}, \{c, ac, bc, ac^2\}, \ldots$
Size 5 :$\{c, ac, bc, abc, c^2\}, \{c, ac, bc, abc, ac^2\}, \{c, ac, bc, abc, bc^2\}, \ldots$
Size 6 :$\{c, ac, bc, abc, c^2, ac^2\}, \{c, ac, bc, abc, c^2, bc^2\}, \{c, ac, bc, abc, c^2, abc^2\}, \ldots$
Size 7 :$\{c, ac, bc, abc, c^2, ac^2, bc^2\}, \{c, ac, bc, abc, c^2, ac^2, abc^2\}, \ldots$
Size 8 :$\{c, ac, bc, abc, c^2, ac^2, bc^2, abc^2\}$.

Therefore, $\alpha(\Gamma) = 8$ since the largest independent set is $\{c, ac, bc, abc, c^2, ac^2, bc^2, abc^2\}$. Other groups that are neither $p-$group of order p^n nor $G = D_n$ where $n \geq 3$ and n is odd is proved as above.

The complete results of the co-prime graph for nonabelian metabelian groups of order less than 24 can be referred to Table 1. The table below indicates the number of edges, the types of graph, the dominating number and the independent number which used the definitions, some propositions and theorem by Ma *et al.* [6].

4 Conclusion

In this paper, the co-prime probability for nonabelian metabelian groups of order less than 24 and their related graphs whereby the number of edges, the types of graph together with its properties such as the dominating number and independent number are found. It can be

concluded that the co-prime probability of dihedral groups or $p-$ group are the same when G has the same order for nonabelian metabelian groups of order less than 24. Same goes to the number of edges and the independent number. When G is neither dihedral group nor $p-$ group for nonabelian metabelian groups of order less than 24, the results for the co-prime probability, the number of edges and the independent number varies. Next, the dominating number for all groups for nonabelian metabelian groups of order less than 24 for the co-prime graph are equal to one. As for the types of graph, it can be concluded that when G is a $p-$ group of order p^n, where $n \geq 1$, the types of graph are star, bipartite and planar. When $G = D_n$ with order $2n$ where $n \geq 3$ and n is odd, the types of graph are either complete tripartite or not bipartite, not planar, not Eulerian, but Hamiltonian. As for the other groups that are neither $p-$ group nor dihedral group, the types of graph are either not bipartite or complete tripartite

Table 1: Table of Co-prime Graph for Nonabelian Metabelian Groups of Order Less than 24

No	G	$|G|$	Number of Edges	$\alpha(\Gamma)$	$\gamma(\Gamma)$	Type of graph
1	D_3	6	11	1	3	Complete tripartite
2	D_4	8	7	1	7	Star, bipartite, planar
3	$Q = Quarternion$	8	7	1	7	Star, bipartite, planar
4	D_5	10	29	1	5	Complete tripartite
5	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	12	25	1	9	Not bipartite
6	A_4	12	35	1	8	Complete tripartite
7	D_6	12	25	1	9	Not bipartite
8	D_7	14	55	1	7	Complete tripartite
9	D_8	16	15	1	15	Star, bipartite, planar
10	$Quasihedral-16$	16	15	1	15	Star, bipartite, planar
11	Q_8	16	15	1	15	Star, bipartite, planar
12	$D_4 \times \mathbb{Z}_2$	16	15	1	15	Star, bipartite, planar
13	$Q \times \mathbb{Z}_2$	16	15	1	15	Star, bipartite, planar
14	$Modular-16$	16	15	1	15	Star, bipartite, planar
15	B	16	15	1	15	Star, bipartite, planar
16	K	16	15	1	15	Star, bipartite, planar
17	$G_{4,4}$	16	15	1	15	Star, bipartite, planar
18	D_9	18	89	1	9	Not planar Not Eulerian, not planar but Hamiltonian Not bipartite
19	$S_3 \times \mathbb{Z}_3$	18	41	1	14	Not bipartite
20	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	18	89	1	9	Complete tripartite
21	D_{10}	20	63	1	15	Nor bipartite
22	$Fr_{20} \cong (\mathbb{Z}_5 \rtimes \mathbb{Z}_2)$	20	63	1	15	Nor bipartite
23	$\mathbb{Z}_5 \rtimes \mathbb{Z}_5$	20	63	1	15	Nor bipartite
24	$Fr_{21} \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$	21	104	1	14	Complete tripartite
25	D_{11}	22	134	1	11	Complete tripartite

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